Unified form for parallel ion viscous stress in magnetized plasmas

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In this work a unified form for the parallel ion viscous stress in a magnetized plasma is presented. Approximately valid for arbitrary collisionality, the integral nature of this generalized closure results from assuming the maximal ordering between collisional pitch-angle scattering and free streaming effects and from taking a Chapman–Enskog-type approach which includes the parallel ion viscous stress itself as a drive. The ion drift kinetic equation is solved in a sheared slab using an expansion in eigenfunctions of the Lorentz scattering operator. Integrating the coefficient equations in space and taking the proper velocity space moments couples the parallel viscous stress closure to an integral momentum restoring term, thus generalizing the concept of momentum conservation for simplified Coulomb collision operators. The integral closure involves following ions along magnetic field lines which are the ideal, time-independent characteristics of the perturbed distribution function. The fact that the viscous stress and momentum restoring term appear in the kernels of these field line integrals means that in general the closure has the form of coupled Volterra equations with an inhomogeneous term supplied by the traditional flow gradient drive. It is shown that the unified closure agrees both qualitatively and quantitatively with previous results and hence represents a generalized physical form for the parallel ion viscous stress in magnetized plasmas. [DOI: 10.1063/1.1626681]

I. INTRODUCTION

Recent efforts in deriving closures for plasma fluid equations address the need to move beyond the collisional forms of Braginskii[1] to relevant expressions for nearly collisionless fusion and astrophysical plasmas of interest. One approach moves entirely to the collisionless limit by positing Landau-fluid-type closures in order to match the response function of the linearized Vlasov equation.[2,3] Although appropriate in the collisionless limit, many plasmas of interest have juxtaposed regions of nearly collisionless and collisional behavior. For example, a single helicity magnetic island has $k_\| v_{\rm th} \geqslant \nu$ inside the island but $k_\| v_{\rm th} \leqslant \nu$ near the X-and O-points. Here $k_\|$, $v_{\rm th}$, and ν are the parallel wave number, thermal speed, and collision frequency, respectively.

Similarly, there is interest in deriving closure relations that apply both in the hot core and in the cool edge of magnetized fusion plasmas. A Chapman–Enskog-type (CEL) procedure has been developed to derive such closures that allow for arbitrary collisionality.[4,5] In order to simplify the algebra, however, this approach Fourier transforms in space and time; hence the integral nature of the closures is not evident. Also, their Fourier form is not amenable to incorporation into nonlinear plasma fluid simulations characterized by a spectrum of overlapping magnetic perturbations where diagnosing $k_\|$ becomes difficult. Furthermore, although the Fourier forms may be transformed into real space expressions in the limit of $\omega \rightarrow 0$, the closures will not assume the proper form of coupled integral equations as presented in this work. This is due to the simplifying assumptions used there to solve the kinetic equation: a moment expansion for the distribution function and a simplified, constant ion momentum restoring term for the collision operator. In contrast, the approach presented here retains the complicated dependence of the momentum restoring term on the distribution function which is solved for by inverting the collision/free-streaming operator directly. It is this inversion that couples the closure and restoring terms in the form of integral equations.

The purpose of this work is to present a novel, integral (nonlocal) closure for the parallel ion viscous stress, $\pi_\| \equiv m \int d^3 v (v_\|^2 - v_\perp^2/2) F$, where $v_\|$ and $v_\perp$ are the particle speeds parallel and perpendicular to the magnetic field, respectively. In this generalized treatment, the closure couples to a nonlocal momentum restoring term which accounts for the fact that particles see spatially varying fluid gradient drives prior to scattering significantly in pitch angle, an effect that is enhanced in regimes of moderate to low collisionality. By *integral* or *nonlocal* closure, we mean analytic forms that involve integrations along characteristics of the perturbed distribution function, F. In this work, we employ a CEL approach that orders time-dependent effects small compared to parallel free streaming and collisional effects; hence the characteristics of F are simply magnetic field lines. Integration along characteristics couples the nonlocal parallel viscous stress closure to a nonlocal momentum restoring term, thus generalizing the notion of simplified, momentum conserving Coulomb collision operators.[6,7] In the nearly collisionless limit, integration along such characteristics incorporates the purely kinetic effect of Landau wave–particle interactions into the parallel ion viscous stress. In the collisional limit, collisional processes localize the field line integration and reduce the closure to the *local*, Braginskii form.[1]

It is important to point out that nonlocal forms for the

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parallel heat flow have appeared in previous work.[2,3,8–11] In addition to fundamentally advancing the theory of integral closures by stating the proper real space form for $\pi_\parallel$ coupled to a momentum restoring term, this work also provides a practical means for incorporating the parallel ion viscous stress into plasma fluid codes. Presently, many plasma fluid simulations of macroscopic phenomena in high temperature plasmas (Lundquist number, $S \geqslant 10^6$) confined by slowly evolving magnetic fields address the enormous anisotropy in parallel versus perpendicular heat conduction, $\kappa_\parallel/\kappa_\perp \geqslant 10^6$, but ignore similar anisotropy in momentum transport by assuming an isotropic viscosity.[12,13] Incorporation of a nonlocal parallel ion viscous stress would allow plasma fluid codes to properly account for the anisotropic nature of momentum transport in the ion flow evolution equation as well as account for ion viscous heating in the ion temperature evolution equation, thus significantly advancing the realism of macroscopic, nonlinear plasma fluid simulations. It is also important to point out that low collisionality plasma turbulence calculations[2,3] have incorporated the physics of anisotropic momentum transport by evolving separate parallel and perpendicular pressure moments. Although, in principle, macroscopic fluid codes could use this approach, because they typically evolve a single, scalar pressure, it is often easier to incorporate such physics in the stress tensor closure directly rather than expanding the set of evolved variables. This is the approach envisioned in this work.

The remainder of this work is organized as follows: in Sec. II we state the CEL ion drift kinetic equation and the momentum restoring form for the collision operator. In Sec. III we show how the parallel ion viscous stress closure couples to the nonlocal momentum restoring term in the form of integral equations. In Sec. IV we explore the solution to these coupled equations for sinusoidal parallel flow perturbations, and show agreement with previous results and conclude.

II. ION DRIFT KINETIC EQUATION AND COLLISION OPERATOR

In order to derive an integral form for the parallel ion viscous stress, $\pi_\parallel$, we begin with a Chapman–Enskog-type (CEL) ansatz which posits that the distribution function, f, is the sum of a dynamic Maxwellian, f_M, plus a kinetic distortion, F:

$$f = f_M + F = n\left(\frac{m}{2\pi T}\right)^{3/2} \exp\left(-\frac{m(\mathbf{v}-\mathbf{u})^2}{2T}\right) + F, \tag{1}$$

where the density, temperature and flow moments of f_M are n, T, and $\mathbf{u}$, respectively. Using this ansatz along with the moment equations for n, $\mathbf{u}$, and T, one can derive the following CEL drift kinetic equation in sheared slab geometry for the lowest order (in a small gyroradius expansion) kinetic distortion, $\tilde{F}$:[4]

$$\begin{aligned}
&\left(\frac{\partial}{\partial t} + \mathbf{v}_\parallel \cdot \boldsymbol{\nabla}\right)\tilde{F} - \langle C(\tilde{F} + \tilde{f}_M)\rangle \\
&= -\frac{m}{T^0}\left(\hat{\mathbf{b}}\hat{\mathbf{b}} - \frac{\mathbf{I}}{\mathbf{3}}\right) : \boldsymbol{\nabla}_\parallel \tilde{\mathbf{u}}_\parallel \left(v_\parallel^2 - \frac{v_\perp^2}{2}\right) f_M^0 \\
&\quad + v_\parallel(\hat{\mathbf{b}} \cdot \boldsymbol{\nabla} \cdot \tilde{\boldsymbol{\Pi}}_\parallel - \tilde{R}_\parallel)\frac{f_M^0}{p^0} + \frac{2}{3}\left(\frac{v^2}{v_{\mathrm{th}}^2} - \frac{3}{2}\right) \\
&\quad \times [\nabla_\parallel \tilde{q}_\parallel - \tilde{Q}]\frac{f_M^0}{p^0} - \left(\frac{v^2}{v_{\mathrm{th}}^2} - \frac{5}{2}\right) v_\parallel \nabla_\parallel \tilde{T}\frac{f_M^0}{T^0},
\end{aligned} \tag{2}$$

and show that $\tilde{F}$ satisfies $\int d^3v\{\mathbf{v}^0,\mathbf{v}^1,\mathbf{v}^2\}\tilde{F}=0$ for the first three velocity moments. Here $\tilde{\boldsymbol{\Pi}}_\parallel = (\hat{\mathbf{b}}\hat{\mathbf{b}} - \mathbf{I}/3)\tilde{\pi}_\parallel$ is the parallel ion viscous stress tensor, where $\hat{\mathbf{b}}$ is a unit vector parallel to the magnetic field and $\mathbf{I}$ is the unit dyad. Also, $\mathbf{v}_\parallel$ is the parallel component of the guiding center motion, $\langle C(\tilde{F} + \tilde{f}_M)\rangle$ is the gyroaveraged collision operator which represents binary, collisional scattering events, and f_M^0 and $\tilde{f}_M$ represent the lowest order, uniform background and first order perturbed Maxwellians, respectively. Note that the parallel viscous stress, $\tilde{\pi}_\parallel$, and the parallel collisional friction, $\tilde{R}_\parallel$, appear as drives on the right-hand side of Eq. (2) along with the traditional flow gradient drive, $\boldsymbol{\nabla}_\parallel \tilde{\mathbf{u}}_\parallel$. The fact that $\tilde{\pi}_\parallel$ is a moment of $\tilde{F}$, which is to be solved for, leads necessarily to an expression for $\tilde{\pi}_\parallel$ in the form of integral equations. The heat flow terms, $\nabla_\parallel \tilde{q}_\parallel - \tilde{Q}$ and $\nabla_\parallel \tilde{T}$, which are relevant when deriving heat flow closures,[4,10,11] will subsequently be ignored in this work for simplicity. Although this implies that the off diagonal term proportional to $\nabla_\parallel \tilde{T}$, which provides for Onsager symmetry in the collisional limit, will be absent, it will not affect how the parallel viscous stress depends on the flow gradient drive and hence will allow for comparison with other models[1,4] that incorporate heat flow effects.

For a unified parallel viscous stress, we seek an ordering scheme that is approximately valid for arbitrary collisionality. In the interest of identifying the dominant parallel free streaming and collisional physics, we order $\mathbf{v}_\parallel \cdot \boldsymbol{\nabla}\tilde{F} \sim \langle C(\tilde{F} + \tilde{f}_M)\rangle \gg \partial\tilde{F}/\partial t \sim \delta$, and expand the perturbed fluid quantities, $\tilde{n}$, $\tilde{\mathbf{u}}$, $\tilde{T}$, $\tilde{\pi}_\parallel$, and $\tilde{R}$, in Eq. (2) as

$$\tilde{M} = \sum_{i=1} \delta^i M^i(\mathbf{x}), \tag{3}$$

and $\tilde{F}$ and $\tilde{f}_M$ as

$$\tilde{F} = \sum_{i=1} \delta^i F^i(\mathbf{x},\mathbf{v}),$$

and

$$\begin{aligned}
\tilde{f}_M = f_M^0 &\left[\frac{n^1}{n^0} - \left(\frac{3}{2} - \frac{mv^2}{2T^0}\right)\frac{T^1}{T^0} + \frac{m}{T^0}\mathbf{v}\cdot\mathbf{u}^1\right] \\
&+ \sum_{i=2} \delta^i f_M^i(\mathbf{x},\mathbf{v}).
\end{aligned} \tag{4}$$

Here $f_M^0 = [n/(\pi^{3/2} v_{\rm th}^3)]\exp[-(v^2/v_{\rm th}^2)]$ and $\mathbf{v}$ represents suitable guiding center coordinates which will be chosen later. Quantities with the superscript 0 are constant in space. Ignoring time-dependent and heat flow effects yields the first order kinetic equation,

$$\mathbf{v}_\parallel \cdot \nabla F^1 - \langle C(F^1+f_M^1)\rangle = -\frac{m}{T^0}\left(\hat{\mathbf{b}}\hat{\mathbf{b}} - \frac{\mathbf{I}}{\mathbf{3}}\right):\nabla_\parallel \mathbf{u}_\parallel^1 \left(v_\parallel^2 - \frac{v_\perp^2}{2}\right) f_M^0 + v_\parallel(\hat{\mathbf{b}}\cdot\nabla\cdot\tilde{\Pi}_\parallel^1 - R_\parallel^1)\frac{f_M^0}{p^0}. \quad (5)$$

This equation captures the dominant physics of the parallel ion dynamics needed to relate the parallel viscous stress to the flow gradient drive in plasmas confined by slowly evolving magnetic fields.

We now specify the collision operator for Eq. (5). The dominant collisional mechanism in studies of plasma mass flow is the scattering of directed momentum due to multiple small-angle Coulomb collisions. With this in mind, we introduce the following form for the linearized ion collision operator:

$$\langle C(F^1+f_M^1)\rangle = \nu(v)[\mathcal{L}(F^1+f_M^1) + v_\parallel \mathcal{N}_\parallel f_M^0], \quad (6)$$

where the first term represents Lorentz, pitch-angle scattering and the second term provides for momentum conservation between the test particle and background ion distributions. The speed dependent collision frequency is $\nu \equiv \nu_{ii}(v_{\rm th}^3/v^3)G(v/v_{\rm th})$, where the reference ion–ion collision frequency, ν_{ii}, and the function, G, have been stated in previous work.[4] It is shown later that, in general, $\mathcal{N}_\parallel$ is a complicated moment of F^1. The Lorentz scattering operator is given by

$$\mathcal{L}(F^1+f_M^1) = \frac{1}{2}\frac{\partial}{\partial\xi}(1-\xi^2)\frac{\partial F^1}{\partial \xi} - 2\frac{v_\parallel u_\parallel^1}{v_{\rm th}^2} f_M^0, \quad (7)$$

where the pitch-angle-type variable, $\xi \equiv \pm\sqrt{1-\mu B/E}$. Here $E = mv^2/2$ is the ion kinetic energy, B is the magnetic field strength, and μ is the magnetic moment.

In order to determine $\mathcal{N}_\parallel$, we set $\int d^3v\, v_\parallel \langle C(F^1+f_M^1)\rangle = 0$ to ensure momentum conservation within the ion species:

$$\mathcal{N}_\parallel = \int d^3v\, \nu v_\parallel (2(v_\parallel u_\parallel^1/v_{\rm th}^2) f_M^0 - \mathcal{L}(F^1)) \Big/ \int d^3v\, \nu v_\parallel^2 f_M^0. \quad (8)$$

Here the dependence of $\mathcal{N}_\parallel$ on F^1, which is a complicated function of $\mathbf{x}$ and $\mathbf{v}$, is made apparent. Combining this result with Eq. (6) yields

$$\langle C(F^1+f_M^1)\rangle = \nu\mathcal{L}(F^1) - \frac{\nu v_\parallel}{t_\parallel} U_\parallel f_M^0, \quad (9)$$

where $t_\parallel \equiv \int d^3v\, \nu v_\parallel^2 f_M^0$ and $U_\parallel \equiv \int d^3v\, \nu v_\parallel \mathcal{L}(F^1)$. The momentum conserving property of the collision operator means that $\tilde{R}_\parallel = \int d^3v\, v_\parallel \langle C(F^1+f_M^1)\rangle = 0$ in Eq. (5), hence the kinetic equation of interest becomes

$$v\xi\frac{\partial F}{\partial L} - \frac{\nu}{2}\frac{\partial}{\partial\xi}(1-\xi^2)\frac{\partial F}{\partial\xi} = vP_1(\xi)\left(\frac{2}{3p}\frac{\partial \pi_\parallel}{\partial L} - \frac{\nu}{t_\parallel}U_\parallel\right) \times f_M - v^2 P_2(\xi)\frac{2m}{3T}\frac{\partial u_\parallel}{\partial L} f_M, \quad (10)$$

where $\mathbf{v}_\parallel\cdot\nabla$ has been written $v\xi\partial/\partial L$. Here the superscripts 1 and 0 have been dropped for ease and the pitch-angle dependence has been made explicit in the drives by writing it in terms of Legendre polynomials $P_1(\xi)=\xi$ and $P_2(\xi) = (3\xi^2-1)/2$. The following approximations have been made in arriving at Eq. (10):

(1) Sheared slab magnetic geometry (ignores particle drifting and trapping);
(2) Steady-state limit ($\omega \to 0$);
(3) Simplified Coulomb collision operator (pitch-angle scattering with momentum restoring term);
(4) Omission of heat flow term and associated temperature gradient drive (focus on flow gradient drive).

We now expand F in Eq. (10) in a set of N Legendre polynomials, which are the eigenfunctions of the collision operator,

$$\mathcal{L}(F) = \mathcal{L}\sum_{n=1}^{N} F_n(v,L)P_n(\xi) = \sum_{n=1}^{N} F_n(v,L)\lambda_n P_n(\xi), \quad (11)$$

with the associated eigenvalues, λ_n. By writing $\mathbf{F} = [F_1, F_2, \ldots, F_N]$ as a column vector of coefficients, we can insert the expanded form for F into Eq. (10) and apply orthogonality to write the following system of equations,

$$\mathbf{IF} + \frac{v}{\bar{\nu}}\mathbf{A}\frac{\partial \mathbf{F}}{\partial L} = -\frac{\mathbf{G}}{\bar{\nu}}, \quad (12)$$

where $\bar{\nu} = \nu/2$. Here $\mathbf{I}$ is the identity matrix, the matrix $\mathbf{A}$ contains the free streaming couplings between the different eigenfunctions, and $\mathbf{G}$ is the projection of the drives onto the eigenfunctions. Equation (12) is solved in the next section and the velocity moments needed to cast the parallel ion viscous stress in integral form are calculated.

III. UNIFIED PARALLEL VISCOUS STRESS

In this section a unified form for the parallel ion viscous stress is constructed. The solution of the set of equations in Eq. (12) is straightforward and has been discussed in previous work.[10,11] Here we simply state the form for the coefficients of the expansion in Legendre polynomials:

$$F_i = \sum_{i=1}^{N}\sum_{j=1}^{N}\int^L dL'\left[a_{i,j}\left(\frac{2}{3p}\frac{\partial \pi_\parallel}{\partial L'} - \frac{\nu U_\parallel}{t_\parallel}\right) + b_{i,j}\frac{4v}{3v_{\rm th}^2}\frac{\partial u_\parallel}{\partial L'}\right] f_M e^{-k_{\parallel j}(L-L')}, \quad (13)$$

where the coefficients $a_{i,j}$ and $b_{i,j}$ and the effective inverse collision lengths $k_{\parallel j} \equiv \bar{\nu}/(\gamma_i v)$ are generated upon inverting the ODE operator in Eq. (12). The γ_i's are the eigenvalues of $\mathbf{A}$ in Eq. (12). The limits of integration in Eq. (13) are $L' \in (-\infty, L]$ for $k_{\parallel j} > 0$ and $L' \in [L, \infty)$ for $k_{\parallel j} < 0$. The fact that the $k_{\parallel j}$'s come in positive-negative pairs will be used below to simplify the closure relations. It is important to note again that moments of F, namely, $\pi_\parallel$ and $U_\parallel$, appear as drives in Eq. (13).

In order to proceed, we employ the following moment definitions:

$$\pi_\parallel \equiv m \int d^3v \left(v_\parallel^2 - \frac{v_\perp^2}{2} \right) F = m \int d^3v v^2 P_2(\xi) F,$$

$$u_{\parallel F} \equiv \frac{1}{n} \int d^3v v_\parallel F = \frac{1}{n} \int d^3v v P_1(\xi) F = 0, \tag{14}$$

where the second moment enforces the requirement that the parallel flow moment of F vanish in accordance with the CEL ansatz. Writing $\int d^3v = 2\pi \int_0^\infty dv v^2 \int_{-1}^1 d\xi$, we integrate over the ξ variable to write

$$\pi_\parallel = \frac{4\pi m}{5} \int_0^\infty dv v^4 f_M \sum_{j=1}^N \int^L dL' \left[a_{2,j} \left(\frac{2}{3p} \frac{\partial \pi_\parallel}{\partial L'} - \frac{\nu U_\parallel}{t_\parallel} \right) + b_{2,j} \frac{4v}{3v_{\mathrm{th}}^2} \frac{\partial u_\parallel}{\partial L'} \right] e^{-k_{\parallel j}(L-L')},$$

$$u_{\parallel F} = \frac{4\pi}{3n} \int_0^\infty dv v^3 f_M \sum_{j=1}^N \int^L dL' \left[a_{1,j} \left(\frac{2}{3p} \frac{\partial \pi_\parallel}{\partial L'} - \frac{\nu U_\parallel}{t_\parallel} \right) + b_{1,j} \frac{4v}{3v_{\mathrm{th}}^2} \frac{\partial u_\parallel}{\partial L'} \right] e^{-k_{\parallel j}(L-L')} = 0. \tag{15}$$

At this point, it is evident how the nonlocal, parallel ion viscous stress closure, $\pi_\parallel$, couples to the nonlocal momentum restoring term, $U_\parallel$. The nonlocality of both of these terms is the necessary result of deriving closures for arbitrary collisionality. For completeness, the Fourier transform of Eq. (15), which may be calculated by applying the convolution theorem, is stated in the Appendix. Here we will concentrate on real space forms for the closure.

A more symmetric form for the closure may be achieved by integrating the $U_\parallel$ term in the $u_{\parallel F}$ moment by parts and interchanging the order of integration over v and L' to write:

$$\pi_\parallel = \frac{4\pi m}{5} \int^L dL' \int_0^\infty dv v^4 f_M \sum_{j=1}^N \left[a_{2,j} \left(\frac{2}{3p} \frac{\partial \pi_\parallel}{\partial L'} - \frac{\nu U_\parallel}{t_\parallel} \right) + b_{2,j} \frac{4v}{3v_{\mathrm{th}}^2} \frac{\partial u_\parallel}{\partial L'} \right] e^{-k_{\parallel j}(L-L')},$$

$$U_\parallel = A \int^L dL' \int_0^\infty dv v^3 f_M \sum_{j=1}^N \left[a_{1,j} \left(\frac{2}{3p} \frac{\partial \pi_\parallel}{\partial L'} + \frac{\nu}{k_{\parallel j} t_\parallel} \frac{\partial U_\parallel}{\partial L'} \right) + b_{1,j} \frac{4v}{3v_{\mathrm{th}}^2} \frac{\partial u_\parallel}{\partial L'} \right] e^{-k_{\parallel j}(L-L')}, \tag{16}$$

where the coefficient A is

$$A = \left(\int_0^\infty dv v^3 f_M \sum_{j=1}^N a_{1,j} \frac{\nu}{|k_{\parallel j}| t_\parallel} \right)^{-1}. \tag{17}$$

In addition to providing a unified form for $\pi_\parallel$, the above closure introduces the concept of nonlocal momentum conservation necessitated by the fact that in many plasmas of interest, collision lengths are longer than gradient scale lengths along magnetic field lines. The appearance of $\pi_\parallel$ on the right-hand side of the closure expression represents a novel aspect this work. It is the necessary result of taking a more general (as opposed to the traditionally collisional) CEL approach, which uses the lower-order moment equations to rewrite the drives but also retains $\pi_\parallel$ in order to allow for arbitrary collisionality. In order to understand the appearance of $\pi_\parallel$ in this closure, it is helpful to think of an alternate approach of writing a fluid evolution equation for $\pi_\parallel$ and then taking the steady-state limit. The result of this exercise is an equation in which $\pi_\parallel$ appears multiple times. In effect, the closure in Eq. (16) allows for analogous physics with $\pi_\parallel$ acting back on itself.

For macroscopic, fluid simulations of plasmas confined by slowly evolving magnetic fields, Eq. (16) provides a means for updating $\pi_\parallel$ in terms of the fluid quantities ($u_\parallel, U_\parallel$, and $\pi_\parallel$) at the previous time step. Kernels for the field line integrals may be generated ahead of time by integrating over the speed, v, and stored for interpolation in L'. Upon integrating the right sides along magnetic field lines throughout the simulation volume, the divergence of $\mathbf{\Pi}_\parallel = (\hat{\mathbf{b}}\hat{\mathbf{b}} - \mathbf{I}/3)\pi_\parallel$ may be inserted into the ion momentum equation.

Such an approach has already been implemented successfully for the integral form of the parallel heat flow[10] in the plasma fluid code NIMROD.[12,13] This massively parallel computational approach has proven numerically efficient for integration lengths of individual closure calculations exceeding several kilometers. Furthermore, because the integral closure terms are the result of slow magnetic evolution, they remain approximately constant over several time steps in the fluid calculation. These results bode well for the explicit calculation of the $\pi_\parallel$ closure as described above using fluid variables at the previous time step. Similarly, the numerical stabilization of the fluid velocity advance with the explicit $\nabla \cdot \mathbf{\Pi}_\parallel$ term on the right-hand side will be treated in an analogous fashion to the parallel heat flow closure. The successful, semi-implicit stabilization of the temperature evolution equation uses a collisional, anisotropic heat conduction operator to approximate and stabilize the physically correct integral closure on the right-hand side. We advocate using an analogous, anisotropic, self-adjoint stress tensor operator to stabilize the velocity advance.

A form for the parallel viscous stress closure involving coupled Volterra equations is also of interest. We interchange the order of integration and integrate the differentiated terms in Eq. (15) by parts to write,

$$\mathbf{K}_{11}(U_\parallel)+\mathbf{K}_{12}(\pi_\parallel)=\int_0^\infty d\overline{L}(u_\parallel(L+\overline{L})+u_\parallel(L-\overline{L}))\times\frac{\partial K_1(\overline{L})}{\partial\overline{L}}+B_1u_\parallel(L),$$

$$\mathbf{K}_{21}(U_\parallel)+(1+B_2)\pi_\parallel+\mathbf{K}_{22}(\pi_\parallel)=\int_0^\infty d\overline{L}(u_\parallel(L+\overline{L})-u_\parallel(L-\overline{L}))\frac{\partial K_2(\overline{L})}{\partial\overline{L}},\tag{18}$$

where the boundary terms are

$$B_1=\sum_{k_{\parallel j}>0}2b_{1,j}^+\int d^3vP_1^2\frac{4v^2}{3v_{\rm th}^2}f_M,$$

and

$$B_2=\sum_{k_{\parallel j}>0}2|a_{2,j}^+|\int d^3vP_2^2\frac{2v^2}{3p}f_M,\tag{19}$$

and the remaining terms are

$$\frac{\partial K_i}{\partial\overline{L}}=\sum_{k_{\parallel j}<0}\int d^3vv^iP_i^2b_{i,j}^-|k_{\parallel j}|\frac{4v}{3v_{\rm th}^2}f_Me^{-|k_{\parallel j}|\overline{L}},$$

$$\mathbf{K}_{i1}(U_\parallel)=\int_0^\infty d\overline{L}(U_\parallel(L-\overline{L})+(-1)^{i+1}U_\parallel(L+\overline{L}))\times\sum_{k_{\parallel j}>0}a_{i,j}^+\int d^3vv^iP_i^2\frac{\nu}{t_\parallel}f_Me^{-k_{\parallel j}\overline{L}},$$

and

$$\mathbf{K}_{i2}(\pi_\parallel)=\int_0^\infty d\overline{L}(\pi_\parallel(L-\overline{L})+(-1)^i\pi_\parallel(L+\overline{L}))\times\sum_{k_{\parallel j}>0}a_{i,j}^+\int d^3vv^iP_i^2k_{\parallel j}\frac{2}{3p}f_Me^{-k_{\parallel j}\overline{L}}.\tag{20}$$

Here we have reverted to writing d^3v for the velocity space integration to condense the notation. The kernels, $\partial K_i/\partial\overline{L}$, on the right-hand side of Eq. (18) are analogous to the kernels of the collisionless[2,3] and unified[10,11] heat flow closures that have appeared in previous work. Equation (18) has the form of coupled, singular Volterra equations.[14] Their singular nature becomes apparent when mapping onto a finite interval using the transformation $L\pm\overline{L}\rightarrow 1/x$. A numerical solution using product integration,[15] a form of numerical quadrature to handle singular kernels, could be used to implement this form of the closures in plasma fluid codes. Such an approach may be desirable in nearly collisionless situations if the truncation of the field line integrations in Eq. (18) is prohibitively slow.

This approach will be explored in subsequent work that concentrates on the numerical implementation of nonlocal closures. This work will compare the promising, explicit approach described following Eq. (16) with the approach that assumes the fluid quantities on the left-hand side of Eq. (18) are understood to be at the present time step. Such studies will be carried out in slab magnetic island geometry by examining the $u_\parallel$ profiles that result from imposing a sheared flow profile across the magnetic island and evolving the flow evolution equation to steady-state. This future work will also permit a comparison of the physical predictions of the $\pi_\parallel$ closure presented in this work with those resulting from a more exact Fokker–Planck-type calculation. This would address the limitations of the simplified collision operator used in this work (see discussion regarding Fig. 1 in Sec. IV). Furthermore, this future work will evaluate the effectiveness of using a semi-implicit operator to stabilize the velocity advance when the anisotropic viscous stress appears explicitly on the right-hand side.

In order to show that Eqs. (16) and (18) represent a unified analytic form for the parallel ion viscous stress we consider the collisional limit of Eq. (16) and the nearly collisionless limit of Eq. (18). In the collisional limit, Eq. (16) simplifies to

$$\pi_\parallel\approx\frac{4\pi m}{5}\int^L dL'\frac{\partial u_\parallel}{\partial L'}\sum_{j=1}^N\int_0^\infty dvv^4\times f_Mb_{2,j}\frac{4v}{3v_{\rm th}^2}e^{-k_{\parallel j}(L-L')}.\tag{21}$$

Since collisions localize the kernels of the field line integrations, we define the collisional limit as the regime in which $\pi_\parallel$ converges in a distance over which $\partial u_\parallel/\partial L'$ may be considered a constant. This is tantamount to an application of the mean value theorem which reduces the parallel viscous stress closure to the following form:

$$\pi_\parallel=-nm\mu_\parallel\frac{\partial u_\parallel}{\partial L},\tag{22}$$

where the viscosity, $\mu_\parallel$, is

$$\mu_\parallel=\left(\frac{32}{15\sqrt{\pi}}\int_0^\infty dss^5e^{-s^2}\sum_{k_{\parallel j}>0}\frac{|b_{2,j}|\nu_{ii}}{k_{\parallel j}v_{\rm th}}\right)\frac{v_{\rm th}^2}{\nu_{ii}}=2.75\frac{v_{\rm th}^2}{\nu_{ii}}.\tag{23}$$

Here the speed variable, $s\equiv v/v_{\rm th}$. The coefficient 2.75 lies between the collisional viscosity coefficients of Braginskii,[1] 1.81, and Chang/Callen,[4] 3.13, for reasons that will be explained in the next section.

In the nearly collisionless limit, Eq. (18) reduces to the form of Eq. (22) with the viscosity

$$\mu_\parallel=1.04\frac{v_{\rm th}^2}{k_\parallel v_{\rm th}}.\tag{24}$$

Here the viscosity is due solely to wave–particle Landau interactions. In the Chang/Callen paper the coefficient in front of this expression is $(3/5)\sqrt{\pi}=1.06$; hence there is good agreement between the closures in the nearly collisionless limit.

It has been shown here that the parallel ion viscous stress closures given in Eqs. (16) and (18) provide a unified analytic form that is approximately valid for arbitrary collisionality. Although similar physics is contained in the Fourier

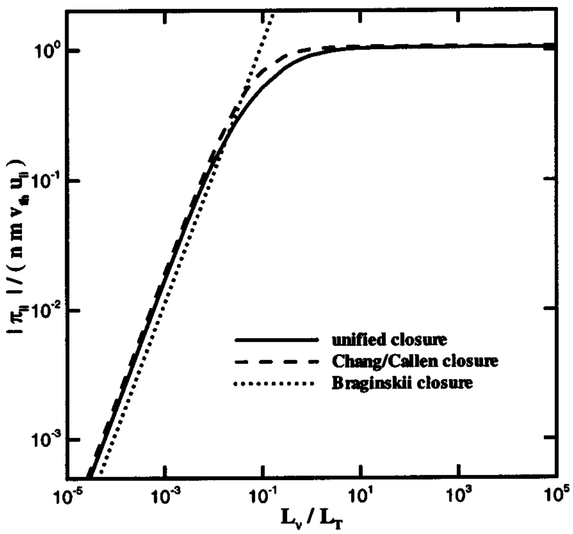

FIG. 1. Plot of $|\pi_{\|}|/(nmv_{\rm th}u_{\|})$ vs $L_\nu/L_{\|}$ shows agreement in the real space and Fourier form for the parallel viscous stress for sinusoidal parallel flow perturbations. In the collisional limit, the unified closure in this work lies between the Chang/Callen and Braginskii results.

form given in the Chang/Callen reference, Eqs. (16) and (18) generalize the notion of momentum conserving collision operators as well as provide a more straightforward means for incorporating the physics of these closures in nonlinear plasma fluid simulations characterized by a spectrum of overlapping magnetic perturbations. This last fact is the result of integrating up the coefficient equations along magnetic field lines. Such integration is readily performed in plasma fluid codes that supply the total magnetic field.[10]

IV. SAMPLE CALCULATION AND CONCLUSION

In order to further show agreement with previous forms, we consider the resultant parallel viscous stress for sinusoidal flow perturbations of scale length, $L_{\|}$, namely,

$$\tilde{u}_{\|}(L)=u_{\|}\sin\left(\frac{2\pi L}{L_{\|}}\right). \tag{25}$$

For such perturbations, the field line integrations in Eqs. (16) or (18) may be carried out analytically with subsequent integration over the speed variable, v, achieved via numerical quadrature. The parallel viscous stress for such flow perturbations is given by $\tilde{\pi}_{\|}=\pi_{\|}\cos(2\pi L/L_{\|})$. Figure 1 plots the normalized coefficient $|\pi_{\|}|/(nmv_{\rm th}u_{\|})$ vs the ratio, $L_\nu/L_{\|}=v_{\rm th}/(\nu_{ii}L_{\|})$. Comparisons of the results for the unified, real space form for $\pi_{\|}$ from this work with the Fourier form of Chang/Callen in the limit $\omega\rightarrow 0$ and the Braginskii result are included. In the collisional limit, the parallel viscous stress increases linearly with $L_\nu/L_{\|}$ as was shown in Eq. (22). In the nearly collisionless limit, the collisional response vanishes with $\pi_{\|}$ taking on a constant value determined by Landau resonance effects. The constant nature of the closure in the nearly collisionless limit is also seen in Eq. (22) where $\pi_{\|}\sim(1/k_{\|})\partial u_{\|}/\partial L$. Note the general agreement between the results of this work and the Chang/Callen closure for all regimes of collisionality.

The discrepancy between the closures in the collisional limit, $(L_\nu/L_{\|}\leqslant 1)$ may be attributed to differences in collision operators. The collisional analysis of Braginskii accounts for the full collision operator by using a moment approach that includes corrections due to momentum and heat flow restoring terms as well as finite flow, heat flow, and energy weighted heat flow effects. In this work and in the Chang/Callen reference, the ion collision operator is simplified to include momentum restoring and finite flow effects only. The overshoot is analogous to the reduction in electrical conductivity when going from the Lorentz to the Spitzer model.[16] To further quantitative comparisons with Braginskii, incorporating additional effects could be achieved by the addition of a model term for speed diffusion of the form, $(1/v^2)\partial_v v^4\nu_s\partial_v F$. This self-adjoint operator could, in principle, be treated in an analogous fashion to the pitch-angle scattering operator via an expansion in speed eigenfunctions. The effect would be to augment the system of equations in Eq. (12). In addition, a heat flow restoring term could also be added further complicating the statement of the integral closures with the additional coupling to a heat flow restoring term. Such complications, including the addition of heat flow effects, will be the subject of future research.

More subtle differences in derivations exist between this work and Chang/Callen in the collisional to moderately collisional limit, $L_\nu/L_{\|}\leqslant 1$. There the collision operator is simplified in two ways. First, the expansion for F is truncated at $P_2(\xi)$ in contrast to the expansion used here which required 1024 Legendre polynomials in order to resolve the ξ dependence in the nearly collisionless regime. Second, as stated previously, the ion momentum restoring term in Chang/Callen is assumed to be independent of velocity and space in contrast to the function $\mathcal{N}_{\|}$ in this work with its complicated dependence on the perturbed distribution function. For these reasons, the unified closure presented here lies between the results of Braginskii and Chang/Callen in the collisional limit. It is important to note that despite differences in derivation, the deviation between the closures for this simple case of a single sinusoidal flow perturbation is at most 25%, with the largest discrepancy occurring in the regime $10^{-2}\leqslant L_\nu/L_{\|}\leqslant 10^0$. Figure 2 shows convergence of $|\pi_{\|}|/(nmv_{\rm th}u_{\|})$ as the number of pitch-angle basis functions, N, increases in the expansion for the regime, $10^{-3}\leqslant L_\nu/L_{\|}\leqslant 10^1$. Note that the closure in this work converges from above to the nearly collisionless limit as eigenfunctions are added. Comparison with the Chang/Callen closure, which uses $N=2$ but relies on a one-pole method for the collisionless response, suggests that using $N=4$ or $N=8$ in the Chang/Callen approach would bring the closures into better agreement in this moderately collisional regime. This discrepancy will be the subject of future work that compares the various closures with the result from the kinetic, Fokker–Planck calculation described following Eq. (18).

Figure 3 shows how the momentum restoring term $|U_{\|}|(L_\nu/nv_{\rm th}u_{\|})$ varies with collisionality, where $\tilde{U}_{\|}=U_{\|}\sin(2\pi L/L_{\|})$. In the collisional limit, $U_{\|}$ plays the role of restoring momentum to the background Maxwellian plasma. In the moderately collisional to nearly collisionless limits

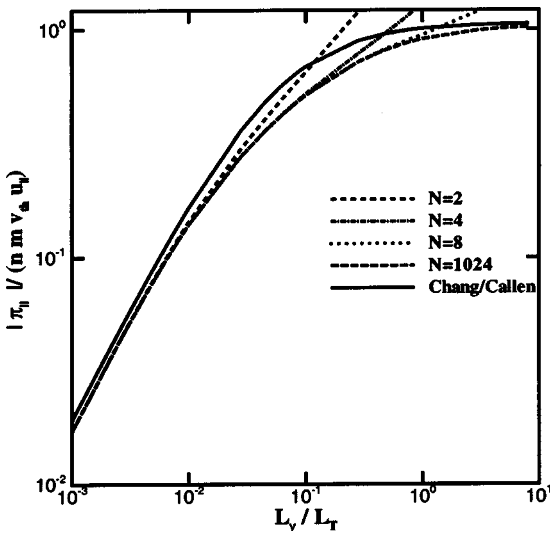

FIG. 2. Plot of $|\pi_\||/(nmv_{\text{th}}u_\|)$ vs $L_\nu/L_\|$ shows convergence of closure as the number of pitch-angle basis functions, N, increases in the expansion for the regime, $10^{-3} \leqslant L_\nu/L_\| \leqslant 10^1$. Comparison with the Chang/Callen closure, which uses $N=2$ but relies on a one-pole method for the collisionless response, suggests that using $N=4$ or $N=8$ in the Chang/Callen approach would likely decrease their result bringing the closures into better agreement in this moderately collisional regime.

$(L_\nu/L_\| \gg 1)$, $U_\|$ enforces the requirement that the parallel flow moment of F vanishes, $u_{\|F}=0$. This is in contrast to the approach of Chang/Callen which uses a moment expansion for F that automatically satisfies this constraint, namely, $F \equiv (2s^2/3p)P_2(\xi)\pi_\| f_M$. Recall that in this work the only assumption imposed on F is the existence of an expansion in separable pitch angle eigenfunctions, $P_n(\xi)$. In contrast to the moment approach, the complicated speed dependence of F [see Eq. (13)] is determined by an exact solution to the coefficient equations. In this more general approach, forcing

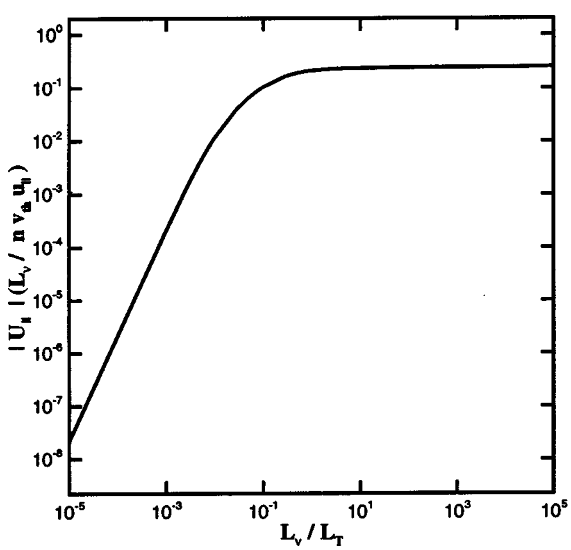

FIG. 3. Plot of $|U_\||(L_\nu/nv_{\text{th}}u_\|)$ vs $L_\nu/L_\|$ shows variation of momentum restoring term as collisionality is varied for sinusoidal parallel flow perturbations. In the limit, $L_\nu/L_\| \ll 1$, $U_\|$ provides for momentum conservation and in the limit, $L_\nu/L_\| \gg 1$, its nearly constant value enforces the constraint $u_{\|F}=0$.

$u_{\|F}=0$ couples the restoring term to the $\pi_\|$ moment and provides for the same collisionless response.

In conclusion, this work presents a novel, integral (nonlocal) form for the parallel ion viscous stress. This closure relates the parallel viscous stress to the flow gradient drive based on the following assumptions: sheared slab geometry, steady-state, and a simplified Coulomb collision operator, which addresses some important collisional effects in regimes of moderate to high collisionality. The closures in Eqs. (16) and (18) were derived by allowing for arbitrary collisionality in the ion drift kinetic equation and by incorporating an integral form for the momentum restoring term for the first time. Comparisons with previous forms for the parallel ion viscous stress closure show both qualitative and quantitative agreement for all regimes of collisionality. In contrast to previous Fourier forms, however, the incorporation of this closure into plasma fluid codes is more straightforward; thus it may be used to advance the realism of nonlinear plasma fluid simulations by providing a quantitative calculation of the anisotropy in parallel vs perpendicular ion momentum transport.

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APPENDIX: FOURIER FORM OF CLOSURE

The Fourier transform of Eq. (15) is stated here for completeness. Terms in the integral coefficients, F_i, in Eq. (13) may be manipulated into the following convolution:

$$\mathcal{F}\left(\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dL' f(L-L')g(L')\right)$$

$$=\mathcal{F}\left(\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} g(L')\sum_{k_{\|j}>0} e^{-k_{\|j}|L-L'|}\right)$$

$$=\sum_{k_{\|j}>0}\sqrt{\frac{2}{\pi}}\frac{k_{\|j}}{k_{\|j}^2+k_\|^2}\tilde{g}(k_\|), \tag{A1}$$

where $\tilde{g}(k_\|)=\mathcal{F}(g(L))$. Here integration by parts is necessary for terms with $a_{i,j}^+=-a_{i,j}^-$ and $b_{i,j}^+=-b_{i,j}^-$ in order to apply the convolution theorem. Using this result, the Fourier form of the closure in Eq. (15) may be written

$$\tilde{\pi}_\| = \frac{8\pi m}{5}\int_0^\infty dv\, v^4 f_M \sum_{k_{\|j}>0}\left[a_{2,j}^+\left(\frac{2}{3p}k_\|^2\tilde{\pi}_\| - \frac{\nu}{t_\|}ik_\|\tilde{U}_\|\right)\right.$$

$$\left. - b_{2,j}^+ k_{\|j}\frac{4v}{3v_{\text{th}}^2}ik_\|\tilde{u}_\|\right] \Big/ (k_{\|j}^2+k_\|^2),$$

$$\tilde{u}_{\|F} = \frac{8\pi}{3n}\int_0^\infty dv\, v^3 f_M \sum_{k_{\|j}>0}\left[a_{1,j}^+ k_{\|j}\left(\frac{2}{3p}(-ik_\|\tilde{\pi}_\|) - \frac{\nu}{t_\|}\tilde{U}_\|\right)\right.$$

$$\left. + b_{1,j}^+\frac{4v}{3v_{\text{th}}^2}k_\|^2\tilde{u}_\|\right]/(k_{\|j}^2+k_\|^2)=0. \tag{A2}$$

The orderings $k_{\|j} \gg k_{\|}$ and $k_{\|} \gg k_{\|j}$ may be used to derive the collisional and nearly collisionless forms for the closure, respectively. For example, applying the ordering $k_{\|j} \gg k_{\|}$ reduces the Fourier form for the closure to

$$\tilde{\pi}_{\|} = -\frac{8\pi m}{5} \int_0^\infty dv v^4 f_M \sum_{k_{\|j}>0} \frac{b_{2,j}^+}{k_{\|j}} \frac{4v}{3v_{\mathrm{th}}^2} i k_{\|} \tilde{u}_{\|}, \tag{A3}$$

or in real space

$$\pi_{\|} = -nm \left(\frac{32}{15\sqrt{\pi}} \int_0^\infty ds s^5 e^{-s^2} \sum_{k_{\|j}>0} \frac{|b_{2,j}^+| \nu_{ii}}{k_{\|j} v_{\mathrm{th}}} \right) \frac{v_{\mathrm{th}}^2}{\nu_{ii}} \frac{\partial u_{\|}}{\partial L}, \tag{A4}$$

which is the result presented in Eqs. (22) and (23). Although the Fourier form of the closure in Eq. (A2) is not particularly useful for nonlinear, electromagnetic plasma fluid simulations where diagnosing $k_{\|}$ is difficult, it may prove useful in spectral, electrostatic calculations where $k_{\|}$ may be determined algebraically.[17]

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