Fourier Transforms and the Wave Equation

Overview and Motivation: We first discuss a few features of the Fourier transform (FT), and then we solve the initial-value problem for the wave equation using the Fourier transform.

Key Mathematics: More Fourier transform theory, especially as applied to solving the wave equation.

I. Some Properties of the Fourier Transform
In the last lecture we introduced the FT \( h(k) \) of a function \( f(x) \) through the two equations

\[
\begin{align*}
  f(x) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} h(k) e^{ikx} \, dk, \quad (1a) \\
  h(k) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-ikx} \, dx. \quad (1b)
\end{align*}
\]

Here we wish to point out a few useful properties of the Fourier transform.

A. Translation
The first property has to do with translation of the function \( f(x) \). Let's say we are interested in \( f(x + x_0) \), which corresponds to translation of \( f(x) \) by \( -x_0 \). Then, using Eq. (1a) we can write

\[
\begin{align*}
  f(x + x_0) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} h(k) e^{ik(x + x_0)} \, dk \\
  &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left[ h(k) e^{ikx_0} \right] e^{ikx} \, dk.
\end{align*}
\]

Thus, we see that the FT of \( f(x + x_0) \) is \( h(k) e^{ikx_0} \). In other words, translation of \( f(x) \) by \( -x_0 \) corresponds to multiplying the FT \( h(k) \) by \( e^{ikx_0} \).

B. Differentiation
The second property has to do with the FT of \( f'(x) \), the derivative of \( f(x) \). Again, using Eq. (1a) we have
\[ f'(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} [ik h(k)] e^{ikx} \, dk. \]  

(3)

So we see that FT of \( f'(x) \) is \( ik h(k) \). That is, differentiation of \( f(x) \) corresponds to multiplying \( h(k) \) by \( ik \).

C. Integration

Let's consider the definite integral of \( f(x) \),

\[ \int_{x_1}^{x_2} dx \, f(x) = \frac{1}{\sqrt{2\pi}} \int_{x_1}^{x_2} dx \left[ \int_{-\infty}^{\infty} dk \, h(k) e^{ikx} \right]. \]  

(4)

Switching the order of integration on the rhs produces

\[ \int_{x_1}^{x_2} dx \, f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dk \left[ \int_{x_1}^{x_2} dx \, e^{ikx} \right] h(k) \] 

\[ = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dk \frac{h(k)}{ik} \left( e^{ikx_2} - e^{ikx_1} \right). \]  

(5)

So if we define \( If(x) \) to be the indefinite integral of \( f(x) \), we can rewrite Eq. (5) as

\[ If(x_2) - If(x_1) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dk \frac{h(k)}{ik} \left( e^{ikx_2} - e^{ikx_1} \right). \]  

(6)

This equation tells us that integration of \( f(x) \) essentially corresponds to dividing the Fourier transform \( h(k) \) by \( ik \).\(^1\)

\(^{1}\) You might think that Eq. (6) could be simplified to \( If(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dk \frac{h(k)}{ik} e^{ikx} \), but this cannot be done because indefinite integration produces an undetermined integration constant. The constant does not appear in Eq. (6) because it is an equation for the difference of \( If(x_2) \) and \( If(x_1) \).
D. Convolution

The last property concerning the a function and its FT has to do with convolution. Because you may not be familiar with convolution, let's first define it. Simply put, the convolution of two functions $f(x)$ and $g(x)$, which we denote $(f * g)(x)$, is defined as

$$ (f * g)(x) = \int_{-\infty}^{\infty} f(x-x')g(x')dx' $$  \hspace{1cm} (7)

Perhaps the most common place that convolution arises is in spectroscopy, where $g(x)$ is some intrinsic spectrum that is being measured, and $f(x)$ is the resolution function of the spectrometer that is being used to measure the spectrum.\(^2\) The convolution $(f * g)(x)$ is then the measured spectrum.

Note that $(f * g)(x)$ is indeed a function of $x$, and so we can calculate its FT, which we denote $(f \hat{ } * g)(k)$. Using Eq. (1b) we can write

$$ (f \hat{ } * g)(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dx \left[ \int_{-\infty}^{\infty} dx' f(x-x')g(x') \right] e^{-ikx} \hspace{1cm} (8)$$

Now this may not look too simple, but we can change variables of integration on the $x$ integral, $x-x' = x''$, $dx = dx''$. Then Eq. (8) becomes

$$ (f \hat{ } * g)(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dx'' \left[ \int_{-\infty}^{\infty} dx' f(x'')g(x') \right] e^{-ik(x''+x'')} \hspace{1cm} (9)$$

which can be rearranged as

$$ (f \hat{ } * g)(k) = \sqrt{2\pi} \left[ \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dx'' f(x'')e^{-ikx''} \right] \left[ \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dx' g(x')e^{-ikx'} \right] \hspace{1cm} (10)$$

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\(^2\) The resolution function is often quite close to a Gaussian of a particular, fixed width.
Notice that Eq. (10) is simply the product of the FT of \( f(x) \) and the FT of \( g(x) \) (along with the factor of \( \sqrt{2\pi} \)). Denoting these FT's as \( \hat{f}(k) \) and \( \hat{g}(k) \), respectively, we thus have\(^3\)

\[
(f \ast g)(k) = \sqrt{2\pi} \hat{f}(k) \hat{g}(k). \tag{11}
\]

So we see that the FT of the convolution is the product of the FT's of the individual functions. One way you may hear this result expressed is that convolution in real space \((x)\) corresponds to multiplication in \(k\) space.

**II. Solution to the Wave Equation Initial Value Problem**

Way back in Lecture 8 we discussed the initial value problem for the wave equation

\[
\frac{\partial^2 q(x,t)}{\partial t^2} = c^2 \frac{\partial^2 q(x,t)}{\partial x^2} \tag{12}
\]

on the interval \(-\infty < x < \infty\). For the initial conditions

\[
q(x,0) = a(x), \tag{13a}
\]

\[
\frac{\partial q}{\partial t}(x,0) = b(x), \tag{13b}
\]

we found that the solution to Eq. (12) can be written as

\[
q(x,t) = \frac{1}{2} \left[ a(x + ct) + a(x - ct) + \frac{1}{c} \int_{x-ct}^{x+ct} b(x') dx' \right]. \tag{14}
\]

With the help of the Fourier transform we are now going to rederive this solution, and along the way we will learn something very interesting about the FT of \( q(x,t) \).

We start by defining the (spatial) FT of \( q(x,t) \) as

\[
\hat{q}(k,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} q(x,t) e^{-ikx} \, dx, \tag{15a}
\]

\(^3\) Although we have not done it up to this point, it is fairly common practice to denote the FT of a function such as \( f(x) \) by \( \hat{f}(k) \). You will even find the practice of denoting the FT of \( f(x) \) as \( f(k) \). At least we won’t be doing that here!
so that we also have

\[ q(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{q}(k,t) e^{ikx} dk. \]  

(15b)

We also define the FT of Eq. (13), the initial conditions,

\[ \hat{q}(k,0) = \hat{a}(k), \]  

(16a)

\[ \hat{q}(k,0) = \hat{b}(k). \]  

(16b)

Now each side of the Eq. (12) is a function of \( x \) and \( t \), so we can calculate the FT of both sides of Eq (12),

\[ \int_{-\infty}^{\infty} \frac{\partial^2 q(x,t)}{\partial t^2} e^{-ikx} dx = \frac{c^2}{\pi} \int_{-\infty}^{\infty} \frac{\partial^2 q(x,t)}{\partial x^2} e^{-ikx} dx \]  

(17)

On the lhs of this equation we can pull the time derivative outside the integral. The lhs is then just the second time derivative of \( \hat{q}(k,t) \). The rhs can be simplified by remembering that the FT of the \( x \) derivative of a function is \( ik \) times the FT of the original function. Thus the FT of \( \partial^2 q(x,t)/\partial x^2 \) is just \( -k^2 \) times \( \hat{q}(k,t) \), the FT of \( q(x,t) \). Thus we can rewrite Eq. (17) as

\[ \frac{\partial^2 \hat{q}(k,t)}{\partial t^2} = -k^2 c^2 \hat{q}(k,t) \]  

(18)

This equation should look very familiar to you. What equation is it? None other than the harmonic oscillator equation! What does this tell us about \( \hat{q}(k,t) \)? It tells us that \( \hat{q}(k,t) \) (for a fixed value of \( k \)) oscillates harmonically at the frequency \( \omega = kc \). Thus we can interpret the function \( \hat{q}(k,t) \) as the set of normal modes coordinates for this problem. This further means that the FT has decoupled the equations of motion for this system [as represented by Eq. (12), the wave equation.] Notice also that the dispersion relation \( \omega = kc \) has also fallen into our lap by considering the FT of Eq. (12).

As we should know by this point, the solution to Eq. (18) can be written as

\[ \hat{q}(k,t) = A(k)e^{ikt} + B(k)e^{-ikt}, \]  

(19)
where $A(k)$ and $B(k)$ are functions of $k$. And as you should suspect, these two functions are determined by the initial conditions, as follows. First, setting $t = 0$ in Eq. (19) and using Eq. (16a) produces

$$\hat{a}(k) = A(k) + B(k), \quad (20a)$$

and calculating the time derivative of Eq. (19), setting $t = 0$, and using Eq. (16b) gives us

$$\frac{\hat{b}(k)}{ikc} = A(k) - B(k) \quad (20b)$$

We can solve Eqs. (20a) and (20b) for $A(k)$ and $B(k)$ by taking their sum and difference, which yields

$$A(k) = \frac{1}{2} \left[ \hat{a}(k) + \frac{\hat{b}(k)}{ikc} \right], \quad (21a)$$

$$B(k) = \frac{1}{2} \left[ \hat{a}(k) - \frac{\hat{b}(k)}{ikc} \right], \quad (21b)$$

which gives us the solution for $\hat{q}(k,t)$ in terms of the initial conditions

$$\hat{q}(k,t) = \frac{1}{2} \left[ \hat{a}(k) + \frac{\hat{b}(k)}{ikc} \right] e^{ikct} + \frac{1}{2} \left[ \hat{a}(k) - \frac{\hat{b}(k)}{ikc} \right] e^{-ikct} \quad (22)$$

We are essentially done. We have now expressed the FT of $q(x,t)$ in terms of the FT's of the initial conditions for the problem. The solution $q(x,t)$ is just the inverse FT of Eq. (22) [see Eq. (15b)],

$$q(x,t) = \frac{1}{\sqrt{2\pi}} \frac{1}{2} \int_{-\infty}^{\infty} \left\{ \hat{a}(k) + \frac{\hat{b}(k)}{ikc} \right\} e^{i(kx+ct)} + \left[ \hat{a}(k) - \frac{\hat{b}(k)}{ikc} \right] e^{i(kx-ct)} \right\} dk. \quad (23)$$

This is the initial-value-problem solution. We can make it look exactly like Eq. (14) with a little bit more manipulation. To see this let's first rewrite Eq. (23) as

\[\text{Notice that } q(x,t) \text{ as expressed in Eq. (23) is the sum of two functions, } f(x+ct) \text{ and } g(x-ct)!\]
\[ q(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left\{ \hat{a}(k) e^{ik(x+ct)} + \hat{\dot{a}}(k) e^{ik(x-ct)} + \frac{1}{c} \left[ \hat{b}(k) \left( e^{ik(x+ct)} - e^{ik(x-ct)} \right) \right] \right\} dk. \] (24)

The first two terms we recognize as \((1/2)[a(x+ct) + a(x-ct)]\), while we can use Eq. (5) to recognize the second half of the rhs of Eq. (24) as \((1/2c)\int_{x-ct}^{x+ct} b(x')dx'\). Thus Eq. (24) can be re-expressed as

\[ q(x,t) = \frac{1}{2} \left[ a(x+ct) + a(x-ct) + \frac{1}{c} \int_{x-ct}^{x+ct} b(x')dx' \right], \] (25)

which is identical to Eq. (14).

**Exercises**

*17.1 FT Properties. If the FT of \(f(x)\) is \(h(k)\),

(a) show that the FT of \(e^{ikx} f(x)\) is \(h(k - k_0)\);
(b) show that the FT of \(xf(x)\) is \(ih'(k)\);
(c) show that the FT of \(x^2 f(x)\) is \(-h''(k)\).

*17.2 Show that \((f * g)(x) = (g * f)(x)\) by

(a) directly by manipulating Eq. (7), the definition of convolution;
(b) by using Eq. (11), the result for the FT of \((f * g)(x)\).

*17.3 Convolution and the Gaussian. The function that has the same form as its Fourier transform is the Gaussian. Specifically if \(f(x) = e^{-x^2/\sigma^2}\), its FT is given by

\[ h(k) = \frac{\sigma}{\sqrt{2}} e^{-k^2(\sigma^2/4)}. \]

Using this fact, show that the convolution \((f_1 * f_2)(x)\) of two Gaussian functions \(f_1(x) = e^{-x^2/\sigma_1^2}\) and \(f_2(x) = e^{-x^2/\sigma_2^2}\) is proportional to the Gaussian function \(e^{-x^2/(\sigma_1^2 + \sigma_2^2)}\). [Hint: you need not calculate any integrals to do this problem.]

**17.4 The Rectangular Pulse.

Consider the function \(f(x)\) which is a rectangular pulse of height \(H\) and width \(2L\) centered at \(x = 0\).

(a) Graph \(f(x)\)
(b) Find \(\hat{f}(k)\), the FT of \(f(x)\).
(c) Graph \( \hat{f}(k) \).

(d) The function \((f \ast f)(x)\), the convolution of \(f(x)\) with itself, is a triangle function of height \(2LH^2\) and base \(4L\) centered at zero. Graph this function.

(e) Write down the functional form of \((f \ast f)(x)\) that you graphed in (d). Then directly calculate \(\hat{f}(\hat{k})\) using your functional form for \((f \ast f)(x)\). (Do not set \(H\) and \(L\) to specific values.)

(f) Graph your calculated transform \(f(\hat{k})\).

(g) Lastly, use \(\hat{f}(k)\) and the convolution theorem to find \((f \ast f)(\hat{k})\). Show that this is equal to the result in part (e).

**17.5 FT Solution to the 1D Wave Equation.** Eq. (23),

\[ q(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left[ \hat{a}(k) + \hat{b}(k) \right] e^{ik(\pi - ct)} + \left[ \hat{a}(k) - \hat{b}(k) \right] e^{ik(\pi + ct)} dk , \]  

is the formal solution to the initial-value problem.

(a) What kind of wave is described by the function \(\exp[ik(x + ct)]\) (traveling wave, standing wave, etc.) Be as specific as possible. What kind of wave is described by the function \(\exp[ik(x - ct)]\)? From a vector-space point of view, what are these functions?

(b) From a vector-space point of view (where functions are viewed as elements of a vector space), what do the terms \(1/2\left[ \hat{a}(k) - \frac{i\hat{b}(k)}{kv} \right]\) and \(1/2\left[ \hat{a}(k) + \frac{i\hat{b}(k)}{kv} \right]\) represent?

(c) Given your answers in (a) and (b), how would you describe the solution \(q(x,t)\) as written above? [Hint: the term linear combination should appear in your answer.]

(d) How are \(\hat{a}(k)\) and \(\hat{b}(k)\) related to the initial conditions \(q(x,0)\) and \(\dot{q}(x,0)\)? That is, write down expressions for \(\hat{a}(k)\) and \(\hat{b}(k)\) in terms of \(q(x,0)\) and \(\dot{q}(x,0)\).

(e) Assume that the initial conditions \(q(x,0)\) and \(\dot{q}(x,0)\) are real. Using your answer to (d), show that \(\hat{a}(k) = \hat{a}^*(-k)\) and \(\hat{b}(k) = \hat{b}^*(-k)\).

(f) Using the results from (e) you can now show that \(q(x,t)\) is real if the initial conditions \(q(x,0)\) and \(\dot{q}(x,0)\) are both real, as follows. First, in Eq. (23) replace \(\hat{a}(k)\) and \(\hat{b}(k)\) by \(\hat{a}^*(-k)\) and \(\hat{b}^*(-k)\), respectively. Then make the change of variable \(k \rightarrow -k\), \(dk \rightarrow -dk\) in the integral (taking care with the limits of integration). Then compare Eq. (23) with your new expression for \(q(x,t)\) and notice how they are related. From your comparison you should be able to conclude that \(q(x,t)\) is real.