Intermediate Lab

PHYS 3870

Lecture 3

Distribution Functions

References: Taylor Ch. 5 (and Chs. 10 and 11 for Reference)
Taylor Ch. 6 and 7
Also refer to "Glossary of Important Terms in Error Analysis"
"Probability Cheat Sheet"

Intermediate Lab

PHYS 3870

Distribution Functions

Introduction Section 0 Lecture 1 Slide 2

Practical Methods to Calculate Mean and St. Deviation

We need to develop a good way to tally, display, and think about a collection of repeated measurements of the same quantity.

Here is where we are headed:

- Develop the notion of a probability distribution function, a distribution to describe the probable outcomes of a measurement
- Define what a distribution function is, and its properties
- Look at the properties of the most common distribution function, the Gaussian distribution for purely random events
- Introduce other probability distribution functions

We will develop the mathematical basis for:

- Mean
- Standard deviation
- Standard deviation of the mean (SDOM)
- Moments and expectation values
- Error propagation formulas
- Addition of errors in quadrature (for independent and random measurements)
- Schwartz inequality (i.e., the uncertainty principle) (next lecture)
- Numerical values for confidence limits (t-test)
- Principle of maximal likelihood
- Central limit theorem

Two Practical Exercises in Probabilities

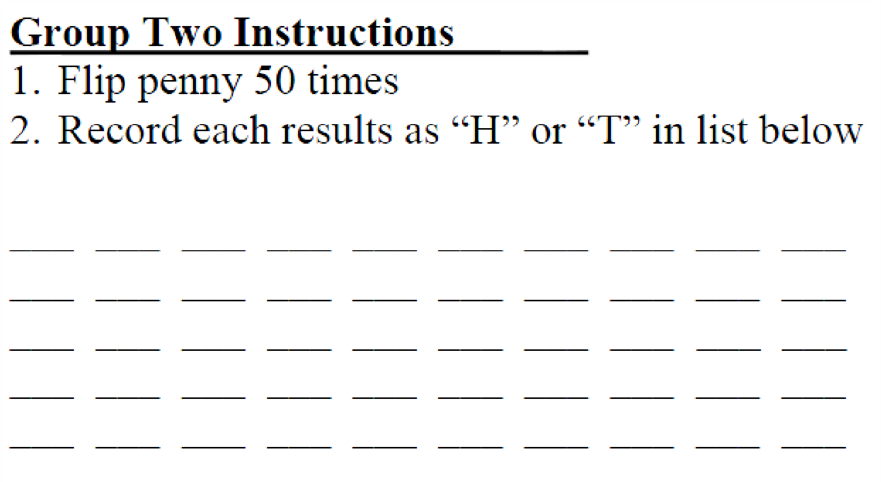
Flip a penny 50 times and record the results

Roll a pair of dice 50 times and record the results

Grab a partner and a set of instructions and complete the exercise.

Two Practical Exercises in Probabilities

Flip a penny 50 times and record the results



Group Two Instructions

1. Flip penny 50 times
2. Record each results as “H” or “T” in list below

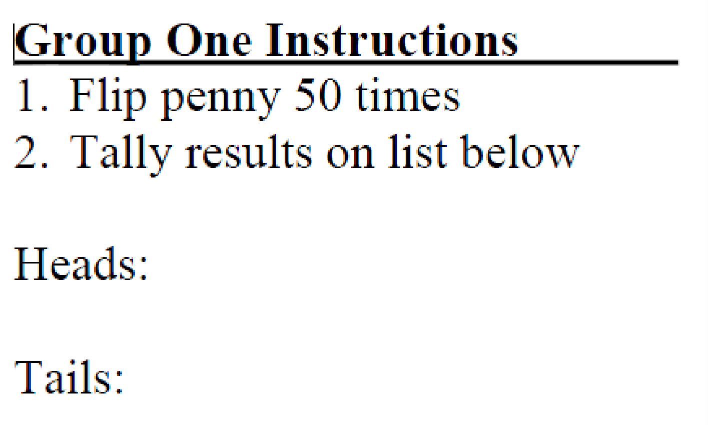
___ ___ ___ ___ ___ ___ ___ ___ ___ ___

___ ___ ___ ___ ___ ___ ___ ___ ___ ___

___ ___ ___ ___ ___ ___ ___ ___ ___ ___

___ ___ ___ ___ ___ ___ ___ ___ ___ ___

___ ___ ___ ___ ___ ___ ___ ___ ___ ___



Group One Instructions

1. Flip penny 50 times
2. Tally results on list below

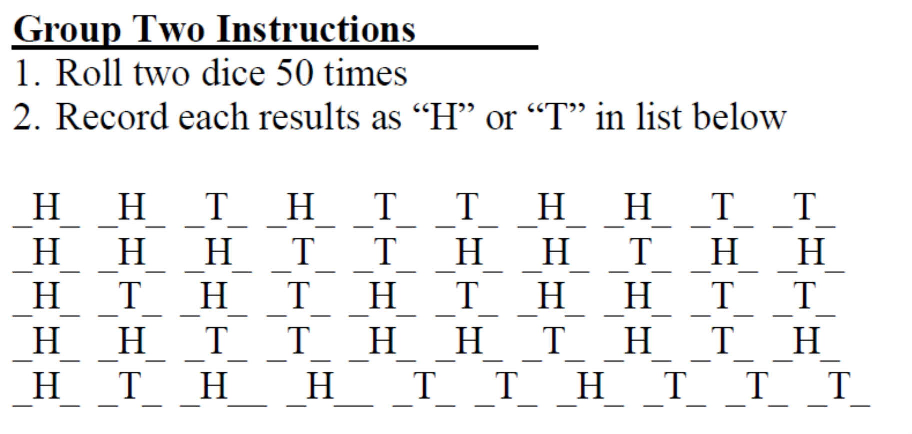
Heads:

Tails:

What is the asymmetry of the results?

Two Practical Exercises in Probabilities

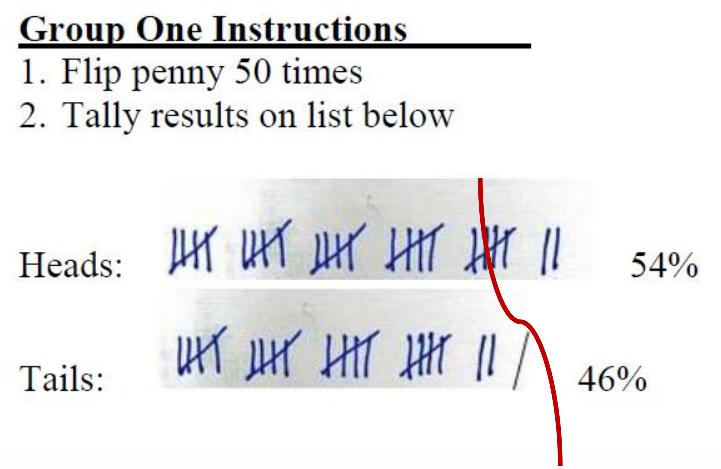
Flip a penny 50 times and record the results



Group Two Instructions

1. Roll two dice 50 times
2. Record each results as "H" or "T" in list below

_H_ _H_ _T_ _H_ _T_ _T_ _H_ _H_ _T_ _T_
_H_ _H_ _H_ _T_ _T_ _H_ _H_ _T_ _H_ _H_
_H_ _T_ _H_ _T_ _H_ _T_ _H_ _H_ _T_ _T_
_H_ _H_ _T_ _T_ _H_ _H_ _T_ _H_ _T_ _H_
_H_ _T_ _H__ _H__ _T_ _T_ _H_ _T_ _T_ _T_



Group One Instructions

1. Flip penny 50 times
2. Tally results on list below

Heads:	𝍸 𝍸 𝍸 𝍸 𝍸 ||	54%
Tails:	𝍸 𝍸 𝍸 𝍸 || /	46%

??% asymmetry

4% asymmetry

What is the asymmetry of the results?

Two Practical Exercises in Probabilities

Roll a pair of dice 50 times and record the results

Group One Instructions

Roll two dice 50 times

Record results on table below, checking one box for each die

2	3	4	5	6	7	8	9	10	11	12

Group Two Instructions

Roll two dice 50 times

Record each result in list below

___ ___ ___ ___ ___ ___ ___ ___ ___ ___

___ ___ ___ ___ ___ ___ ___ ___ ___ ___

___ ___ ___ ___ ___ ___ ___ ___ ___ ___

___ ___ ___ ___ ___ ___ ___ ___ ___ ___

___ ___ ___ ___ ___ ___ ___ ___ ___ ___

What is the mean value?
The standard deviation?

Two Practical Exercises in Probabilities

Roll a pair of dice 50 times and record the results

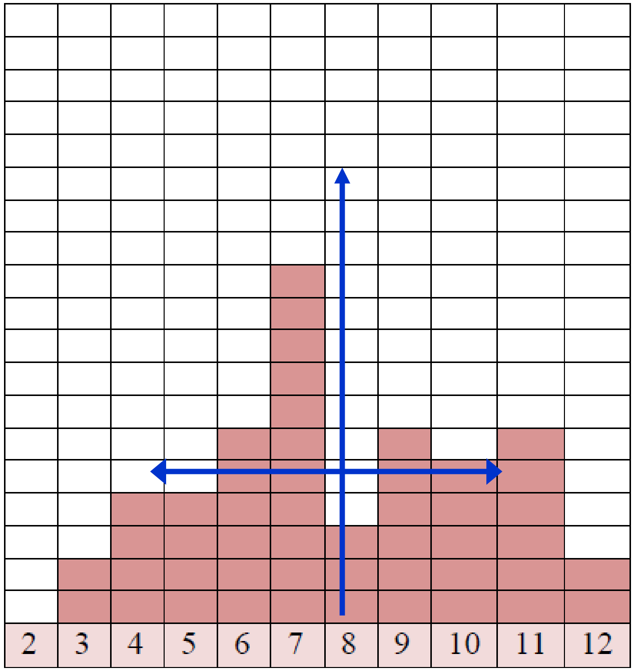
Group Two Instructions

Roll two dice 50 times
Record each result in list below

3	4	10	6	9	7	9	3	11	10
6	6	9	7	11	12	6	11	6	7
7	7	8	4	11	10	12	11	7	4
12	8	9	9	7	10	9	11	4	7
7	6	10	5	8	7	5	5	5	7

Group One Instructions

Roll two dice 50 times
Record results on table below, checking one box for each die



Mean = 7.3
St. Dev. = 2.8

What is the mean value?

The standard deviation?

What is the asymmetry (kurtosis)?

What is the probability of rolling a 4?

Discrete Distribution Functions

A data set to play with

$$26, 24, 26, 28, 23, 24, 25, 24, 26, 25. \qquad (5.1)$$

$$23, 24, 24, 24, 25, 25, 26, 26, 26, 28. \qquad (5.2)$$

The mean value

$$\bar{x} = \frac{\sum_i x_i}{N} = \frac{23 + 24 + 24 + 24 + 25 + \ldots + 28}{10}.$$

This equation is the same as

$$\bar{x} = \frac{23 + (24 \times 3) + (25 \times 2) + \ldots + 28}{10}$$

or in general

$$\sum_k (n_k) = N \quad \rightarrow \quad X = \frac{\sum_k (n_k \cdot x_k)}{N} = \frac{\sum_k (n_k \cdot x_k)}{\sum_k (n_k)}$$

Written in terms of "occurrence" F

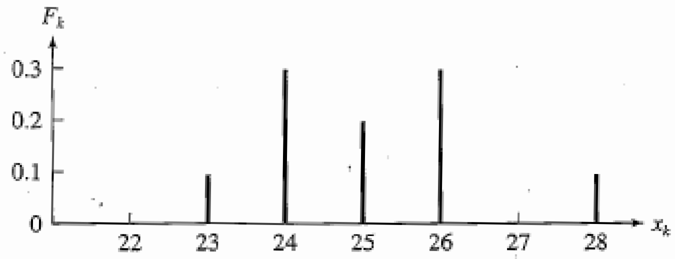


Table 5.1. Measured lengths x and their numbers of occurrences.

Different values, x_k	23	24	25	26	27	28
Number of times found, n_k	1	3	2	3	0	1

In terms of fractional expectations

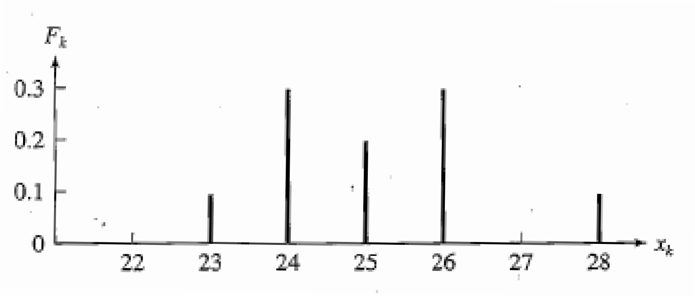
Fractional expectations $$F_k = \frac{n_k}{N}$$

Normalization condition $$1 = \sum_k (F_k)$$

Mean value $$X = \sum_k (F_k \cdot x_k)$$

(This is just a weighted sum.)

Limit of Discrete Distribution Functions



Binned data sets

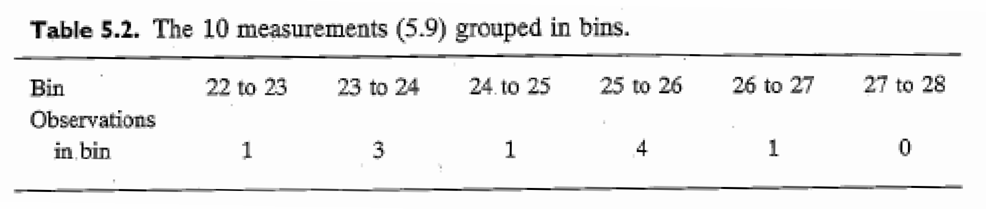
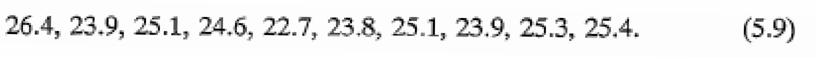
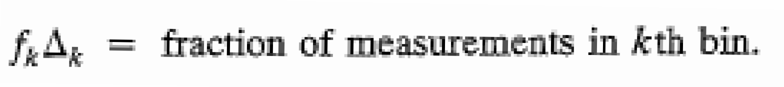


Table 5.2. The 10 measurements (5.9) grouped in bins.

Bin	22 to 23	23 to 24	24 to 25	25 to 26	26 to 27	27 to 28
Observations in bin	1	3	1	4	1	0


26.4, 23.9, 25.1, 24.6, 22.7, 23.8, 25.1, 23.9, 25.3, 25.4. (5.9)

"Normalizing" data sets


$f_k\Delta_k$ = fraction of measurements in kth bin.

f_k ≡ fractional occurrence

Δ_k ≡ bin width

Mean value: $$X = \sum_k F_k\, x_k = \sum_k (f_k\, \Delta_k) x_k$$

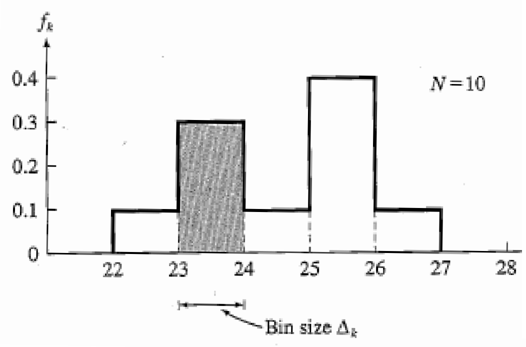
Normalization: $$1 = \frac{\sum_k (f_k\, \Delta_k) x_k}{\sum_k \Delta_k}$$

Expected value: $$Prob(4) = \frac{F_4}{N} = f_4$$

Limits of Distribution Functions

Consider the limiting distribution function as $N \rightarrow \infty$ and $\Delta_k \rightarrow 0$

Larger data sets



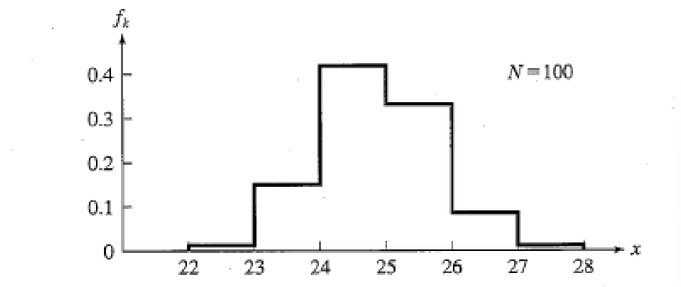
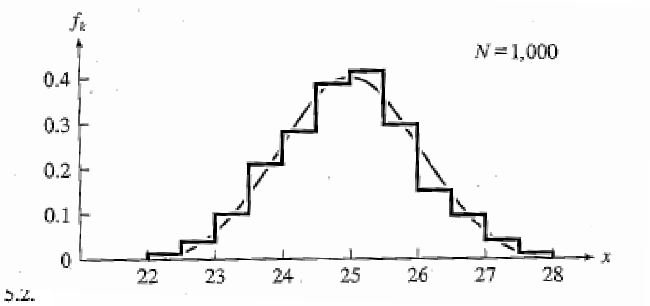
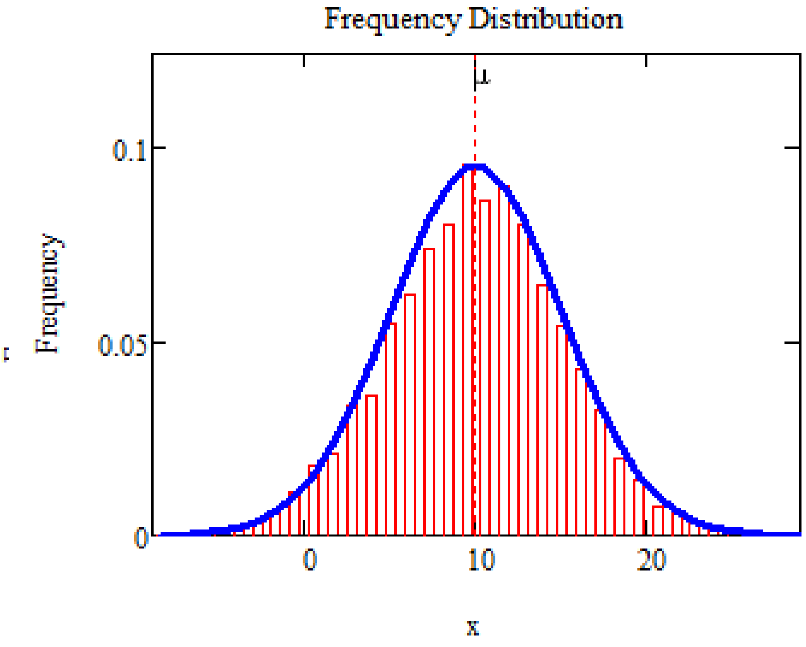


Figure 5.3. Histogram for 100 measurements of the same quantity as in Figure 5.2.





Mathcad Games:

# Data Pts: $N \equiv 6000$

Mean: $\mu \equiv 10$

Std. Dev.: $\sigma \equiv 5$

SDOM: $\frac{\sigma}{\sqrt{N}} = 0.065$

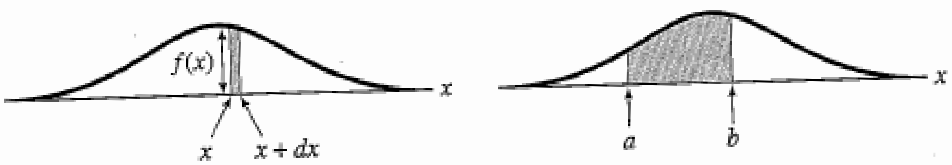
Fractional Error:

$$\left(\frac{\sigma}{\sqrt{N}}\right)\cdot\frac{1}{\mu} = 0.645\,\%$$

Continuous Distribution Functions

Meaning of Distribution Interval

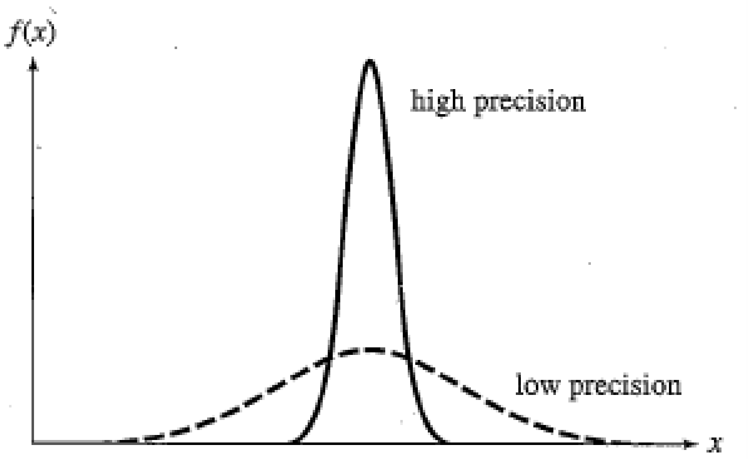
$$f(x)\,dx \;=\; \text{fraction of measurements that fall between } x \text{ and } x + dx. \qquad (5.10)$$



Thus (a) (b)

$$\int_a^b f(x)\,dx$$ = fraction of measurements that fall within a<x<b

and by extension $$\int_{-\infty}^{\infty} f(x)\,dx = 1$$



Normalization of Distribution

$$\sum_k (F_k) = 1 \quad \rightarrow \quad \int_{-\infty}^{+\infty} f(x)\,dx = 1$$

Central (mean) Value

$$\sum_k (F_k \cdot x_k) = X \quad \rightarrow$$

$$\int_{-\infty}^{+\infty} x\, f(x)\,dx = \bar{x} = \langle x \rangle \qquad (5.15)$$

Width of Distribution

$$\sum_k \left[(x_k - x)^2 \cdot F_k \right] = \sigma_X{}^2 \quad \rightarrow$$

$$\int_{-\infty}^{+\infty} (x - \bar{x})^2 f(x)\,dx = \sigma_x{}^2 = \langle (x - \bar{x})^2 \rangle \quad (5.16)$$

Moments of Distribution Functions

The first moment for a **probability distribution function** is

$$\bar{x} \equiv \langle x \rangle = first\ moment = \int_{-\infty}^{+\infty} x\, f(x)\, dx$$

For a **general distribution function**,

$$\bar{x} \equiv \langle x \rangle = first\ moment = \frac{\int_{-\infty}^{+\infty} x\ g(x) dx}{\int_{-\infty}^{+\infty}\ g(x) dx}$$

Generalizing, the nth moment is

$$x_n \equiv \langle x^n \rangle = nth\ moment = \frac{\int_{-\infty}^{+\infty} x^n\ g(x) dx}{\int_{-\infty}^{+\infty}\ g(x) dx} = \int_{-\infty}^{+\infty} x^n\ f(x)\, dx$$

(for a centered distribution)

0th moment ≡ N

1st moment ≡ $\bar{x}$

2nd moment ≡ $\langle (x \cancelto{0}{-\bar{x}})^2 \rangle \rightarrow \langle x^2 \rangle$

3rd moment ≡ kurtosis

Moments of Distribution Functions

Generalizing, the nth moment is

$$x_n \equiv \langle x^n \rangle = nth\ moment = \frac{\int_{-\infty}^{+\infty} x^n\ g(x)dx}{\int_{-\infty}^{+\infty}\ g(x)dx} = \int_{-\infty}^{+\infty} x^n\ f(x)\ dx$$

Oth moment ≡ N

1st moment ≡ $\bar{x}$

2nd moment ≡ $\langle (x - \cancelto{0}{\bar{x}})^2 \rangle \rightarrow \langle x^2 \rangle$

3rd moment ≡ kurtosis

The nth moment about the mean is

$$\mu_n \equiv \langle (x-\bar{x})^n \rangle = nth\ moment\ about\ the\ mean$$

$$= \frac{\int_{-\infty}^{+\infty} (x-\bar{x})^n\ g(x)dx}{\int_{-\infty}^{+\infty}\ g(x)dx} = \int_{-\infty}^{+\infty} (x-\bar{x})^n\ f(x)\ dx$$

The standard deviation (or second moment about the mean) is

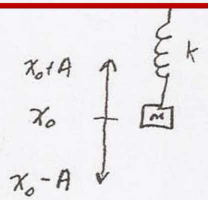
$$\sigma_x^2 \equiv \mu_2 \equiv \langle (x-\bar{x})^2 \rangle = 2nd\ moment\ about\ the\ mean$$

$$= \frac{\int_{-\infty}^{+\infty} (x-\bar{x})^2\ g(x)dx}{\int_{-\infty}^{+\infty}\ g(x)dx} = \int_{-\infty}^{+\infty} (x-\bar{x})^2\ f(x)\ dx$$

Example of Continuous Distribution Functions and Expectation Values

Harmonic Oscillator: Example from Mechanics

Consider a mass on a spring with frequency ω and equilibrium position x_0



$$\omega = \sqrt{k/m} = \frac{2\pi}{T}$$

The equations of motion are:

$$x(t) = A \sin \omega t + x_0$$
$$\dot{x}(t) = -A\omega \cos \omega t$$
$$\ddot{x}(t) = -A\omega^2 \sin \omega t$$

Expected Values ➔

The expectation value of a function R(x) is

$$\langle \mathrm{R(x)} \rangle \equiv \frac{\int_{-\infty}^{+\infty} \mathrm{R(x)}\, g(x)\, dx}{\int_{-\infty}^{+\infty} g(x)\, dx}$$

$$= \int_{-\infty}^{+\infty} \mathrm{R(x)}\, f(x)\, dx$$

The averge time over one period is:

$$\langle t \rangle = \frac{\int_0^T t\,dt}{\int_0^T dt} = \frac{\frac{1}{2} t^2 \big|_0^T}{t \big|_0^T} = \frac{\frac{1}{2} T^2}{T} = \frac{1}{2} T = \frac{\pi}{\omega}$$

The average position over one period (and over all time) is:

$$\langle x \rangle = \frac{\int_0^T x(t)\,dt}{T} = \frac{\int_0^T [A \sin \omega t + x_0]\,dt}{T}$$
$$= \frac{\left[\frac{A}{\omega} \cos \omega t + x_0 t\right]_0^T}{T} = x_0$$

The average of the position squared is:

$$\langle x^2 \rangle = \frac{1}{T} \int_0^T [x(t)]^2 dt = \frac{1}{T} \int_0^T [A \sin \omega t + x_0]^2 dt$$
$$= \frac{1}{T}\left[T\left(x_0^2 + A^2/2\right)\right] = x_0^2 - A^2/2$$

The average of the force is:

$$\langle F \rangle = \langle -k(x - x_0) \rangle = 0$$
$$\langle F \rangle = \langle m\ddot{x} \rangle = -m\omega^2 \langle x - x_0 \rangle = 0$$

The average velocity is:

$$\langle v \rangle = \frac{\int_0^T (-A\omega \cos \omega t)\,dt}{T} = \frac{-A\omega}{T}\left[-\sin \omega t\right]_0^T = 0$$

The average kinetic euergy is:

$$\langle KE \rangle = \frac{1}{2} m \langle v^2 \rangle = \frac{m}{2} A^2 \omega^2 \int_0^T \cos^2 \omega t\, dt = \frac{mA^2\omega^2}{2}\left[\frac{t}{2} + \frac{\sin 2\omega t}{4\omega}\right]_0^T$$
$$= \frac{mA^2\omega^2 T}{4} = \frac{\pi}{2}\sqrt{km}$$

Note: $\langle KE \rangle \neq \frac{1}{2} m \langle v \rangle^2 = 0$

Example of Continuous Distribution Functions and Expectation Values

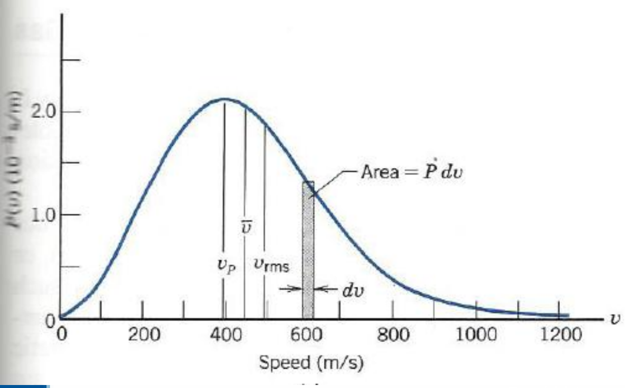
Boltzmann Distribution: Example from Kinetic Theory

Expected Values ➔

The expectation value of a function R(x) is

$$\langle \mathrm{R(x)} \rangle \equiv \frac{\int_{-\infty}^{+\infty} \mathrm{R(x)}\, g(x)\, dx}{\int_{-\infty}^{+\infty} g(x)\, dx}$$

$$= \int_{-\infty}^{+\infty} \mathrm{R(x)}\, f(x)\, dx$$



The Boltzmann distribution function for velocities of particles as a function of temperature, T is:

$$P(v;T) = 4\pi \left(\frac{M}{2\pi\, k_B T}\right)^{3/2} v^2 exp\left[\tfrac{1}{2} M v^2 \Big/ \frac{1}{2} k_B T\right]$$

Then

$$\langle \mathrm{v} \rangle = \int_{-\infty}^{+\infty} v\, P(v)\, dv = \left[{}^{8\, k_B T}\!/_{\pi\, M}\right]^{1/2}$$

$$\langle \mathrm{v}^2 \rangle = \int_{-\infty}^{+\infty} \mathrm{v}^2\, P(v)\, dv = \left[{}^{3\, k_B T}\!/_{M}\right]^{1/2}$$

$$\text{implies } \langle \mathrm{KE} \rangle = \tfrac{1}{2} M \langle \mathrm{v}^2 \rangle = \frac{3}{2} k_B T$$

$$\mathrm{v}_{\mathrm{peak}} = \sqrt{\left[{}^{2\, k_B T}\!/_{M}\right]^{1/2}} = \left[{}^{2}\!/_{3}\right]^{1/2} \langle \mathrm{v}^2 \rangle$$

Example of Continuous Distribution Functions and Expectation Values

Fermi-Dirac Distribution: Example from Kinetic Theory

For a system of identical fermions, the average number of fermions in a single-particle state i, is given by the Fermi–Dirac (F–D) distribution,

$$\bar{n}_i = \frac{1}{e^{(\epsilon_i-\mu)/kT}+1}$$

where k_B is Boltzmann's constant, T is the absolute temperature, ϵ_i is the energy of the single-particle state i, and μ is the total chemical potential.

Since the F–D distribution was derived using the Pauli exclusion principle, which allows at most one electron to occupy each possible state, a result is that $0 < \bar{n}_i < 1$

When a quasi-continuum of energies ϵ has an associated density of states $g(\epsilon)$ (i.e. the number of states per unit energy range per unit volume) the average number of fermions per unit energy range per unit volume is,

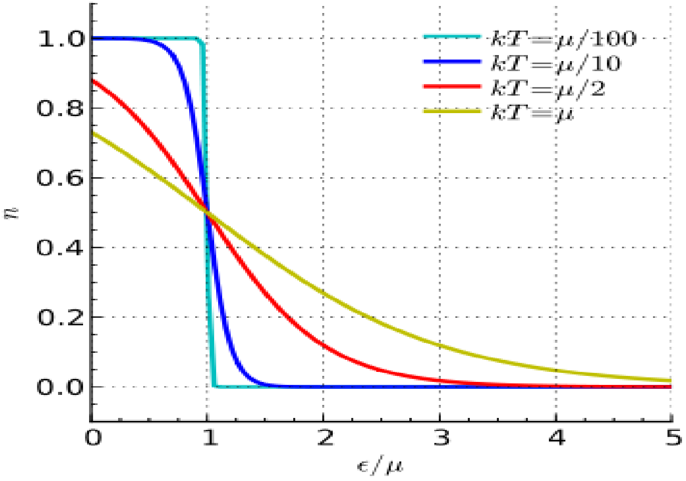
$$\bar{\mathcal{N}}(\epsilon) = g(\epsilon)\; F(\epsilon)$$

where $F(\epsilon)$ is called the Fermi function

$$F(\epsilon) = \frac{1}{e^{(\epsilon-\mu)/kT}+1}$$

so that,

$$\bar{\mathcal{N}}(\epsilon) = \frac{g(\epsilon)}{e^{(\epsilon-\mu)/kT}+1}$$



Example of Continuous Distribution Functions and Expectation Values

Finite Square Well: Example from Quantum Mechanics

Expectation Values

The expectation value of a **QM operator** O(x) is $\langle \mathrm{O(x)} \rangle \equiv \dfrac{\int_{-\infty}^{+\infty} \Psi^*(\mathrm{x}) O(x) \Psi(\mathrm{x}) dx}{\int_{-\infty}^{+\infty} \Psi^*(\mathrm{x}) \Psi(\mathrm{x}) dx}$

For a finite square well of width L, $\Psi_\mathrm{n}(\mathrm{x}) = \sqrt{{}^{2}/_{\mathrm{L}}}\ \sin\left[\dfrac{\mathrm{n}\,\pi\,\mathrm{x}}{\mathrm{L}}\right]$

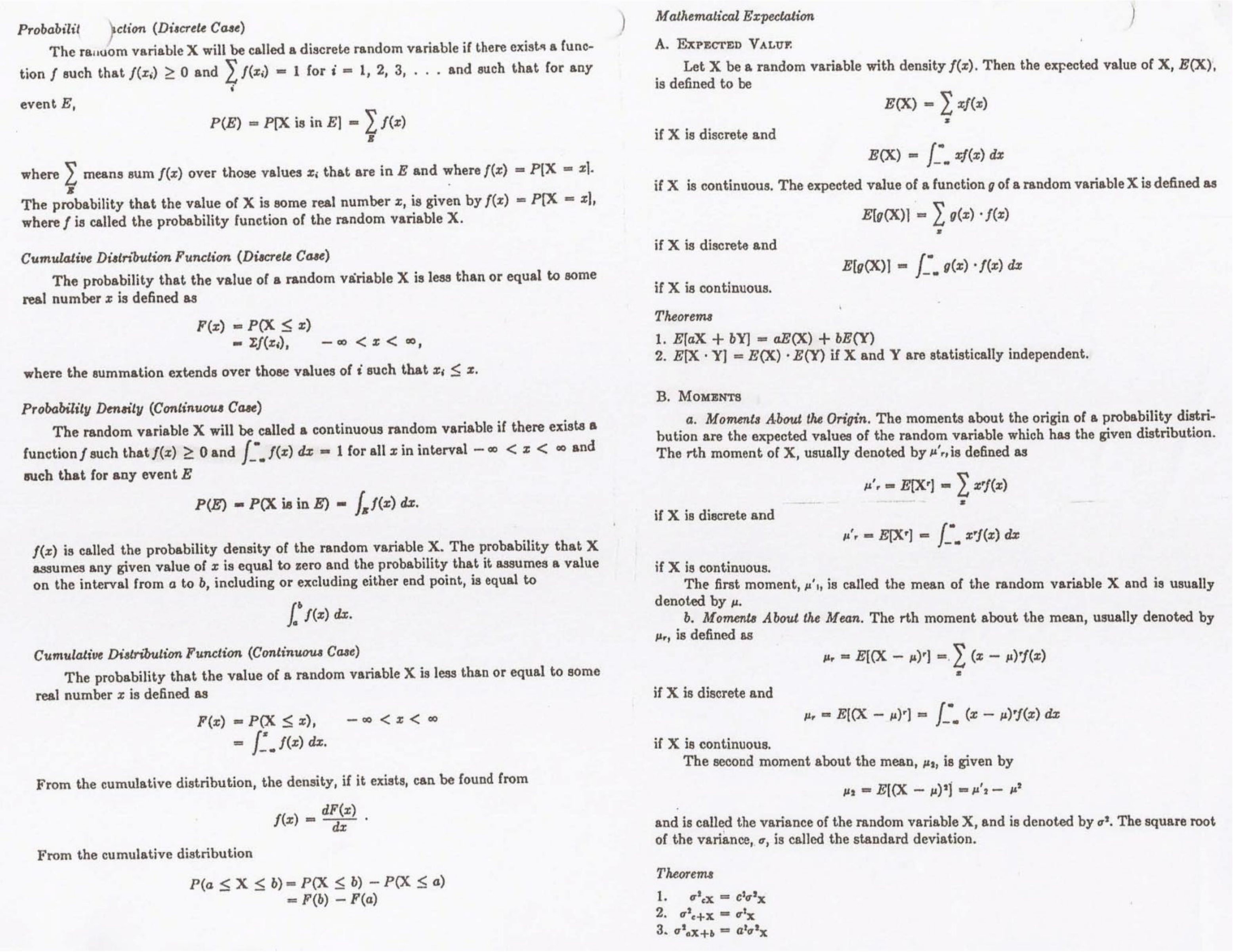
$$\langle \Psi_\mathrm{n}{}^*(\mathrm{x}) | \Psi_\mathrm{n}(\mathrm{x}) \rangle \equiv \frac{\int_{-\infty}^{+\infty} \Psi_\mathrm{n}{}^*(\mathrm{x}) O(x)\, \Psi_\mathrm{n}(\mathrm{x}) dx}{\int_{-\infty}^{+\infty} \Psi_\mathrm{n}{}^*(\mathrm{x}) \Psi_\mathrm{n}(\mathrm{x})\, dx} = 1$$

$$\langle x \rangle = \langle \Psi_\mathrm{n}{}^*(\mathrm{x}) | \mathrm{x} | \Psi_\mathrm{n}(\mathrm{x}) \rangle \equiv \frac{\int_{-\infty}^{+\infty} \Psi_\mathrm{n}{}^*(\mathrm{x}) x\, \Psi_\mathrm{n}(\mathrm{x}) dx}{\int_{-\infty}^{+\infty} \Psi_\mathrm{n}{}^*(\mathrm{x}) \Psi_\mathrm{n}(\mathrm{x})\, dx} = L/2$$

$$\langle p \rangle = \langle \Psi_\mathrm{n}{}^*(\mathrm{x}) | \frac{\hbar}{\mathrm{i}} \frac{\partial}{\partial \mathrm{x}} | \Psi_\mathrm{n}(\mathrm{x}) \rangle \equiv \frac{\int_{-\infty}^{+\infty} \Psi_\mathrm{n}{}^*(\mathrm{x}) \frac{\hbar}{\mathrm{i}} \frac{\partial}{\partial \mathrm{x}} \Psi_\mathrm{n}(\mathrm{x}) dx}{\int_{-\infty}^{+\infty} \Psi_\mathrm{n}{}^*(\mathrm{x}) \Psi_\mathrm{n}(\mathrm{x})\, dx} = 0$$

$$\langle E_n \rangle = \langle \Psi_\mathrm{n}{}^*(\mathrm{x}) | \mathrm{i}\hbar \frac{\partial}{\partial \mathrm{t}} | \Psi_\mathrm{n}(\mathrm{x}) \rangle \equiv \frac{\int_{-\infty}^{+\infty} \Psi_\mathrm{n}{}^*(\mathrm{x}) \mathrm{i}\hbar \frac{\partial}{\partial \mathrm{t}} \Psi_\mathrm{n}(\mathrm{x}) dx}{\int_{-\infty}^{+\infty} \Psi_\mathrm{n}{}^*(\mathrm{x}) \Psi_\mathrm{n}(\mathrm{x})\, dx} = \frac{n^2 \pi^2 \hbar^2}{2\, m\, L^2}$$

Summary of Distribution Functions



Probabilit[illegible]*ction (Discrete Case)*

The random variable X will be called a discrete random variable if there exists a function f such that $f(x_i) \geq 0$ and $\sum_i f(x_i) = 1$ for $i = 1, 2, 3, \ldots$ and such that for any event E,

$$P(E) = P[\mathrm{X} \text{ is in } E] = \sum_E f(x)$$

where $\sum_E$ means sum $f(x)$ over those values x_i that are in E and where $f(x) = P[\mathrm{X} = x]$. The probability that the value of X is some real number x, is given by $f(x) = P[\mathrm{X} = x]$, where f is called the probability function of the random variable X.

Cumulative Distribution Function (Discrete Case)

The probability that the value of a random variable X is less than or equal to some real number x is defined as

$$\begin{aligned} F(x) &= P(\mathrm{X} \leq x) \\ &= \Sigma f(x_i), \qquad -\infty < x < \infty, \end{aligned}$$

where the summation extends over those values of i such that $x_i \leq x$.

Probability Density (Continuous Case)

The random variable X will be called a continuous random variable if there exists a function f such that $f(x) \geq 0$ and $\int_{-\infty}^{\infty} f(x)\,dx = 1$ for all x in interval $-\infty < x < \infty$ and such that for any event E

$$P(E) = P(\mathrm{X} \text{ is in } E) = \int_E f(x)\,dx.$$

$f(x)$ is called the probability density of the random variable X. The probability that X assumes any given value of x is equal to zero and the probability that it assumes a value on the interval from a to b, including or excluding either end point, is equal to

$$\int_a^b f(x)\,dx.$$

Cumulative Distribution Function (Continuous Case)

The probability that the value of a random variable X is less than or equal to some real number x is defined as

$$\begin{aligned} F(x) &= P(\mathrm{X} \leq x), \qquad -\infty < x < \infty \\ &= \int_{-\infty}^{x} f(x)\,dx. \end{aligned}$$

From the cumulative distribution, the density, if it exists, can be found from

$$f(x) = \frac{dF(x)}{dx}.$$

From the cumulative distribution

$$\begin{aligned} P(a \leq \mathrm{X} \leq b) &= P(\mathrm{X} \leq b) - P(\mathrm{X} \leq a) \\ &= F(b) - F(a) \end{aligned}$$

Mathematical Expectation

A. Expected Valuf.

Let X be a random variable with density $f(x)$. Then the expected value of X, $E(\mathrm{X})$, is defined to be

$$E(\mathrm{X}) = \sum_x x f(x)$$

if X is discrete and

$$E(\mathrm{X}) = \int_{-\infty}^{\infty} x f(x)\,dx$$

if X is continuous. The expected value of a function g of a random variable X is defined as

$$E[g(\mathrm{X})] = \sum_x g(x) \cdot f(x)$$

if X is discrete and

$$E[g(\mathrm{X})] = \int_{-\infty}^{\infty} g(x) \cdot f(x)\,dx$$

if X is continuous.

Theorems

1. $E[a\mathrm{X} + b\mathrm{Y}] = aE(\mathrm{X}) + bE(\mathrm{Y})$
2. $E[\mathrm{X} \cdot \mathrm{Y}] = E(\mathrm{X}) \cdot E(\mathrm{Y})$ if X and Y are statistically independent.

B. Moments

a. Moments About the Origin. The moments about the origin of a probability distribution are the expected values of the random variable which has the given distribution. The rth moment of X, usually denoted by μ'_r, is defined as

$$\mu'_r = E[\mathrm{X}^r] = \sum_x x^r f(x)$$

if X is discrete and

$$\mu'_r = E[\mathrm{X}^r] = \int_{-\infty}^{\infty} x^r f(x)\,dx$$

if X is continuous.

The first moment, μ'_1, is called the mean of the random variable X and is usually denoted by μ.

b. Moments About the Mean. The rth moment about the mean, usually denoted by μ_r, is defined as

$$\mu_r = E[(\mathrm{X} - \mu)^r] = \sum_x (x - \mu)^r f(x)$$

if X is discrete and

$$\mu_r = E[(\mathrm{X} - \mu)^r] = \int_{-\infty}^{\infty} (x - \mu)^r f(x)\,dx$$

if X is continuous.

The second moment about the mean, μ_2, is given by

$$\mu_2 = E[(\mathrm{X} - \mu)^2] = \mu'_2 - \mu^2$$

and is called the variance of the random variable X, and is denoted by σ^2. The square root of the variance, σ, is called the standard deviation.

Theorems

1. $\sigma^2_{c\mathrm{X}} = c^2\sigma^2_{\mathrm{X}}$
2. $\sigma^2_{c+\mathrm{X}} = \sigma^2_{\mathrm{X}}$
3. $\sigma^2_{a\mathrm{X}+b} = a^2\sigma^2_{\mathrm{X}}$

Available on web site

Intermediate Lab

PHYS 3870

The Gaussian Distribution Function

References: Taylor Ch. 5

Gaussian Integrals

Factorial Approximations

$n! \approx (2\pi n)^{1/2}\, n^n\, exp\left[-n + \frac{1}{12\,n} + O\left(\frac{1}{n^2}\right)\right]$

$log(n!) \approx \frac{1}{2}\, log(2\pi) + \left(n + \frac{1}{2}\right)\log(n) - n + \frac{1}{12\,n} + O\left(\frac{1}{n^2}\right)$

$log(n!) \approx n\log(n) - n$ (for all terms decreasing faster than linearly with n)

Gaussian Integrals

$I_m = 2\int_0^\infty x^m \exp(-x^2)\; dx$; m>-1

$I_m = 2\int_0^\infty y^n \exp(-y)\; dy \equiv \Gamma(n+1)$; $x^2 \equiv y,\;\; 2\,dx = y^{1/2}\,dy,\quad n \equiv \frac{1}{2}\,(m-1)$

$I_0 = \Gamma\left(n = \frac{1}{2}\right) = \sqrt{\pi}$; m=0, $n = -\frac{1}{2}$

$I_{2\,k} = \Gamma\left(k + \frac{1}{2}\right) = \left(k - {}^{1}/_{2}\right)\left(k - {}^{3}/_{2}\right)\ldots\left({}^{3}/_{2}\right)\left({}^{1}/_{2}\right)\sqrt{\pi}$; even m m=2 k>0, $n = k - \frac{1}{2}$

$I_{2\,k+1} = \Gamma(k+1) = k!$; odd m m=2 k+1>0, $n = k \geq 0$

Gaussian Distribution Function

Distribution Function

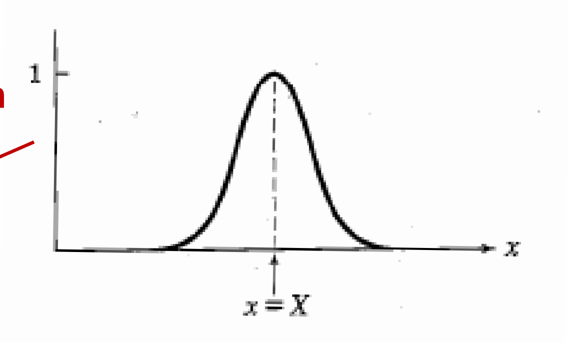
Independent Variable

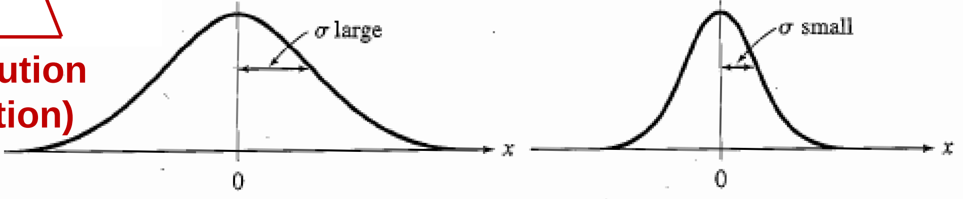
Center of Distribution (mean)

$$G_{X,\sigma}(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-X)^2/2\sigma^2}$$

Normalization Constant

Width of Distribution (standard deviation)





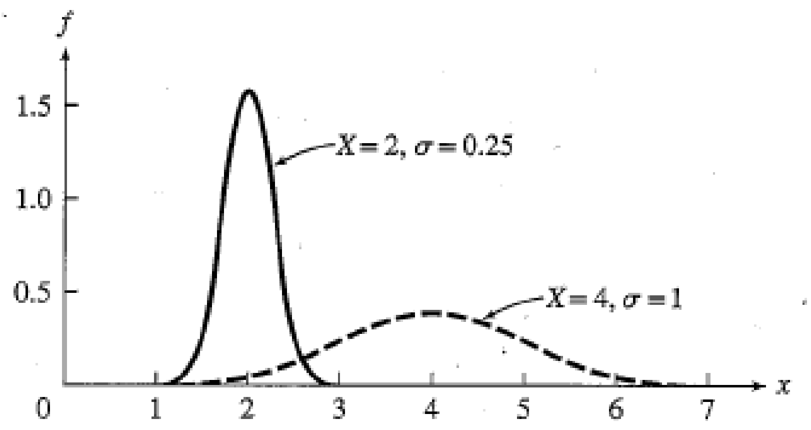
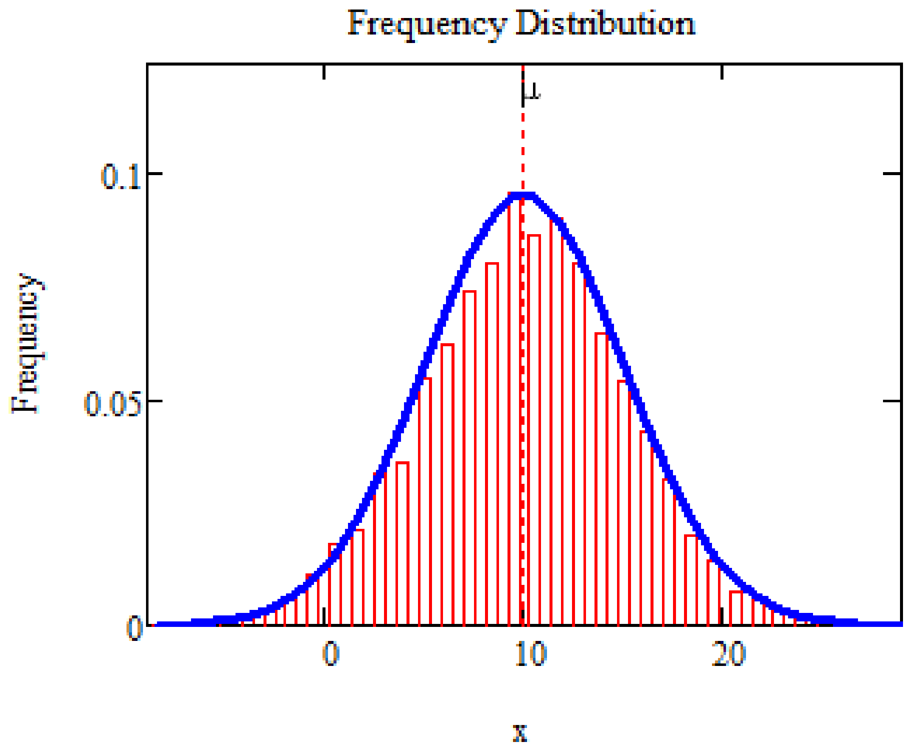


Figure 5.10. Two normal, or Gauss, distributions.

Gaussian Distribution Function

Effects of Increasing N on Gaussian Distribution Functions

Consider the limiting distribution function as N →∞ and dx→0

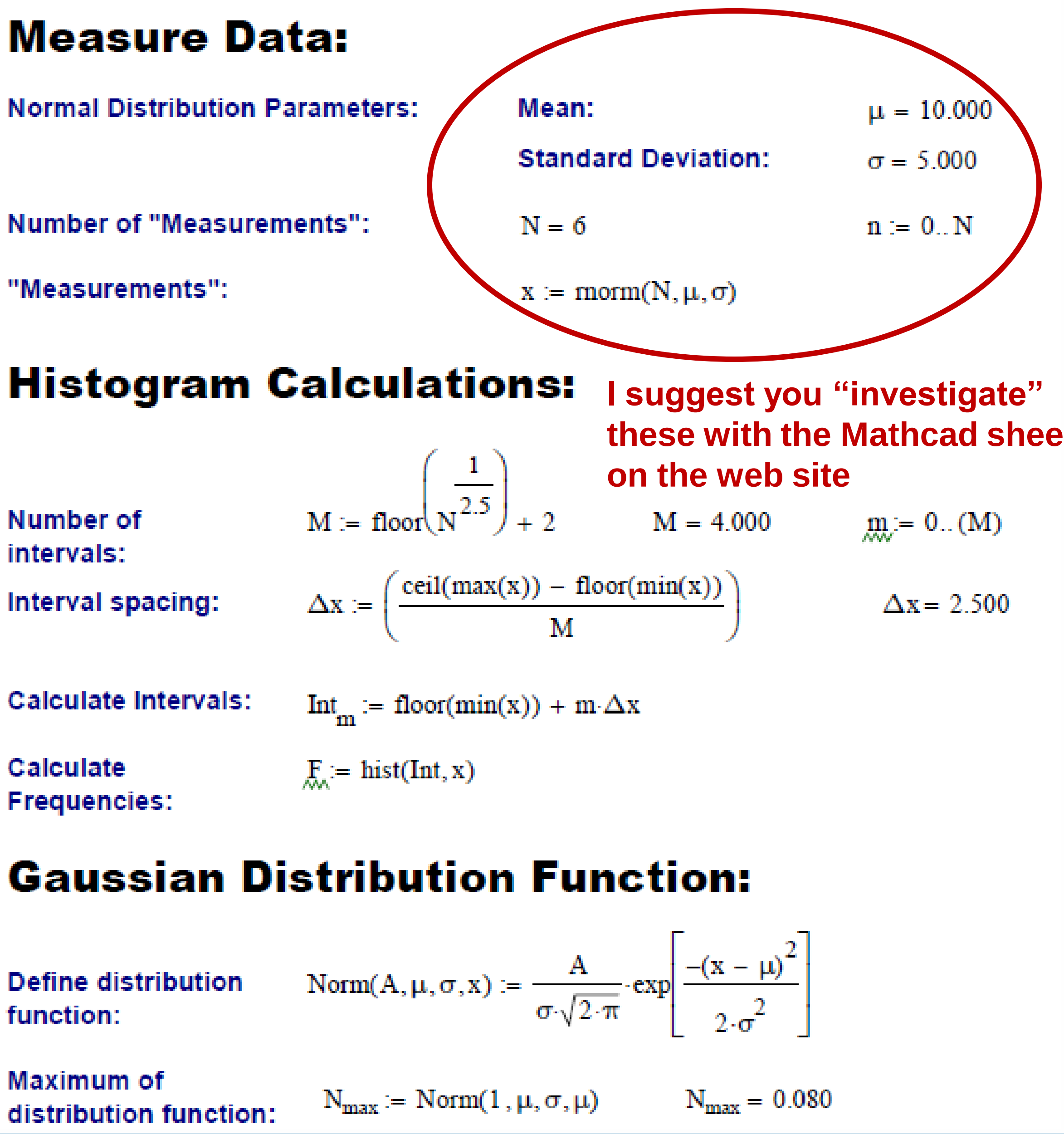


# Data Pts:	$N \equiv 6000$
Mean:	$\mu \equiv 10$
Std. Dev.:	$\sigma \equiv 5$
SDOM:	$\frac{\sigma}{\sqrt{N}} = 0.065$

Fractional Error:

$$\left(\frac{\sigma}{\sqrt{N}}\right)\cdot\frac{1}{\mu} = 0.645\,\%$$

Defining the Gaussian distribution function in Mathcad



Measure Data:

Normal Distribution Parameters: Mean: $\mu = 10.000$

Standard Deviation: $\sigma = 5.000$

Number of "Measurements": $N = 6$ $\quad n := 0..N$

"Measurements": $x := \mathrm{rnorm}(N, \mu, \sigma)$

Histogram Calculations:

I suggest you “investigate” these with the Mathcad sheet on the web site

Number of intervals: $M := \mathrm{floor}\left(N^{\frac{1}{2.5}}\right) + 2$ $\quad M = 4.000$ $\quad m := 0..(M)$

Interval spacing: $\Delta x := \left(\dfrac{\mathrm{ceil}(\max(x)) - \mathrm{floor}(\min(x))}{M}\right)$ $\quad \Delta x = 2.500$

Calculate Intervals: $\mathrm{Int}_m := \mathrm{floor}(\min(x)) + m \cdot \Delta x$

Calculate Frequencies: $F := \mathrm{hist}(\mathrm{Int}, x)$

Gaussian Distribution Function:

Define distribution function: $\mathrm{Norm}(A, \mu, \sigma, x) := \dfrac{A}{\sigma \cdot \sqrt{2 \cdot \pi}} \cdot \exp\left[\dfrac{-(x-\mu)^2}{2 \cdot \sigma^2}\right]$

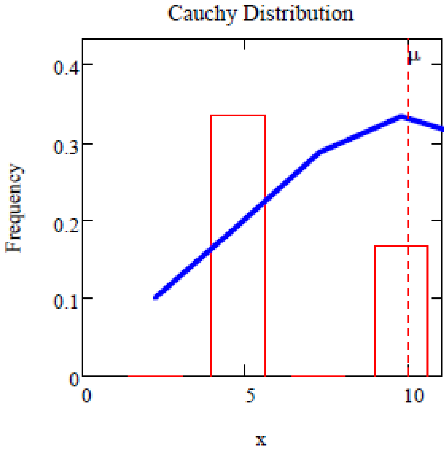
Maximum of distribution function: $N_{max} := \mathrm{Norm}(1, \mu, \sigma, \mu)$ $\quad N_{max} = 0.080$

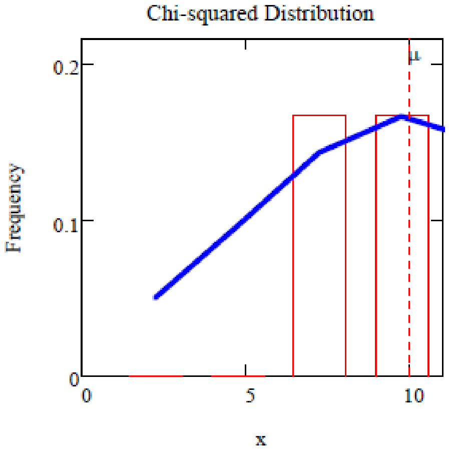
Alternative Distribution Function:

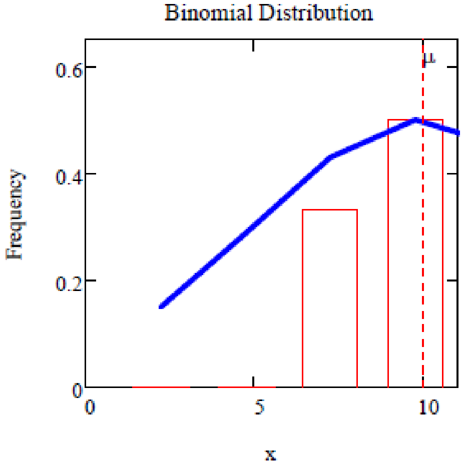
Distribution function:	Normal Distribution Parameters:		"Measurements":	Frequencies :
Binomial	$p := .68$	$\nu := 15$	$x_b := \mathrm{rbinom}(N, \nu, p)$	$F_b := \mathrm{hist}(\mathrm{Int}, x_b)$
Poisson	$\lambda := 10$		$x_p := \mathrm{rpois}(N, \lambda)$	$F_p := \mathrm{hist}(\mathrm{Int}, x_p)$
Cauchy	$l := \mu$	$s := \sigma$	$x_c := \mathrm{rcauchy}(N, 1, s)$	$F_c := \mathrm{hist}(\mathrm{Int}, x_c)$
Chi-squared	$d := N - 4$		$x_\chi := \mathrm{rpois}(N, \lambda)$	$F_\chi := \mathrm{hist}(\mathrm{Int}, x_\chi)$

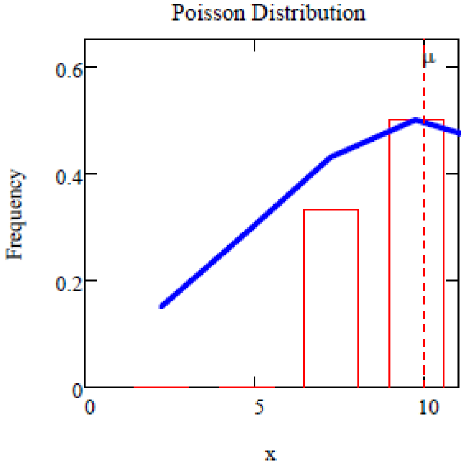
Using Mathcad to define other common distribution functions.

Consider the limiting distribution function as N →ꝏ and Δ_k→0









Gaussian Distribution Moments

Consider the Gaussian distribution function

$$G_{\bar{X}\sigma}(x) = \frac{1}{\sigma\sqrt{2\pi}} exp[(x-\bar{X})^2/2\sigma^2]$$

Use the normalization condition to evaluate the normalization constant (see Taylor, p. 132)

$$1 = \int_{-\infty}^{\infty} G_{\bar{X}\sigma}(x)\,dx = \int_{-\infty}^{\infty} Nexp[-(x-\bar{X})^2/2\sigma^2]\,dx$$

$$1 \xrightarrow{y\equiv x-\bar{X}, dy\equiv dx} \int_{-\infty}^{\infty} Nexp[-y^2/2\sigma^2]\,dx$$

$$1 \xrightarrow{z\equiv y/\sigma, dz\equiv dy/\sigma} \int_{-\infty}^{\infty} Nexp[-z^2/2]\,dz = N\sigma\sqrt{2\pi}$$

$$N = 1/(\sigma\sqrt{2\pi})$$

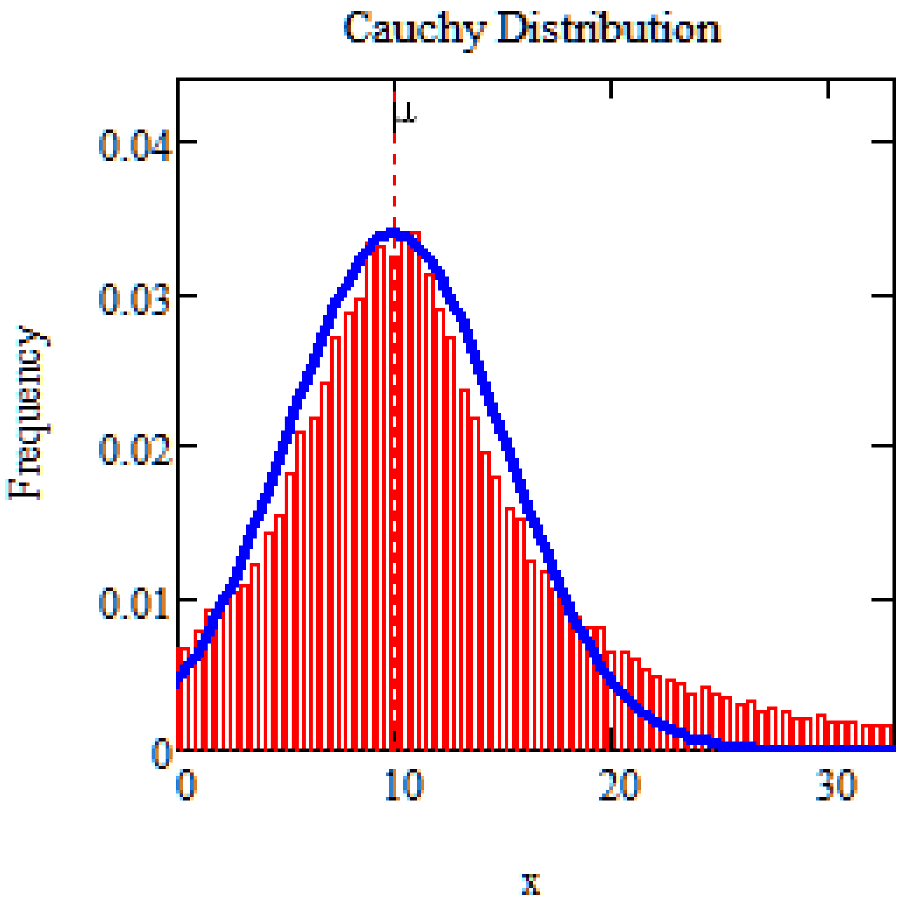
The mean, Ẋ, is the first moment of the Gaussian distribution function (see Taylor, p. 134)

$$\langle x\rangle = \int_{-\infty}^{\infty} x\, G_{\bar{X}\sigma}(x)\,dx = \bar{X}$$

The standard deviation, σ_x, is the standard deviation of the mean of the Gaussian distribution function (see Taylor, p. 143)

$${\sigma_x}^2 = \int_{-\infty}^{\infty} (x-\bar{X})^{\,2} G_{\bar{X}\sigma}(x)\,dx = \sigma^2$$

When is mean x not X_{best}?



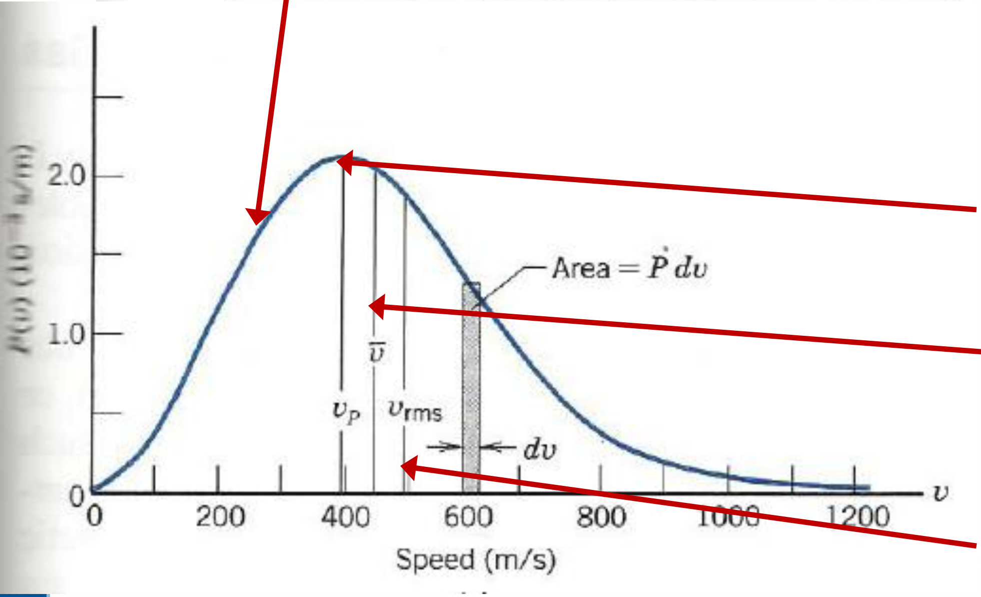
Answer: When the distribution is not symmetric about X.

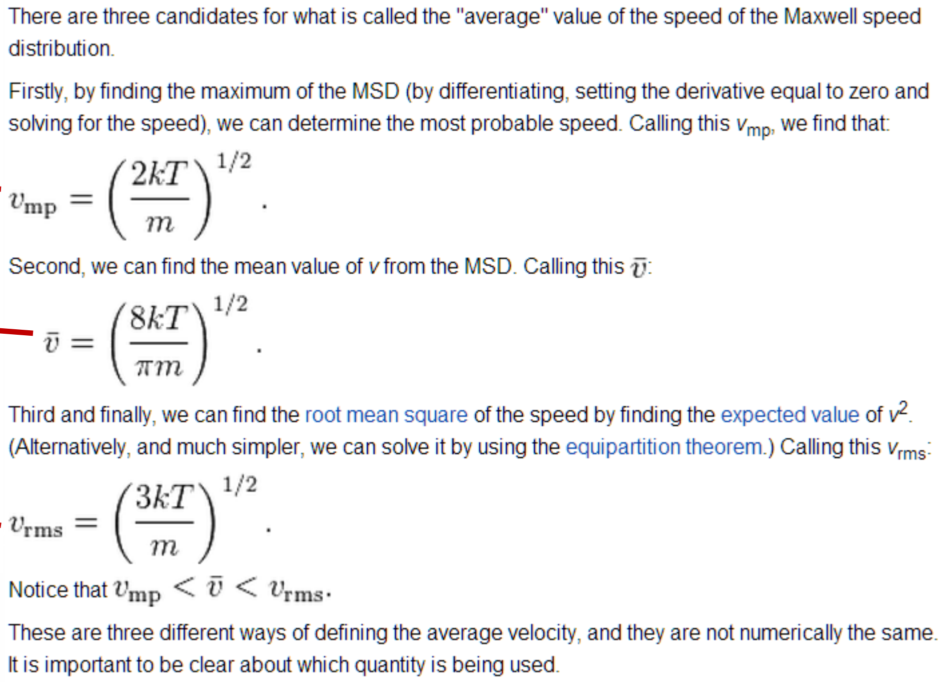
Example: Cauchy Distribution

When is mean x not X_{best}?

Maxwell speed distribution law is

$$\boxed{P(v) = 4\pi\left(\frac{M}{2\pi RT}\right)^{3/2} v^2 e^{-Mv^2/2RT}.}$$





There are three candidates for what is called the "average" value of the speed of the Maxwell speed distribution.

Firstly, by finding the maximum of the MSD (by differentiating, setting the derivative equal to zero and solving for the speed), we can determine the most probable speed. Calling this v_{mp}, we find that:

$$v_{\mathrm{mp}} = \left(\frac{2kT}{m}\right)^{1/2}.$$

Second, we can find the mean value of v from the MSD. Calling this $\bar{v}$:

$$\bar{v} = \left(\frac{8kT}{\pi m}\right)^{1/2}.$$

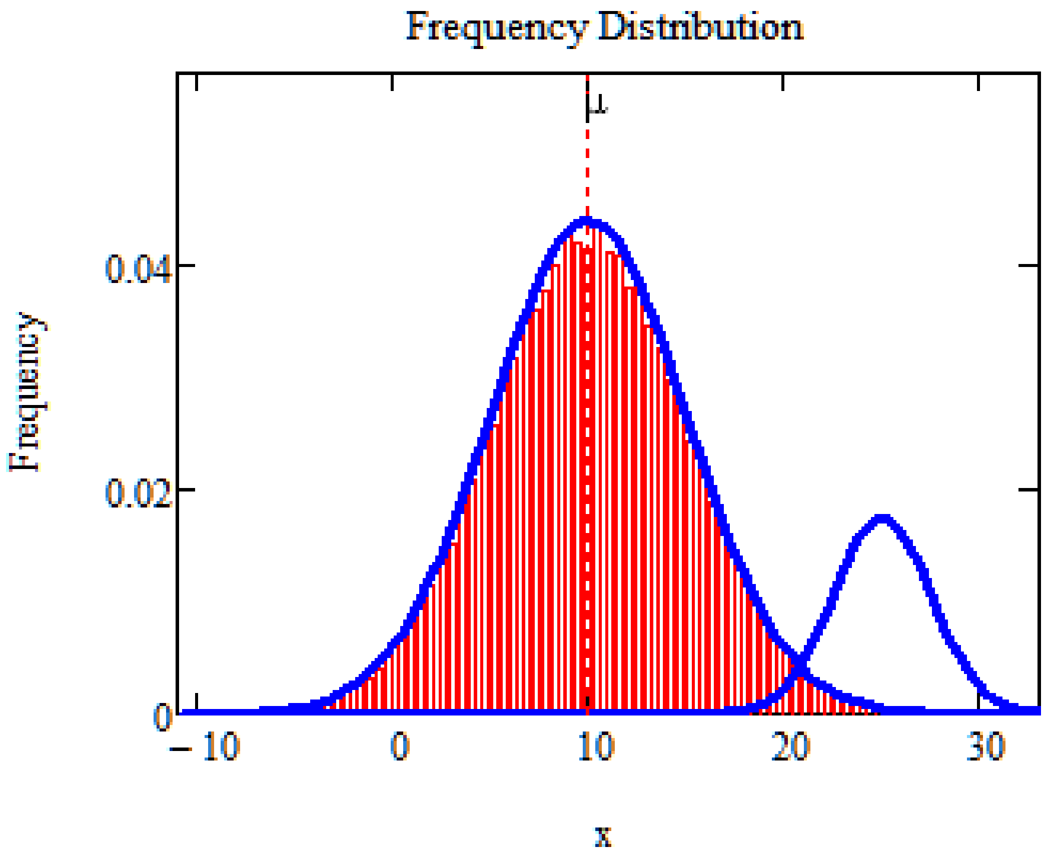
Third and finally, we can find the root mean square of the speed by finding the expected value of v^2. (Alternatively, and much simpler, we can solve it by using the equipartition theorem.) Calling this v_{rms}:

$$v_{\mathrm{rms}} = \left(\frac{3kT}{m}\right)^{1/2}.$$

Notice that $v_{\mathrm{mp}} < \bar{v} < v_{\mathrm{rms}}$.

These are three different ways of defining the average velocity, and they are not numerically the same. It is important to be clear about which quantity is being used.

When is mean x not X_{best}?



Answer: When the distribution is has more than one peak.

Intermediate Lab

PHYS 3870

The Gaussian Distribution Function and Its Relation to Errors

A Review of Probabilities in Combination

1 head AND 1 Four

$P(H,4) = P(H) * P(4)$

1 Six OR 1 Four

$P(6,4) = P(6) + P(4)$

(true for a "mutually exclusive" single role)

1 head OR 1 Four

$P(H,4) = P(H) + P(4) - P(H \text{ and } 4)$

(true for a "non-mutually exclusive" events)

NOT 1 Six

$P(\text{NOT } 6) = 1 - P(6)$

Probability of a data set of N like measurements, $(x_1,x_2,\ldots x_N)$

$$P\,(x_1,x_2,\ldots x_N) = P(x)_1 * P(x_2) * \ldots P(x_N)$$

The Gaussian Distribution Function and Its Relation to Errors

We will use the Gaussian distribution as applied to random variables to develop the mathematical basis for:

- **Mean**
- **Standard deviation**
- **Standard deviation of the mean (SDOM)**
- **Moments and expectation values**
- **Error propagation formulas**
- **Addition of errors in quadrature (for independent and random measurements)**
- **Numerical values for confidence limits (t-test)**
- **Principle of maximal likelihood**
- **Central limit theorem**
- **Weighted distributions and Chi squared**
- **Schwartz inequality (i.e., the uncertainty principle) (next lecture)**

Gaussian Distribution Moments

Consider the Gaussian distribution function

$$G_{\bar{X}\sigma}(x) = \frac{1}{\sigma\sqrt{2\pi}} exp[(x-\bar{X})^2/2\sigma^2]$$

Use the normalization condition to evaluate the normalization constant (see Taylor, p. 132)

$$1 = \int_{-\infty}^{\infty} G_{\bar{X}\sigma}(x)\, dx = \int_{-\infty}^{\infty} N exp[-(x-\bar{X})^2/2\sigma^2]\, dx$$

$$1 \xrightarrow{y\equiv x-\bar{X},dy\equiv dx} \int_{-\infty}^{\infty} N exp[-y^2/2\sigma^2]\, dx$$

$$1 \xrightarrow{z\equiv y/\sigma,dz\equiv dy/\sigma} \int_{-\infty}^{\infty} N exp[-z^2/2]\, dz = N\sigma\sqrt{2\pi}$$

$$N = 1/\left(\sigma\sqrt{2\pi}\right)$$

The mean, Ẋ, is the first moment of the Gaussian distribution function (see Taylor, p. 134)

$$\langle x\rangle = \int_{-\infty}^{\infty} x\, G_{\bar{X}\sigma}(x)\, dx = \bar{X}$$

The standard deviation, σ_x, is the standard deviation of the mean of the Gaussian distribution function (see Taylor, p. 143)

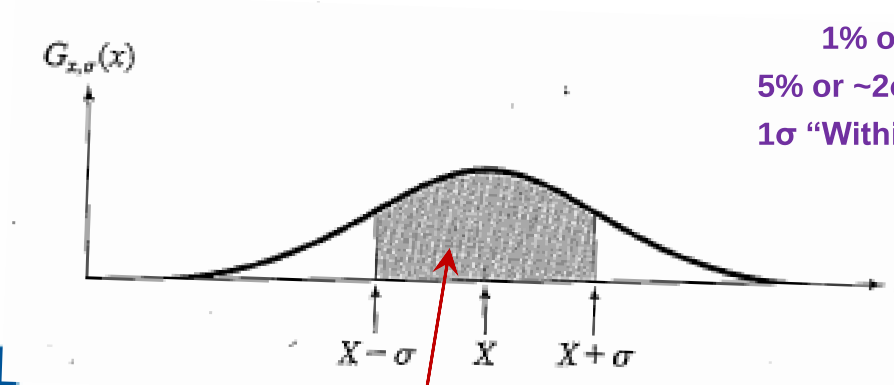
$$\sigma_x{}^2 = \int_{-\infty}^{\infty} (x-\bar{X})^2 G_{\bar{X}\sigma}(x)\, dx = \sigma^2$$

Standard Deviation of Gaussian Distribution

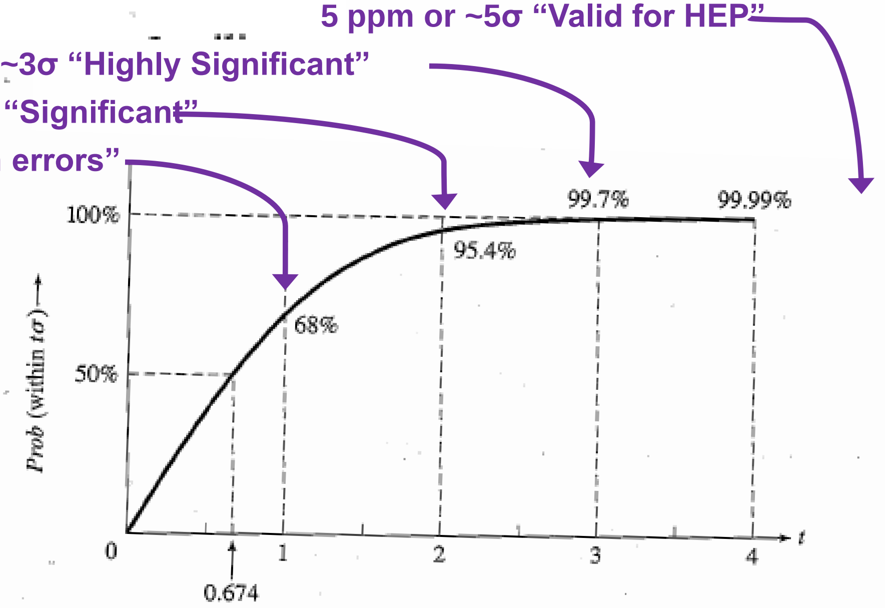
$$Prob(\text{within } \sigma) = \int_{X-\sigma}^{X+\sigma} G_{X,\sigma}(x)\,dx \qquad (5.32)$$

$$= \frac{1}{\sigma\sqrt{2\pi}} \int_{X-\sigma}^{X+\sigma} e^{-(x-X)^2/2\sigma^2}\,dx. \qquad (5.33)$$

See Sec. 10.6: Testing of Hypotheses



Area under curve (probability that –σ<x<+σ) is 68%



5 ppm or ~5σ “Valid for HEP”

1% or ~3σ “Highly Significant”

5% or ~2σ “Significant”

1σ “Within errors”

t	0	0.25	0.5	0.75	1.0	1.25	1.5	1.75	2.0	2.5	3.0	3.5	4.0
$Prob$ (%)	0	20	38	55	68	79	87	92	95.4	98.8	99.7	99.95	99.99

Figure 5.13. The probability $Prob$(within $t\sigma$) that a measurement of x will fall within t standard deviations of the true value $x = X$. Two common names for this function are the *normal error integral* and the *error function*, erf(t).

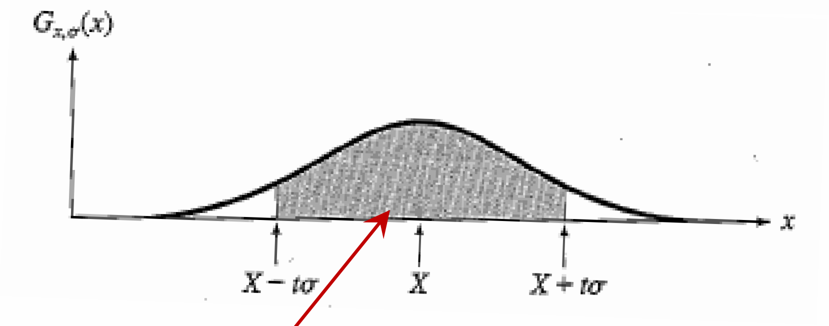
More complete Table in App. A and B

Ah, that’s highly significant!

Error Function of Gaussian Distribution

Error Function: (probability that –tσ<x<+tσ).

$$Prob(\text{within } t\sigma) = \frac{1}{\sqrt{2\pi}} \int_{-t}^{t} e^{-z^2/2}\, dz. \qquad (5.35)$$



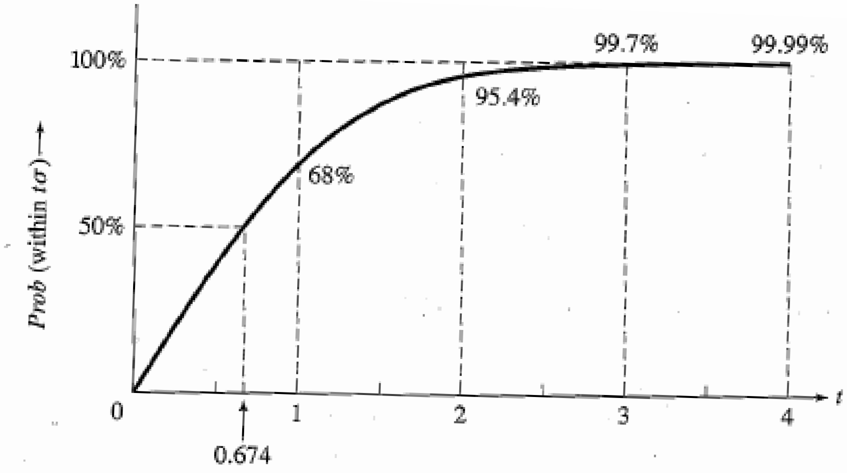
Area under curve (probability that –tσ<x<+tσ) is given in Table at right.

Complementary Error Function: (probability that –x<-tσ AND x>+tσ).

Prob(x outside tσ) = 1 - Prob(x within tσ)

Probable Error: (probability that –0.67σ<x<+0.67σ) is 50%.

Error Function: Tabulated values-see App. A.



t	0	0.25	0.5	0.75	1.0	1.25	1.5	1.75	2.0	2.5	3.0	3.5	4.0
Prob (%)	0	20	38	55	68	79	87	92	95.4	98.8	99.7	99.95	99.99

Figure 5.13. The probability *Prob*(within *tσ*) that a measurement of *x* will fall within *t* standard deviations of the true value *x* = *X*. Two common names for this function are the *normal error integral* and the *error function*, erf(*t*).

More complete Table in App. A and B

Useful Points on Gaussian Distribution

Full Width at Half Maximum

FWHM

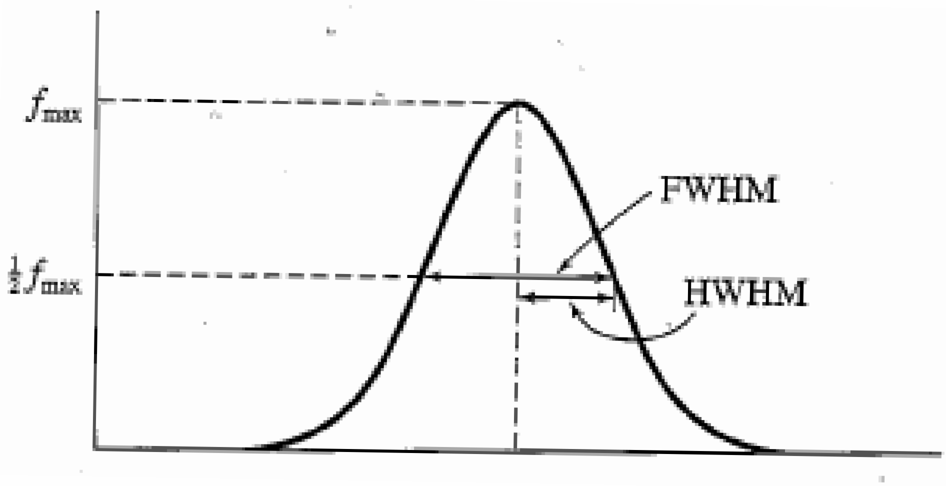
(See Prob. 5.12)

$$\mathrm{FWHM} = 2\sigma\sqrt{2\ln 2} = 2.35\,\sigma.$$

Points of Inflection

Occur at ±σ

(See Prob. 5.13)



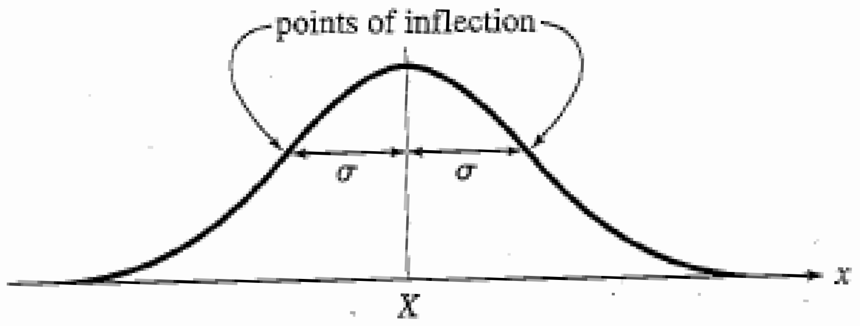
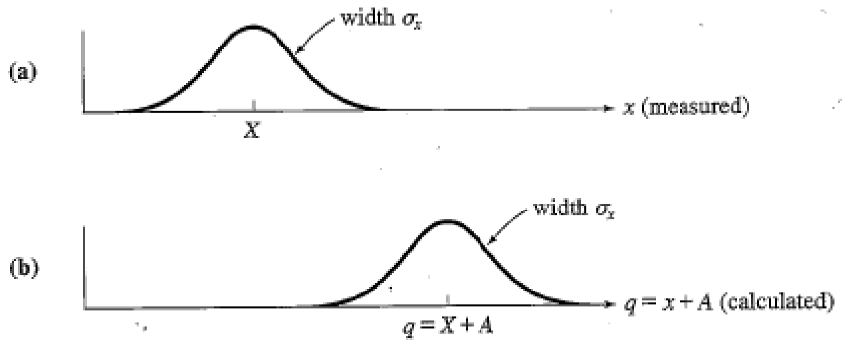


Figure 5.20. The points $X \pm \sigma$ are the points of inflection of the Gauss curve; for Problem 5.13.

Error Analysis and Gaussian Distribution

Adding a Constant

$$q = x + A, \qquad (5.47)$$



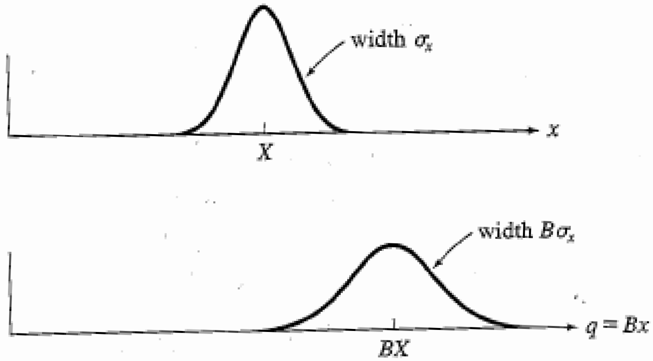
$$G_{X,\sigma}(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-X)^2/2\sigma^2}$$

$$\begin{aligned}(\text{probability of obtaining value } q) &\propto e^{-[(q-A)-X]^2/2\sigma_x^2} \\ &= e^{-[q-(X+A)]^2/2\sigma_x^2}. \qquad (5.49)\end{aligned}$$

X→X+A

Multiplying by a Constant

$$q = Bx,$$



$$G_{X,\sigma}(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-X)^2/2\sigma^2}$$

$$\begin{aligned}(\text{probability of obtaining value } q) &\propto (\text{probability of obtaining } x = q/B) \\ &\propto \exp\left[-\left(\frac{q}{B} - X\right)^2 / 2\sigma_x^2\right] \\ &= \exp[-(q - BX)^2/2B^2\sigma_x^2]. \qquad (5.50)\end{aligned}$$

X→BX and σ→B σ

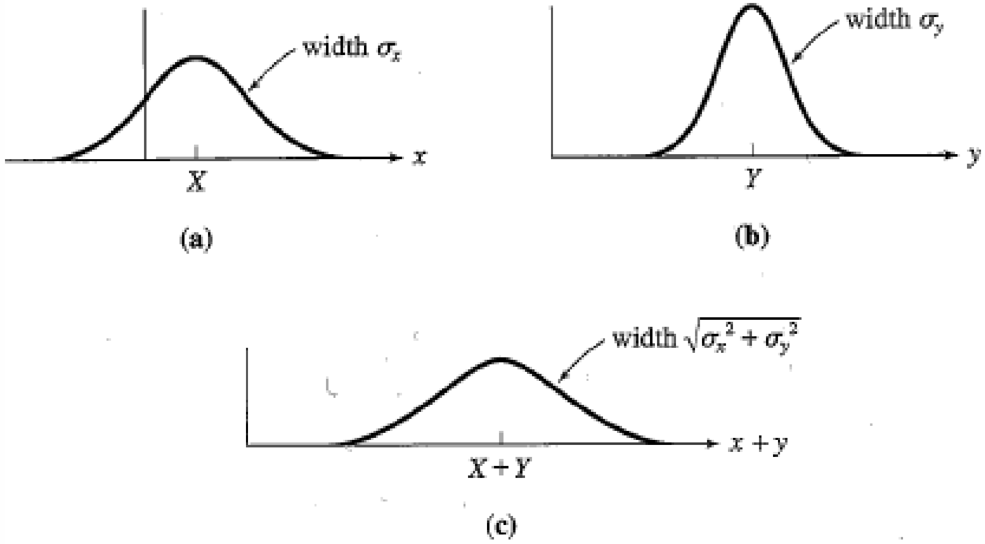
Error Propagation: Addition

Sum of Two Variables

Consider the derived quantity
Z=X + Y
(with X=0 and Y=0)

$$Prob(x) \propto \exp\left(\frac{-x^2}{2\sigma_x^2}\right) \qquad (5.51)$$

$$Prob(y) \propto \exp\left(\frac{-y^2}{2\sigma_y^2}\right). \qquad (5.52)$$



Error in Z:
Multiple two probabilities

$$Prob(x,y) = Prob(x)\cdot Prob(x) \propto exp\left[-\frac{1}{2}\left(\frac{x^2}{\sigma_x{}^2}+\frac{y^2}{\sigma_y{}^2}\right)\right]$$

$$\propto exp\left[-\frac{1}{2}\left(\frac{(x+y)^2}{(\sigma_x{}^2+\sigma_y{}^2)}-z^2\right)\right]$$

ICBST (Eq. 5.53)

$$\propto exp\left[-\frac{1}{2}\left(\frac{(x+y)^2}{(\sigma_x{}^2+\sigma_y{}^2)}\right)\right]\; exp\left[-\frac{z^2}{2}\right]$$

Integrates to √2π

X+Y→Z and $\sigma_x^2 + \sigma_y^2 \rightarrow \sigma_z^2$
(addition in quadrature for random, independent variables!)

General Formula for Error Propagation

How do we determine the error of a derived quantity Z(X,Y,…) from errors in X,Y,…?

General formula for error propagation see [Taylor, Secs. 3.5 and 3.9]

Uncertainty as a function of one variable [Taylor, Sec. 3.5]

1. Consider a graphical method of estimating error
   a) Consider an arbitaray function q(x)
   b) Plot q(x) vs. x.
   c) On the graph, label:
      (1) $q_{best} = q(x_{best})$
      (2) $q_{hi} = q(x_{best} + \delta x)$
      (3) $q_{low} = q(x_{best} - \delta x)$
   d) Making a linear approximation:

$$q_{hi} = q_{best} + slope \cdot \delta x = q_{best} + \left(\frac{\partial q}{\partial x}\right)$$

$$q_{low} = q_{best} - slope \cdot \delta x = q_{best} - \left(\frac{\partial q}{\partial x}\right)$$

   e) Therefore:

$$\delta q = \left|\frac{\partial q}{\partial x}\right| \cdot \delta x$$

Note the absolute value.

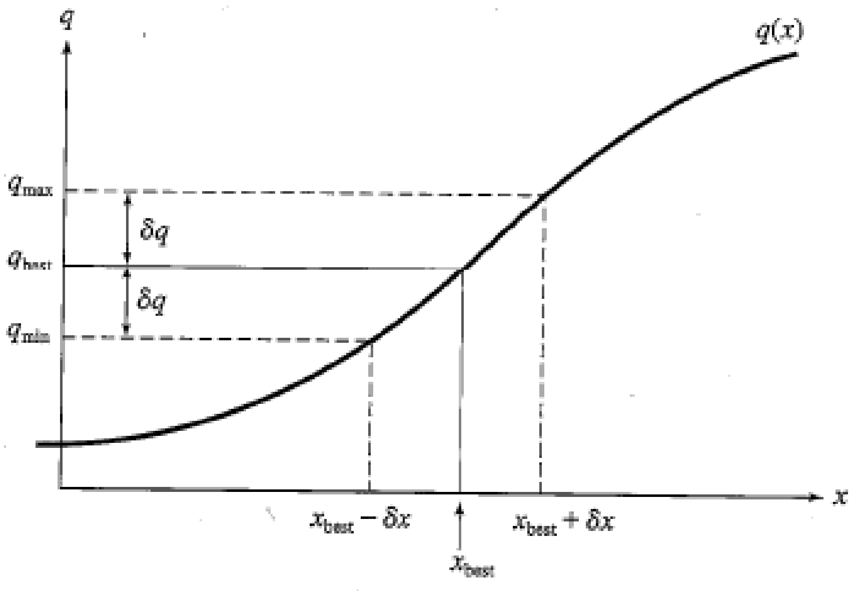


Figure 3.3. Graph of $q(x)$ vs x. If x is measured as $x_{best} \pm \delta x$, then the best estimate for $q(x)$ is $q_{best} = q(x_{best})$. The largest and smallest probable values of $q(x)$ correspond to the values $x_{best} \pm \delta x$ of x.

General Formula for Error Propagation

General formula for uncertainty of a function of one variable

$\delta q = \left|\frac{\partial q}{\partial x}\right| \cdot \delta x$ [Taylor, Eq. 3.23]

Can you now derive for specific rules of error propagation:

1. Addition and Subtraction **[Taylor, p. 49]**
2. Multiplication and Division **[Taylor, p. 51]**
3. Multiplication by a constant (exact number) **[Taylor, p. 54]**
4. Exponentiation (powers) **[Taylor, p. 56]**

General Formula for Multiple Variables

Uncertainty of a function of multiple variables [Taylor, Sec. 3.11]

1. It can easily (no, really) be shown that (see Taylor Sec. 3.11) for a function of several variables

$$\delta q(x,y,z,\ldots)=\left|\frac{\partial q}{\partial x}\right|\cdot\delta x+\left|\frac{\partial q}{\partial y}\right|\cdot\delta y+\left|\frac{\partial q}{\partial z}\right|\cdot\delta z+\ldots$$

[Taylor, Eq. 3.47]

2. More correctly, it can be shown that (see Taylor Sec. 3.11) for a function of several variables

$$\delta q(x,y,z,\ldots)\leq\left|\frac{\partial q}{\partial x}\right|\cdot\delta x+\left|\frac{\partial q}{\partial y}\right|\cdot\delta y+\left|\frac{\partial q}{\partial z}\right|\cdot\delta z+\ldots$$

[Taylor, Eq. 3.47]

where the equals sign represents an upper bound, as discussed above.

3. For a function of several *independent and random* variables

$$\delta q(x,y,z,\ldots)=\sqrt{\left(\frac{\partial q}{\partial x}\cdot\delta x\right)^2+\left(\frac{\partial q}{\partial y}\cdot\delta y\right)^2+\left(\frac{\partial q}{\partial z}\cdot\delta z\right)^2+\ldots}$$

[Taylor, Eq. 3.48]

Again, the proof is left for Ch. 5.

Error Propagation: General Case

How do we determine the error of a derived quantity Z(X,Y,…) from errors in X,Y,…?

Consider the arbitrary derived quantity
q(x,y) of two independent random variables x and y.

Expand q(x,y) in a Taylor series about the expected values of x and y
(i.e., at points near X and Y).

Fixed, shifts peak of distribution

$$q(x,y) = q(X,Y) + \left(\frac{\partial q}{\partial x}\right)\bigg|_X (x-X) + \left(\frac{\partial q}{\partial y}\right)\bigg|_Y (y-Y)$$

Fixed   Distribution centered at X with width σ_X

Product of Two Variables

$$\delta q(x,y) = \sigma_q = \sqrt{q(X,Y) + \left[\left(\frac{\partial q}{\partial x}\right)\bigg|_X \sigma_x\right]^2 + \left[\left(\frac{\partial q}{\partial y}\right)\bigg|_Y \sigma_y\right]^2}$$

0

SDOM of Gaussian Distribution

Standard Deviation of the Mean

Each measurement has similar $\sigma_{Xi=}$ $\sigma_{\dot{X}}$

$$\sigma_{x_1} = \cdots = \sigma_{x_N} = \sigma_x .$$

$$G_{X,\sigma}(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-X)^2/2\sigma^2}$$

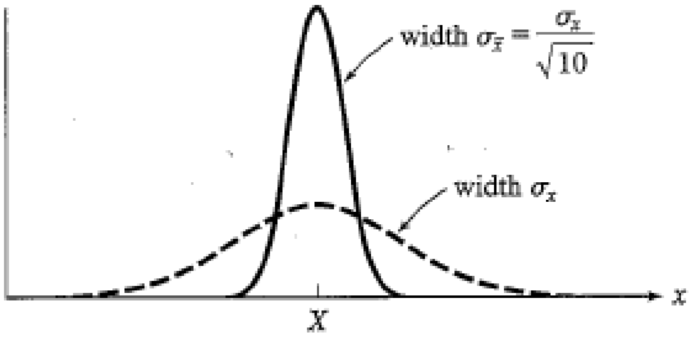
and similar partial derivatives

$$\frac{\partial \bar{x}}{\partial x_1} = \cdots = \frac{\partial \bar{x}}{\partial x_N} = \frac{1}{N} .$$

Thus…

$$\sigma_{\bar{x}} = \sqrt{\left(\frac{1}{N}\sigma_x\right)^2 + \cdots + \left(\frac{1}{N}\sigma_x\right)^2}$$

$$= \sqrt{N\frac{\sigma_x{}^2}{N^2}} = \frac{\sigma_x}{\sqrt{N}} ,$$



(5.66)

The SDOM decreases as the square root of the number of measurements.

That is, the relative width, σ/Ẋ, of the distribution gets narrower as more measurements are made.

Two Key Theorems from Probability

Central Limit Theorem

For random, independent measurements (each with a well-define expectation value and well-defined variance), the arithmetic mean (average) will be approximately normally distributed.

Principle of Maximum Likelihood

Given the N observed measurements, x_1, x_2,…x_N, the best estimates for $\dot{X}$ and σ are those values for which the observed x_1, x_2,…x_N, are most likely.

Mean of Gaussian Distribution as “Best Estimate”

Principle of Maximum Likelihood

To find the most likely value of the mean (the best estimate of ẋ), find X that yields the highest probability for the data set.

Consider a data set $\{x_1, x_2, x_3 \dots x_N\}$

Each randomly distributed with

$$Prob_{X,\sigma}(x_i) = G_{X,\sigma}(x_i) \equiv \frac{1}{\sigma\sqrt{2\pi}} e^{-(x_i-X)^2/2\sigma} \propto \frac{1}{\sigma} e^{-(x_i-X)^2/2\sigma}$$

The combined probability for the full data set is the product

$$Prob_{X,\sigma}(x_1,x_2 \dots x_N) = Prob_{X,\sigma}(x_1) \times Prob_{X,\sigma}(x_2) \times \dots \times Prob_{X,\sigma}(x_N)$$

$$\propto \frac{1}{\sigma} e^{-(x_1-X)^2/2\sigma} \times \frac{1}{\sigma} e^{-(x_2-X)^2/2\sigma} \times \dots \times \frac{1}{\sigma} e^{-(x_N-X)^2/2\sigma} = \frac{1}{\sigma^N} e^{\Sigma-(x_i-X)^2/2\sigma}$$

Best Estimate of X is from maximum probability or minimum summation

Minimize Sum	$\sum_{i=1}^{N} (x_i - X)^2/\sigma$	**Solve for derivative wrst X set to 0**	$\sum_{i=1}^{N} (x_i - X) = 0$	**Best estimate of X**	$X_{best} = {\sum x_i}/{N}$

Uncertainty of "Best Estimates" of Gaussian Distribution

Principle of Maximum Likelihood

To find the most likely value of the standard deviation (the best estimate of the width of the x distribution), find σ_x that yields the highest probability for the data set.

Consider a data set $\{x_1, x_2, x_3 \ldots x_N\}$

The combined probability for the full data set is the product

$$Prob_{X,\sigma}(x_1,x_2 \ldots x_N) = Prob_{X,\sigma}(x_1) \times Prob_{X,\sigma}(x_2) \times \ldots \times Prob_{X,\sigma}(x_N)$$

$$\propto \frac{1}{\sigma} e^{-(x_1-X)^2/2\sigma} \times \frac{1}{\sigma} e^{-(x_2-X)^2/2\sigma} \times \ldots \times \frac{1}{\sigma} e^{-(x_N-X)^2/2\sigma} = \frac{1}{\sigma^N} e^{\Sigma-(x_i-X)^2/2\sigma}$$

Best Estimate of X is from maximum probability or minimum summation

Minimize Sum $\sum_{i=1}^{N} (x_i - X)^2/\sigma$

Solve for derivative wrst X set to 0 $\sum_{i=1}^{N} (x_i - X) = 0$

Best estimate of X $X_{best} = \sum x_i / N$

Best Estimate of σ is from maximum probability or minimum summation

Minimize Sum $\sum_{i=1}^{N} (x_i - X)^2/\sigma$

Solve for derivative wrst σ set to 0

See Prob. 5.26

Best estimate of σ $\sigma_{best} = \sqrt{\frac{1}{N} \sum_{i=1}^{N} (x_i - X)^2/\sigma}$

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Combining Data Sets

Weighted Averages

References: Taylor Ch. 7

Weighted Averages

Question: How can we properly combine two or more separate independent measurements of the same randomly distributed quantity to determine a best combined value with uncertainty?

Weighted Averages

Consider two measurements of the same quantity, described by a random Gaussian distribution

$<x_1> \pm \sigma_{x1}$ and $<x_2> \pm \sigma_{x2}$

Assume negligible systematic errors.

The probability of measuring two such measurements is

$$Prob_x(x_1, x_2) = Prob_x(x_1)\; Prob_x(x_2)$$

$$= \frac{1}{\sigma_1 \sigma_2} e^{-\chi^2/2} \quad where \;\; \chi^2 \equiv \left[\frac{(x_1 - X)}{\sigma_1}\right]^2 + \left[\frac{(x_2 - X)}{\sigma_2}\right]^2$$

To find the best value for χ, find the maximum Prob or minimum χ^2

Note: χ^2 , or Chi squared, is the sum of the squares of the deviations from the mean, divided by the corresponding uncertainty.

Such methods are called "Methods of Least Squares". They follow directly from the Principle of Maximum Likelihood.

Weighted Averages

The probability of measuring two such measurements is

$$Prob_x(x_1,x_2) = Prob_x(x_1)\,Prob_x(x_2)$$

$$= \frac{1}{\sigma_1\sigma_2}e^{-\chi^2/2} \quad where \;\; \chi^2 \equiv \left[\frac{(x_1-X)}{\sigma_1}\right]^2 + \left[\frac{(x_2-X)}{\sigma_2}\right]^2$$

To find the best value for χ, find the maximum Prob or minimum χ^2

Best Estimate of χ is from maximum probibility or minimum summation

Minimize Sum

$$\chi^2 \equiv \left[\frac{(x_1-X)}{\sigma_1}\right]^2 + \left[\frac{(x_2-X)}{\sigma_2}\right]^2$$

Solve for derivative wrst χ set to 0

$$2\left[\frac{(x_1-X)}{\sigma_1}\right] + 2\left[\frac{(x_2-X)}{\sigma_2}\right] = 0$$

Solve for best estimate of χ

$$X_{best} = \left(\frac{x_1}{\sigma_1{}^2} + \frac{x_2}{\sigma_2{}^2}\right) \Big/ \left(\frac{1}{\sigma_1{}^2} + \frac{1}{\sigma_2{}^2}\right)$$

This leads to

$$x_{W\_avg} = \frac{w_1x_1+w_2x_2}{w_1+w_2} = \frac{\sum w_i\, x_i}{\sum w_i} \quad where\; w_i = {}^{1}\!/_{(\sigma_i)^2}$$

Note: If w_1=w_2, we recover the standard result X_{wavg}= (1/2) (x_1+x_2)

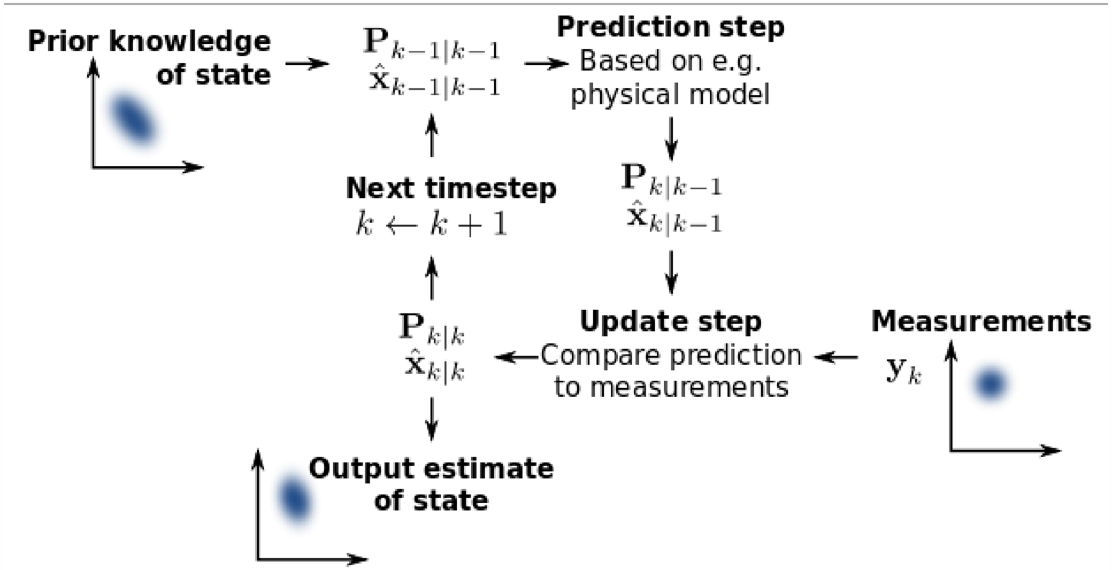
Finally, the width of a weighted average distribution is

$$\sigma_{wieg\,hted\;avg} = \frac{1}{\sum_i w_i}$$

Weighted Averages on Steroids

A very powerful method for combining data from different sources with different methods and uncertainties (or, indeed, data of related measured and calculated quantities) is Kalman filtering.

The Kalman filter, also known as linear quadratic estimation (LQE), is an algorithm that uses a series of measurements observed over time, containing noise (random variations) and other inaccuracies, and produces estimates of unknown variables that tend to be more precise than those based on a single measurement alone.



Prior knowledge of state
$\mathbf{P}_{k-1|k-1}$ $\hat{\mathbf{x}}_{k-1|k-1}$
Prediction step
Based on e.g. physical model
Next timestep
$k \leftarrow k+1$
$\mathbf{P}_{k|k-1}$ $\hat{\mathbf{x}}_{k|k-1}$
$\mathbf{P}_{k|k}$ $\hat{\mathbf{x}}_{k|k}$
Update step
Compare prediction to measurements
Measurements
$\mathbf{y}_k$
Output estimate of state

The Kalman filter keeps track of the estimated state of the system and the variance or uncertainty of the estimate. The estimate is updated using a state transition model and measurements. $x_{k|k-1}$ denotes the estimate of the system's state at time step k before the kth measurement y_k has been taken into account; $P_{k|k-1}$ is the corresponding uncertainty. --Wikipedia, 2013.

Ludger Scherliess, of USU Physics, is a world expert at using Kalman filtering for the assimilation of satellite and ground-based data and the USU GAMES model to predict space weather .

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Rejecting Data

Chauvenet's Criterion

References: Taylor Ch. 6

Rejecting Data

What is a good criteria for rejecting data?

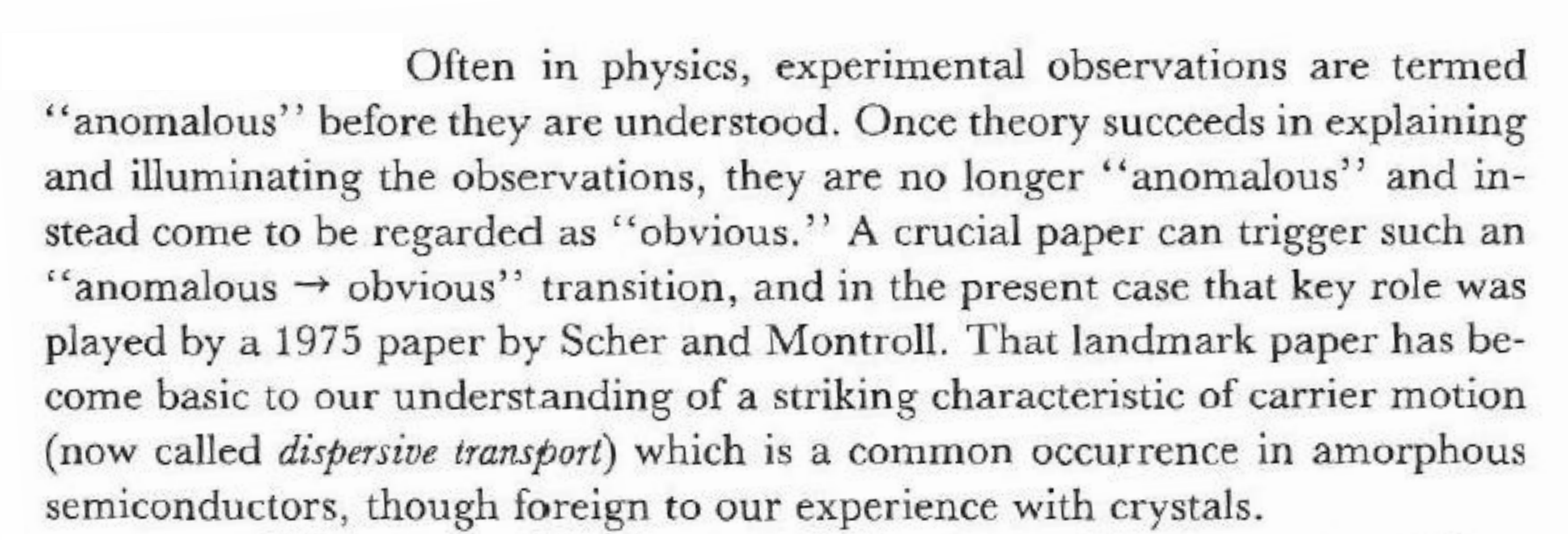
Question: When is it “reasonable” to discard a seemingly “unreasonable” data point from a set of randomly distributed measurements?

- Never
- Whenever it makes things look better
- Chauvenet’s criterion provides a (quantitative) compromise

Rejecting Data

Zallen’s Criterion

Question: When is it “reasonable” to discard a seemingly “unreasonable” data point from a set of randomly distributed measurements?



Often in physics, experimental observations are termed ‘‘anomalous’’ before they are understood. Once theory succeeds in explaining and illuminating the observations, they are no longer ‘‘anomalous’’ and instead come to be regarded as ‘‘obvious.’’ A crucial paper can trigger such an ‘‘anomalous → obvious’’ transition, and in the present case that key role was played by a 1975 paper by Scher and Montroll. That landmark paper has become basic to our understanding of a striking characteristic of carrier motion (now called *dispersive transport*) which is a common occurrence in amorphous semiconductors, though foreign to our experience with crystals.

Rejecting Data

Disney's Criterion

Question: When is it "reasonable" to discard a seemingly "unreasonable" data point from a set of randomly distributed measurements?

- **Whenever it makes things look better**

Disney's First Law

Wishing will make it so.

Disney's Second Law

Dreams are more colorful than reality.

Rejecting Data

Chauvenet's Criterion

Data may be rejected if the expected number of measurements at least as deviant as the suspect measurement is less than 50%.

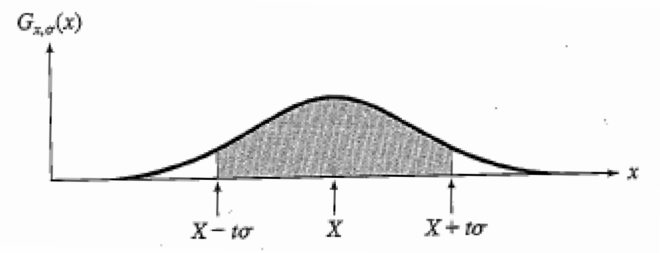
Consider a set of N measurements of a single quantity

$\{ x_1, x_2, \ldots\ldots x_N \}$

Calculate <x> and σ_x and then determine the fractional deviations from the mean of all the points:

$$x_{frac\_dev} = \frac{|x_i - \bar{x}|}{\sigma_x}$$

For the suspect point(s), $x_{suspect}$, find the probability of such a point occurring in N measurements



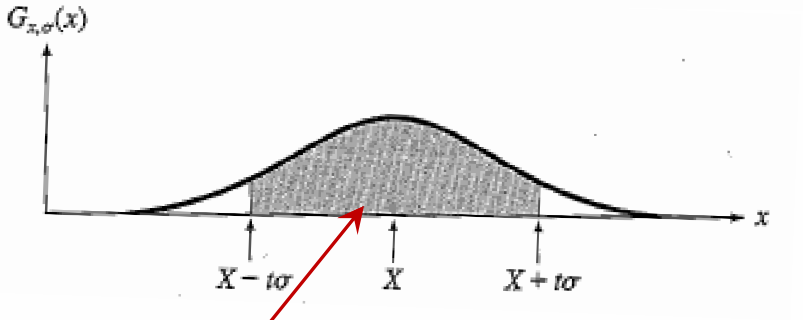
n = (expected number as deviant as $x_{suspect}$)

= N Prob(outside $x_{suspect}\cdot\sigma_x$)

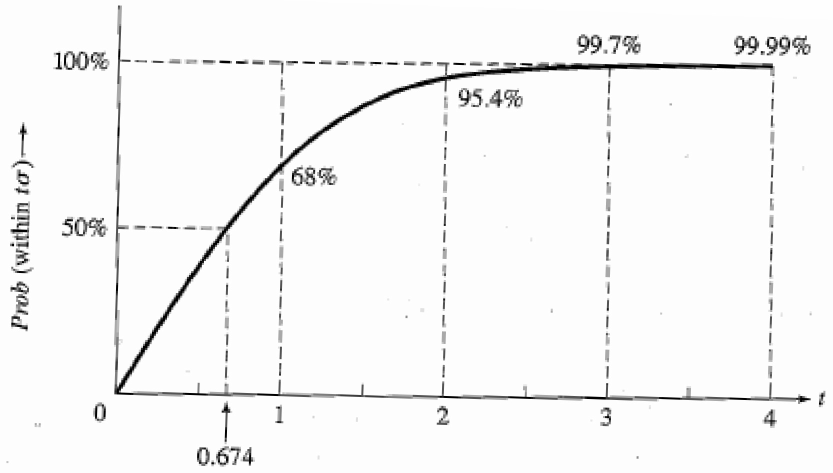
Error Function of Gaussian Distribution

Error Function: (probability that –tσ<x<+tσ).

$$Prob(\text{within } t\sigma) = \frac{1}{\sqrt{2\pi}}\int_{-t}^{t} e^{-z^2/2}\,dz. \qquad (5.35)$$



Error Function: Tabulated values-see App. A.



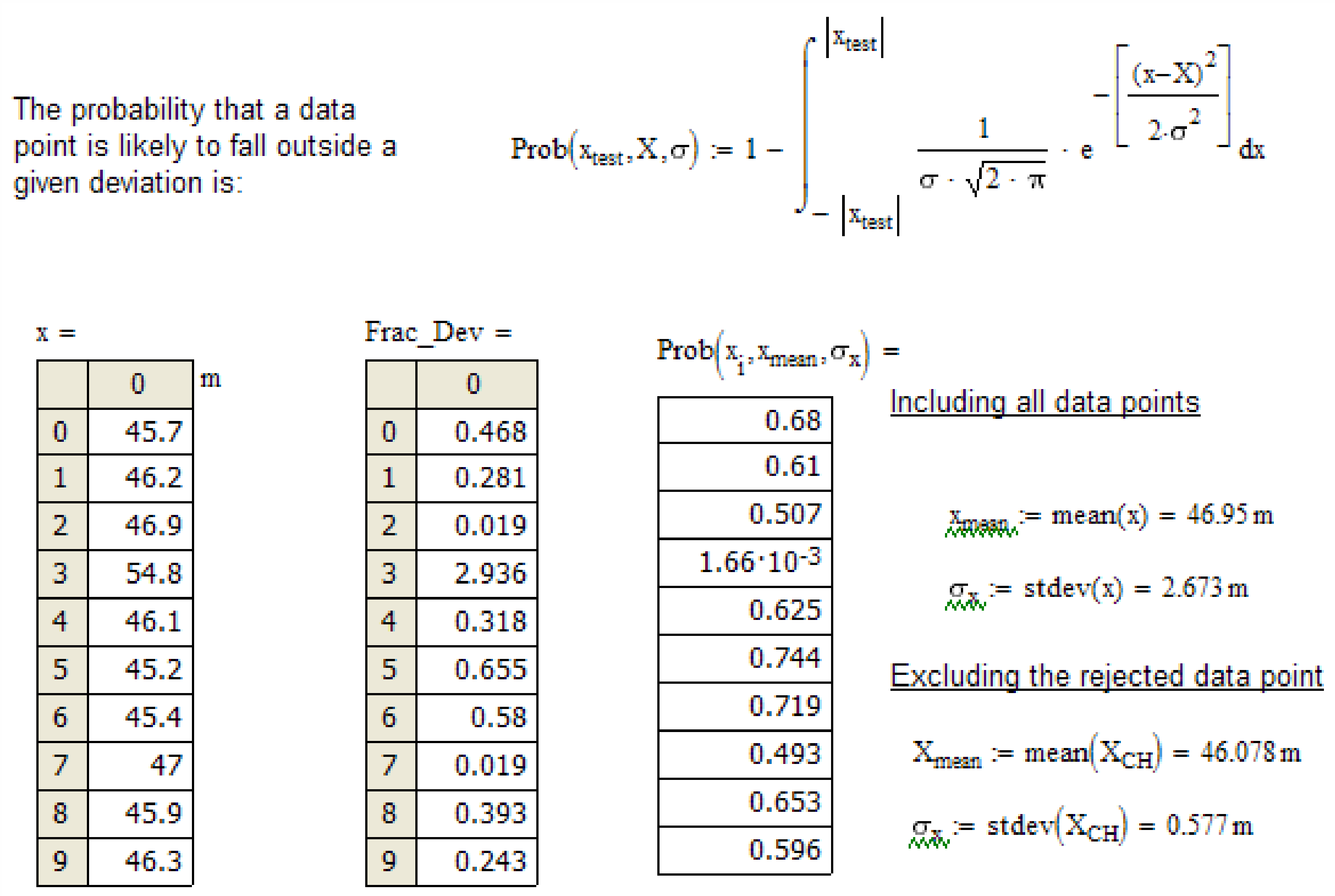
t	0	0.25	0.5	0.75	1.0	1.25	1.5	1.75	2.0	2.5	3.0	3.5	4.0
Prob (%)	0	20	38	55	68	79	87	92	95.4	98.8	99.7	99.95	99.99

Figure 5.13. The probability *Prob*(within $t\sigma$) that a measurement of x will fall within t standard deviations of the true value $x = X$. Two common names for this function are the *normal error integral* and the *error function*, erf(t).

Area under curve (probability that –tσ<x<+tσ) is given in Table at right.

Probable Error: (probability that –0.67σ<x<+0.67σ) is 50%.

Chauvenet’s Criterion



The probability that a data point is likely to fall outside a given deviation is:

$$\mathrm{Prob}\left(x_{test}, X, \sigma\right) := 1 - \int_{-\left|x_{test}\right|}^{\left|x_{test}\right|} \frac{1}{\sigma \cdot \sqrt{2 \cdot \pi}} \cdot e^{-\left[\frac{(x-X)^2}{2 \cdot \sigma^2}\right]} dx$$

$x =$

	0
0	45.7
1	46.2
2	46.9
3	54.8
4	46.1
5	45.2
6	45.4
7	47
8	45.9
9	46.3

m

Frac_Dev =

	0
0	0.468
1	0.281
2	0.019
3	2.936
4	0.318
5	0.655
6	0.58
7	0.019
8	0.393
9	0.243

$\mathrm{Prob}\left(x_i, x_{mean}, \sigma_x\right) =$

0.68
0.61
0.507
$1.66 \cdot 10^{-3}$
0.625
0.744
0.719
0.493
0.653
0.596

Including all data points

$x_{mean} := \mathrm{mean}(x) = 46.95\,\mathrm{m}$

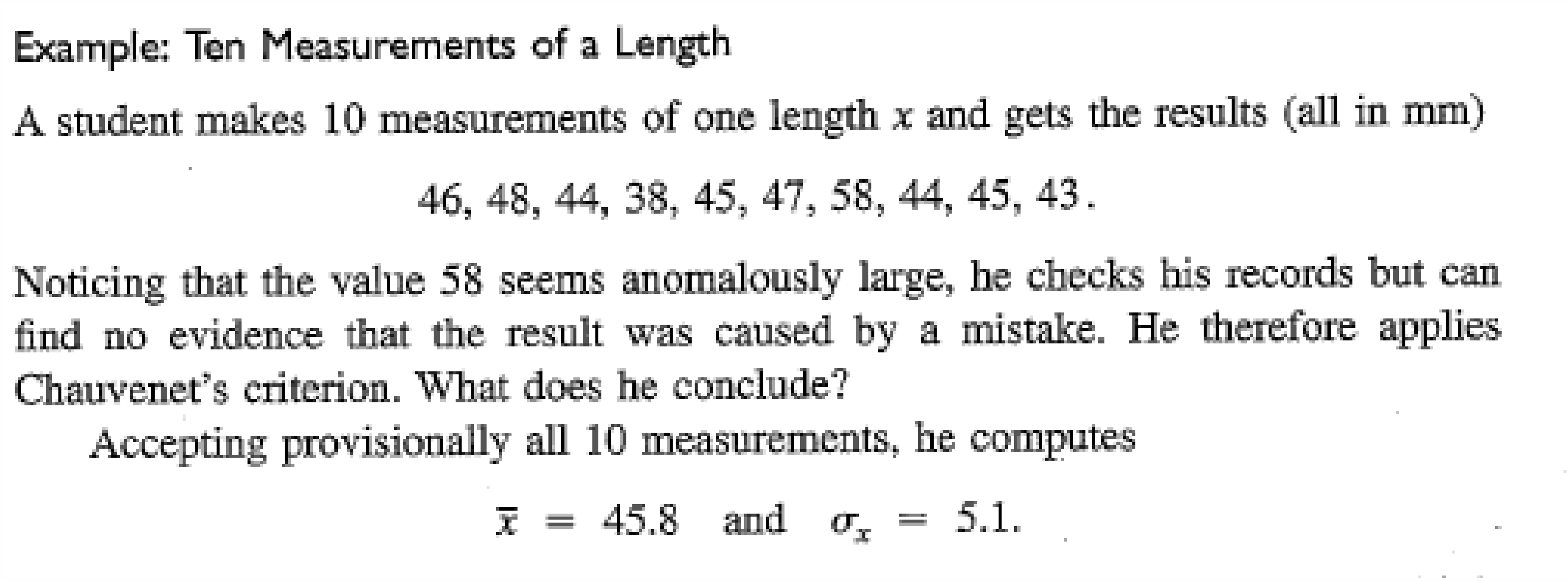
$\sigma_x := \mathrm{stdev}(x) = 2.673\,\mathrm{m}$

Excluding the rejected data point

$X_{mean} := \mathrm{mean}\left(X_{CH}\right) = 46.078\,\mathrm{m}$

$\sigma_x := \mathrm{stdev}\left(X_{CH}\right) = 0.577\,\mathrm{m}$

Chauvenet’s Criterion—Example 1



Example: Ten Measurements of a Length

A student makes 10 measurements of one length x and gets the results (all in mm)

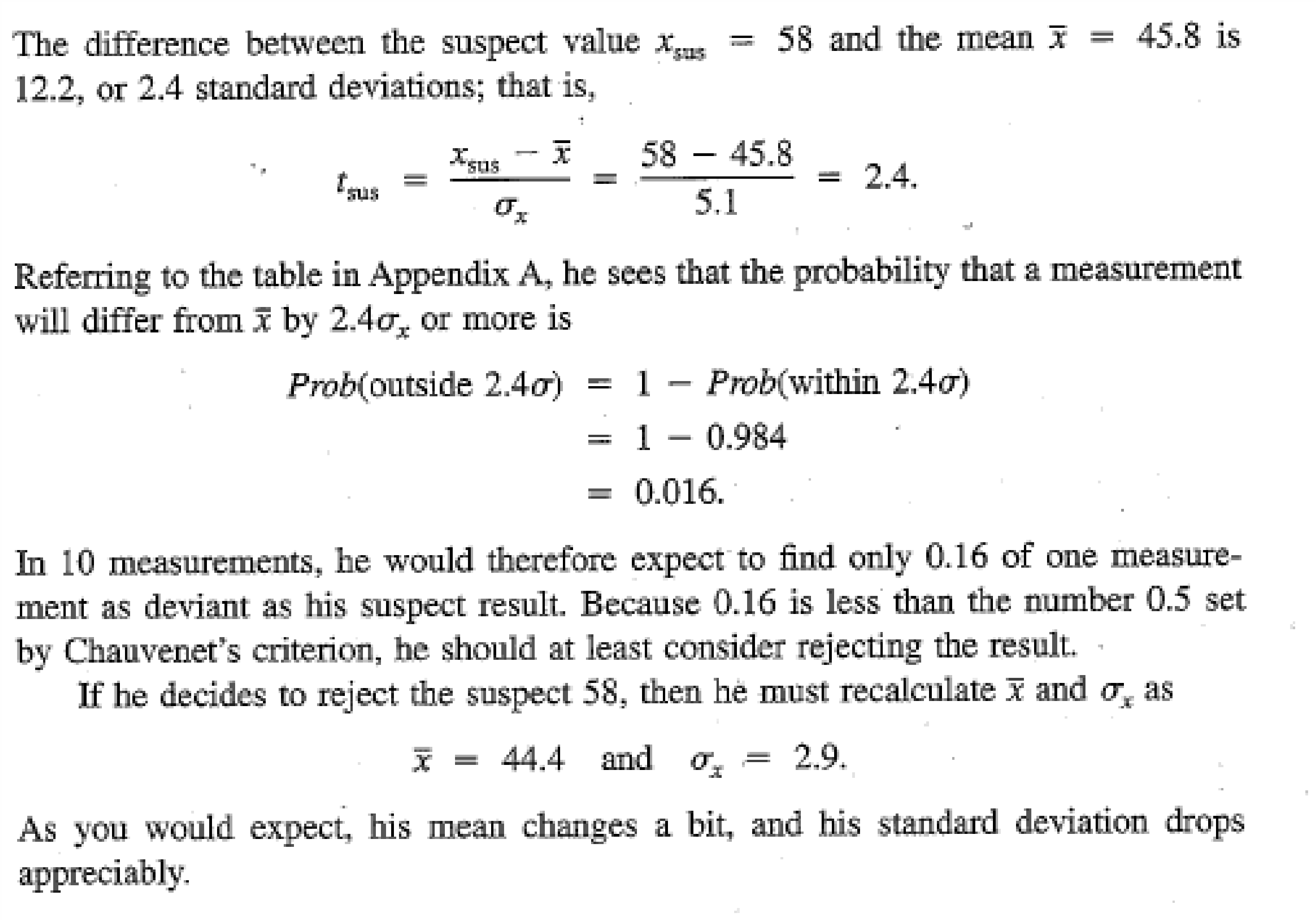
$$46,\ 48,\ 44,\ 38,\ 45,\ 47,\ 58,\ 44,\ 45,\ 43\,.$$

Noticing that the value 58 seems anomalously large, he checks his records but can find no evidence that the result was caused by a mistake. He therefore applies Chauvenet’s criterion. What does he conclude?

Accepting provisionally all 10 measurements, he computes

$$\bar{x} = 45.8 \quad \text{and} \quad \sigma_x = 5.1.$$

Chauvenet's Details (1)



The difference between the suspect value $x_{sus} = 58$ and the mean $\bar{x} = 45.8$ is 12.2, or 2.4 standard deviations; that is,

$$t_{sus} = \frac{x_{sus} - \bar{x}}{\sigma_x} = \frac{58 - 45.8}{5.1} = 2.4.$$

Referring to the table in Appendix A, he sees that the probability that a measurement will differ from $\bar{x}$ by $2.4\sigma_x$ or more is

$$\begin{aligned} Prob(\text{outside } 2.4\sigma) &= 1 - Prob(\text{within } 2.4\sigma) \\ &= 1 - 0.984 \\ &= 0.016. \end{aligned}$$

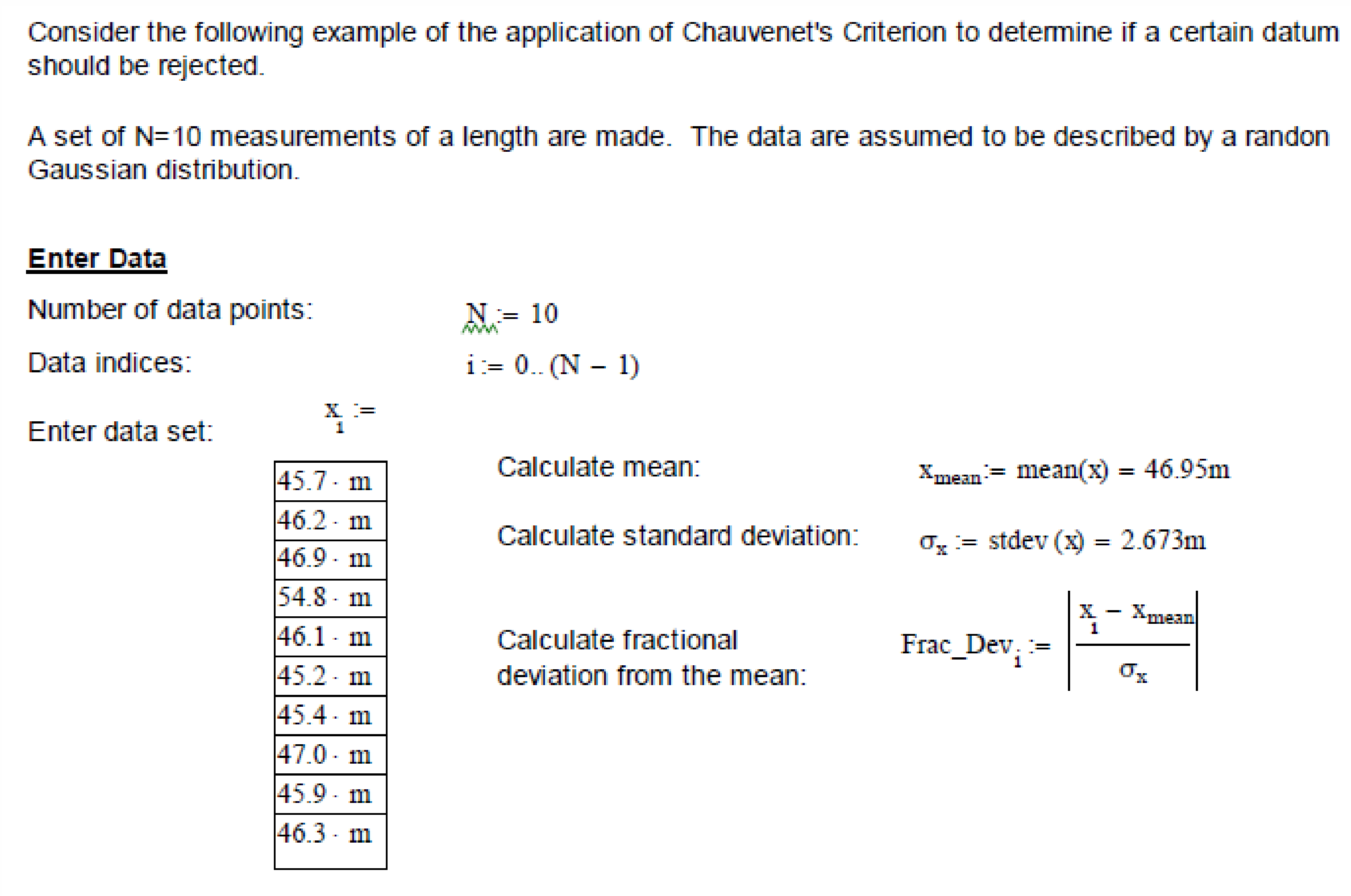
In 10 measurements, he would therefore expect to find only 0.16 of one measurement as deviant as his suspect result. Because 0.16 is less than the number 0.5 set by Chauvenet's criterion, he should at least consider rejecting the result.

If he decides to reject the suspect 58, then he must recalculate $\bar{x}$ and σ_x as

$$\bar{x} = 44.4 \quad \text{and} \quad \sigma_x = 2.9.$$

As you would expect, his mean changes a bit, and his standard deviation drops appreciably.

Chauvenet's Criterion—Details (2)



Consider the following example of the application of Chauvenet's Criterion to determine if a certain datum should be rejected.

A set of N=10 measurements of a length are made. The data are assumed to be described by a randon Gaussian distribution.

Enter Data

Number of data points: $N := 10$

Data indices: $i := 0..(N-1)$

Enter data set: $x_i :=$

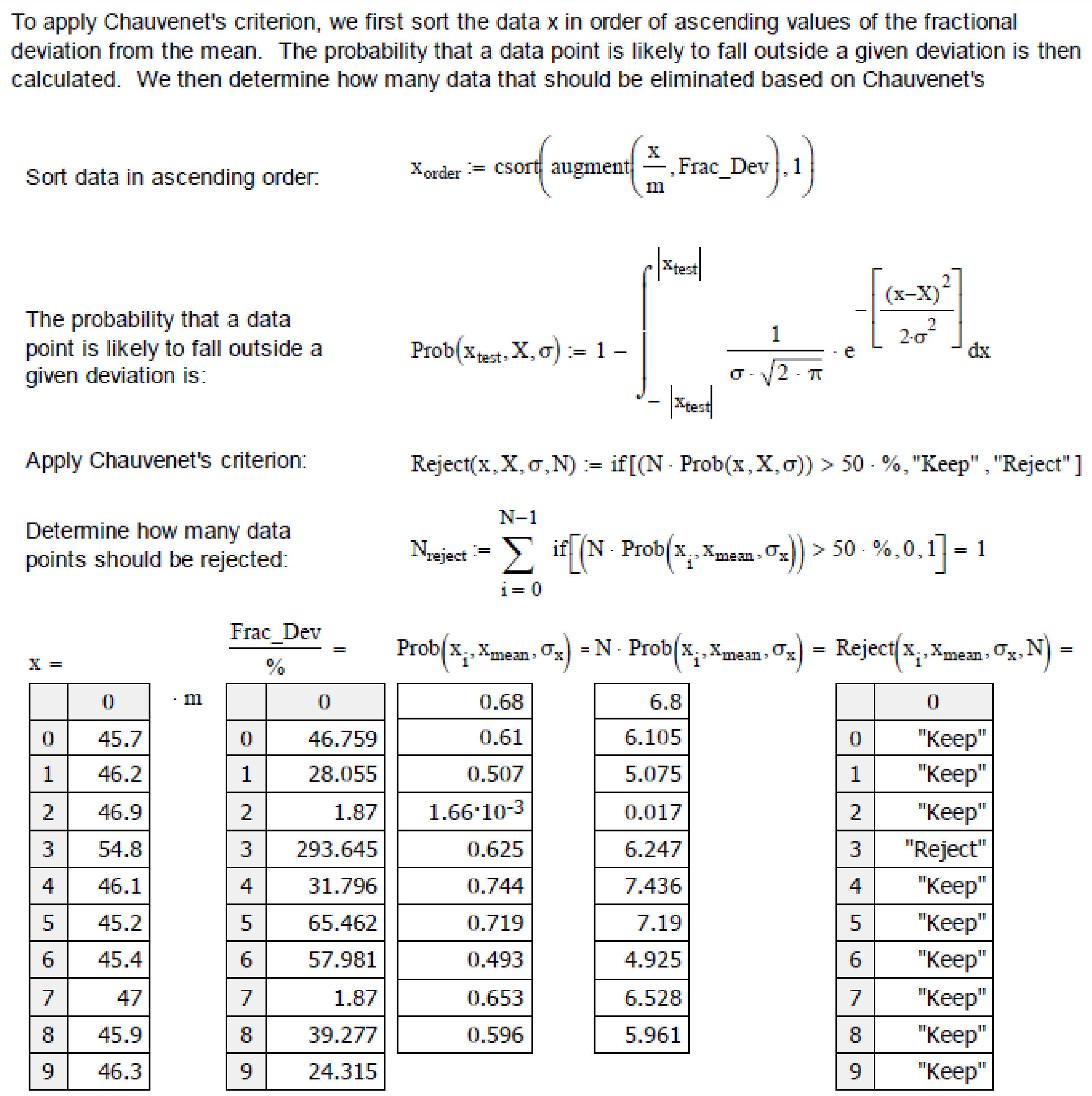
$45.7 \cdot m$
$46.2 \cdot m$
$46.9 \cdot m$
$54.8 \cdot m$
$46.1 \cdot m$
$45.2 \cdot m$
$45.4 \cdot m$
$47.0 \cdot m$
$45.9 \cdot m$
$46.3 \cdot m$

Calculate mean: $x_{mean} := mean(x) = 46.95m$

Calculate standard deviation: $\sigma_x := stdev(x) = 2.673m$

Calculate fractional deviation from the mean: $Frac\_Dev_i := \left|\frac{x_i - x_{mean}}{\sigma_x}\right|$

Chauvenet's Criterion—Details (3)



To apply Chauvenet's criterion, we first sort the data x in order of ascending values of the fractional deviation from the mean. The probability that a data point is likely to fall outside a given deviation is then calculated. We then determine how many data that should be eliminated based on Chauvenet's

Sort data in ascending order:

$$x_{order} := \text{csort}\left(\text{augment}\left(\frac{x}{m}, \text{Frac\_Dev}\right), 1\right)$$

The probability that a data point is likely to fall outside a given deviation is:

$$\text{Prob}(x_{test}, X, \sigma) := 1 - \int_{-|x_{test}|}^{|x_{test}|} \frac{1}{\sigma \cdot \sqrt{2 \cdot \pi}} \cdot e^{-\left[\frac{(x-X)^2}{2 \cdot \sigma^2}\right]} dx$$

Apply Chauvenet's criterion:

$$\text{Reject}(x, X, \sigma, N) := \text{if}[(N \cdot \text{Prob}(x, X, \sigma)) > 50 \cdot \%, \text{"Keep"}, \text{"Reject"}]$$

Determine how many data points should be rejected:

$$N_{reject} := \sum_{i=0}^{N-1} \text{if}\left[\left(N \cdot \text{Prob}(x_i, x_{mean}, \sigma_x)\right) > 50 \cdot \%, 0, 1\right] = 1$$

$x =$ ... $\cdot m$

	0
0	45.7
1	46.2
2	46.9
3	54.8
4	46.1
5	45.2
6	45.4
7	47
8	45.9
9	46.3

$\frac{\text{Frac\_Dev}}{\%} =$

	0
0	46.759
1	28.055
2	1.87
3	293.645
4	31.796
5	65.462
6	57.981
7	1.87
8	39.277
9	24.315

$\text{Prob}(x_i, x_{mean}, \sigma_x) =$

0.68
0.61
0.507
$1.66 \cdot 10^{-3}$
0.625
0.744
0.719
0.493
0.653
0.596

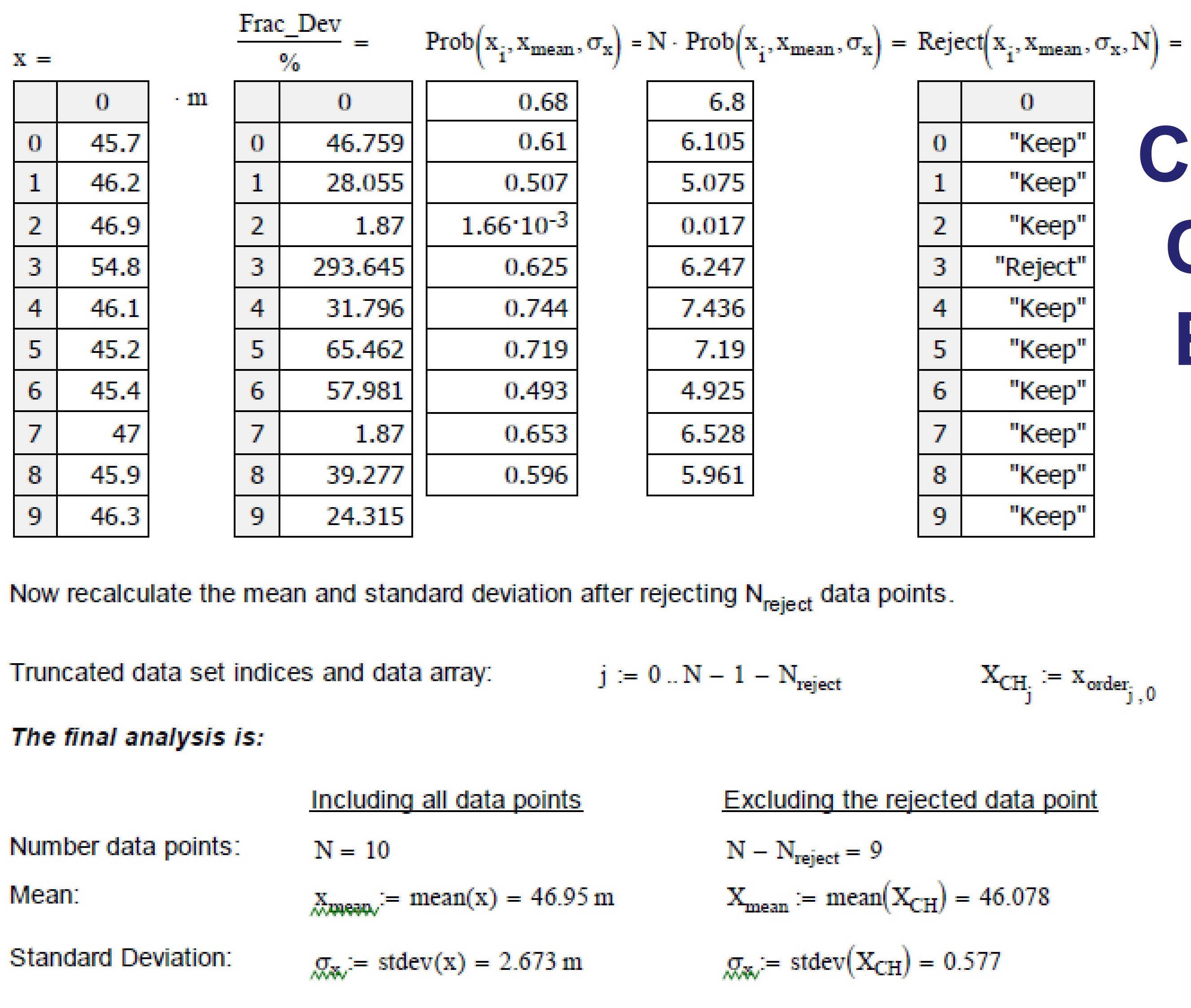
$N \cdot \text{Prob}(x_i, x_{mean}, \sigma_x) =$

6.8
6.105
5.075
0.017
6.247
7.436
7.19
4.925
6.528
5.961

$\text{Reject}(x_i, x_{mean}, \sigma_x, N) =$

	0
0	"Keep"
1	"Keep"
2	"Keep"
3	"Reject"
4	"Keep"
5	"Keep"
6	"Keep"
7	"Keep"
8	"Keep"
9	"Keep"

Chauvenet's Criterion—Example 2



$x =$

	0
0	45.7
1	46.2
2	46.9
3	54.8
4	46.1
5	45.2
6	45.4
7	47
8	45.9
9	46.3

$\cdot \mathrm{m}$

$\frac{\mathrm{Frac\_Dev}}{\%} =$

	0
0	46.759
1	28.055
2	1.87
3	293.645
4	31.796
5	65.462
6	57.981
7	1.87
8	39.277
9	24.315

$\mathrm{Prob}\left(x_i, x_{mean}, \sigma_x\right) =$

0.68
0.61
0.507
$1.66 \cdot 10^{-3}$
0.625
0.744
0.719
0.493
0.653
0.596

$= N \cdot \mathrm{Prob}\left(x_i, x_{mean}, \sigma_x\right) =$

6.8
6.105
5.075
0.017
6.247
7.436
7.19
4.925
6.528
5.961

$\mathrm{Reject}\left(x_i, x_{mean}, \sigma_x, N\right) =$

	0
0	"Keep"
1	"Keep"
2	"Keep"
3	"Reject"
4	"Keep"
5	"Keep"
6	"Keep"
7	"Keep"
8	"Keep"
9	"Keep"

Now recalculate the mean and standard deviation after rejecting N_{reject} data points.

Truncated data set indices and data array: $j := 0 .. N - 1 - N_{reject}$ $\quad X_{CH_j} := x_{order_j, 0}$

The final analysis is:

	Including all data points	Excluding the rejected data point
Number data points:	$N = 10$	$N - N_{reject} = 9$
Mean:	$x_{mean} := \mathrm{mean}(x) = 46.95\ \mathrm{m}$	$X_{mean} := \mathrm{mean}\left(X_{CH}\right) = 46.078$
Standard Deviation:	$\sigma_x := \mathrm{stdev}(x) = 2.673\ \mathrm{m}$	$\sigma_x := \mathrm{stdev}\left(X_{CH}\right) = 0.577$

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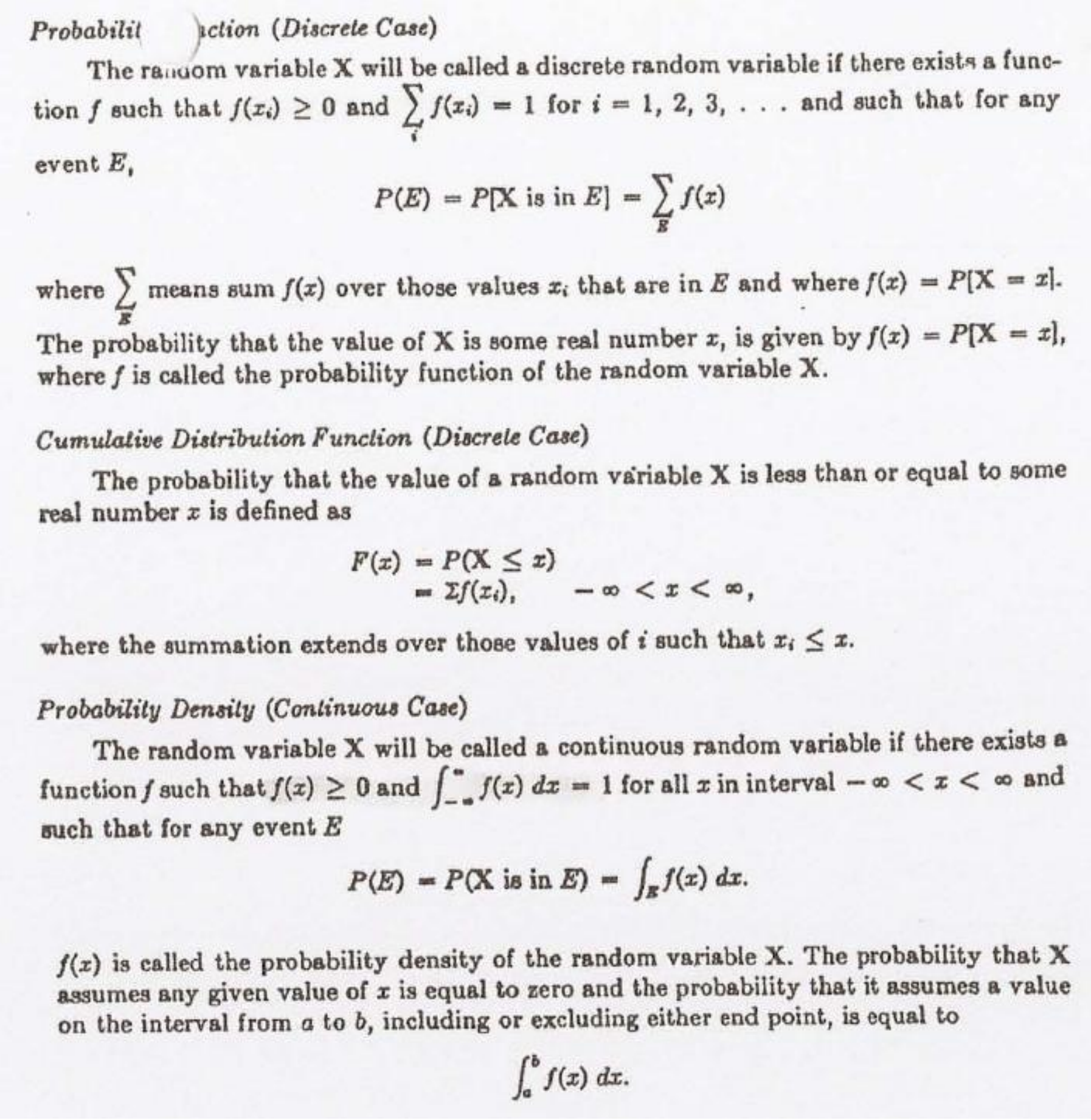
Summary of Probability Theory

Introduction Section 0 Lecture 1 Slide 64

Summary of Probability Theory-I



Probabilit[illegible]ction (*Discrete Case*)

The random variable X will be called a discrete random variable if there exists a function f such that $f(x_i) \geq 0$ and $\sum_i f(x_i) = 1$ for $i = 1, 2, 3, \ldots$ and such that for any event E,

$$P(E) = P[\text{X is in } E] = \sum_E f(x)$$

where $\sum_E$ means sum $f(x)$ over those values x_i that are in E and where $f(x) = P[\text{X} = x]$. The probability that the value of X is some real number x, is given by $f(x) = P[\text{X} = x]$, where f is called the probability function of the random variable X.

Cumulative Distribution Function (*Discrete Case*)

The probability that the value of a random variable X is less than or equal to some real number x is defined as

$$\begin{aligned} F(x) &= P(\text{X} \leq x) \\ &= \Sigma f(x_i), \qquad -\infty < x < \infty, \end{aligned}$$

where the summation extends over those values of i such that $x_i \leq x$.

Probability Density (*Continuous Case*)

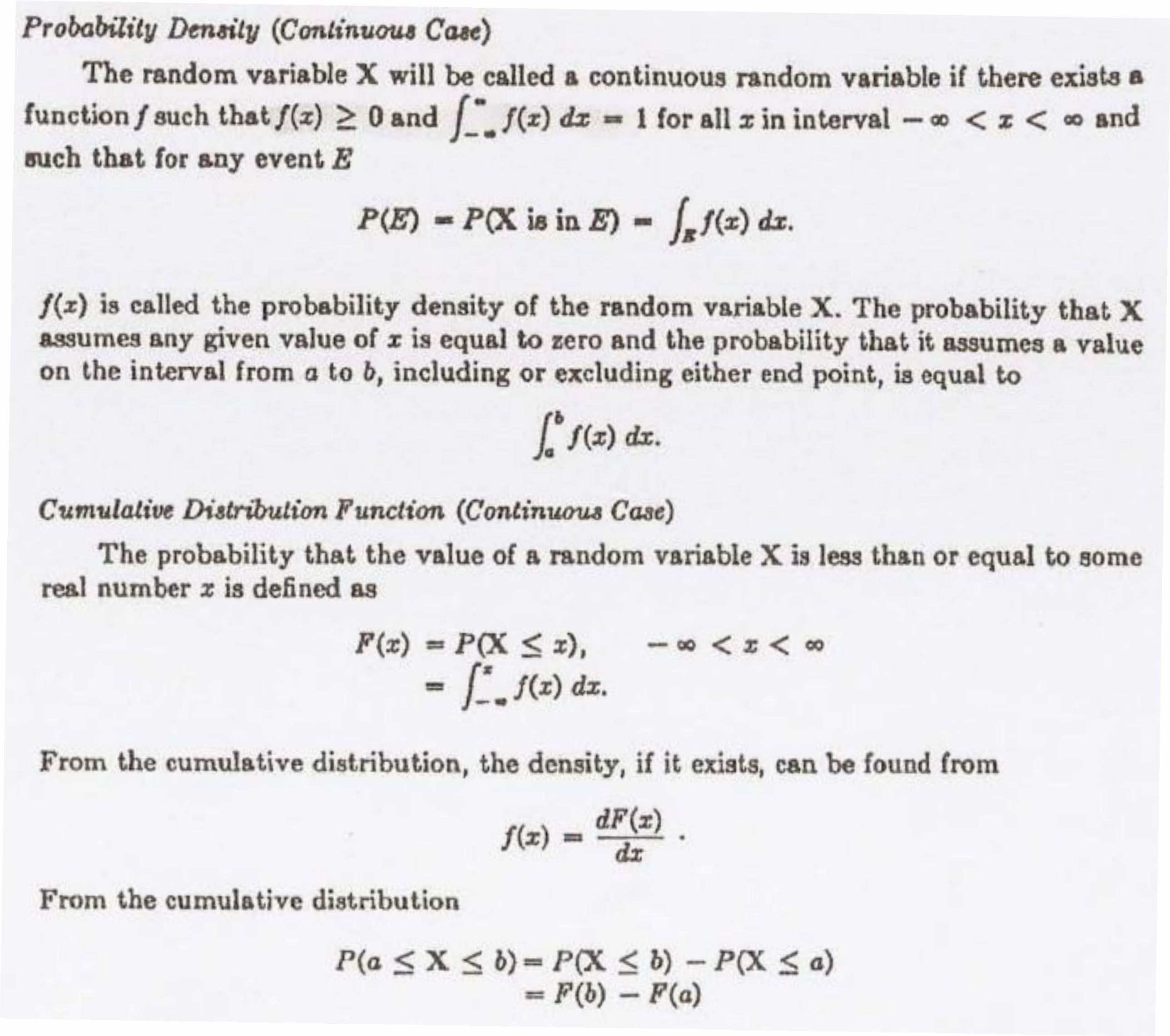
The random variable X will be called a continuous random variable if there exists a function f such that $f(x) \geq 0$ and $\int_{-\infty}^{\infty} f(x)\,dx = 1$ for all x in interval $-\infty < x < \infty$ and such that for any event E

$$P(E) = P(\text{X is in } E) = \int_E f(x)\,dx.$$

$f(x)$ is called the probability density of the random variable X. The probability that X assumes any given value of x is equal to zero and the probability that it assumes a value on the interval from a to b, including or excluding either end point, is equal to

$$\int_a^b f(x)\,dx.$$

Summary of Probability Theory-II



Probability Density (Continuous Case)

The random variable X will be called a continuous random variable if there exists a function f such that $f(x) \geq 0$ and $\int_{-\infty}^{\infty} f(x)\,dx = 1$ for all x in interval $-\infty < x < \infty$ and such that for any event E

$$P(E) = P(\text{X is in } E) = \int_E f(x)\,dx.$$

$f(x)$ is called the probability density of the random variable X. The probability that X assumes any given value of x is equal to zero and the probability that it assumes a value on the interval from a to b, including or excluding either end point, is equal to

$$\int_a^b f(x)\,dx.$$

Cumulative Distribution Function (Continuous Case)

The probability that the value of a random variable X is less than or equal to some real number x is defined as

$$\begin{aligned} F(x) &= P(\text{X} \leq x), \qquad -\infty < x < \infty \\ &= \int_{-\infty}^{x} f(x)\,dx. \end{aligned}$$

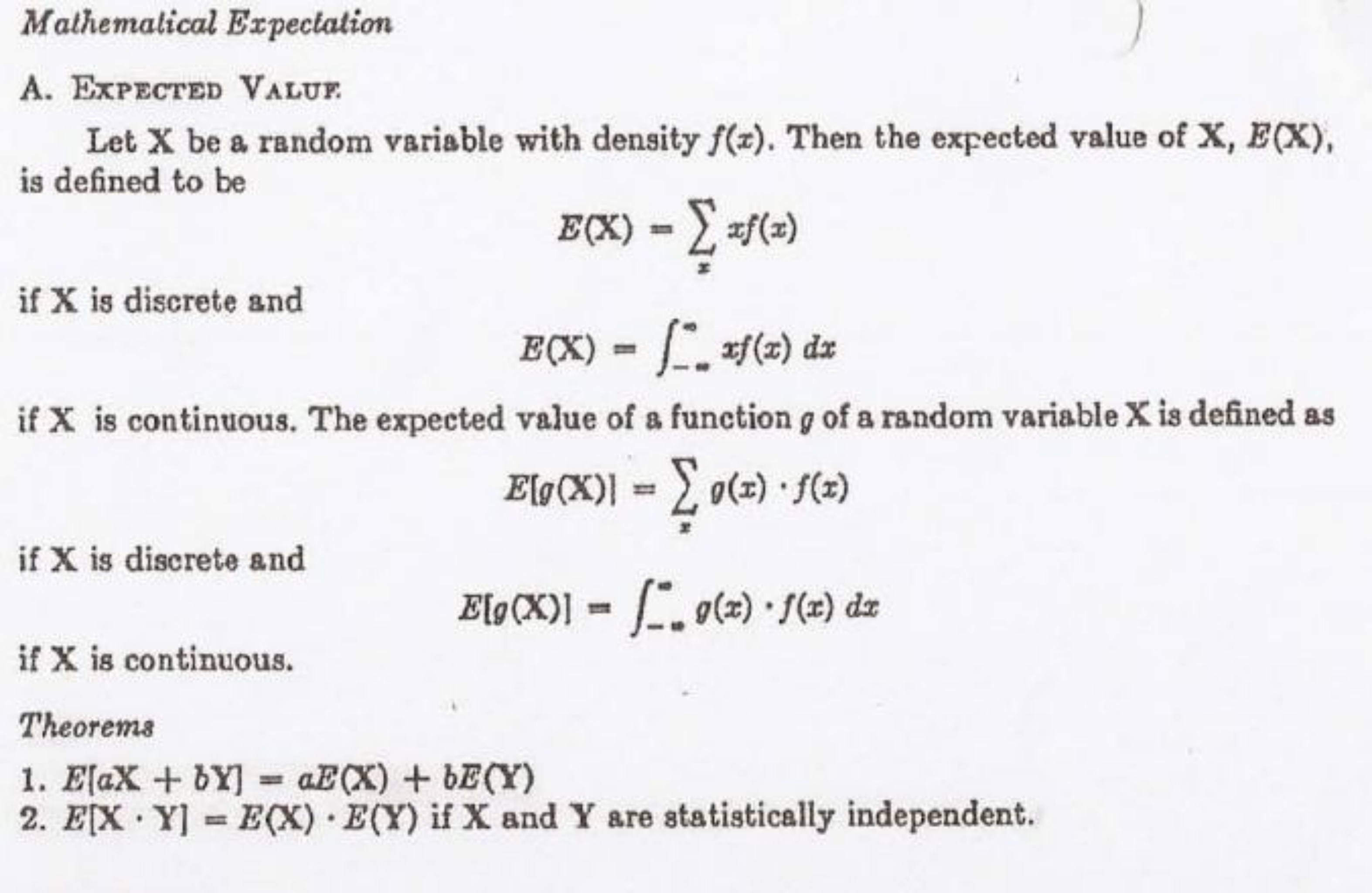
From the cumulative distribution, the density, if it exists, can be found from

$$f(x) = \frac{dF(x)}{dx}\,.$$

From the cumulative distribution

$$\begin{aligned} P(a \leq \text{X} \leq b) &= P(\text{X} \leq b) - P(\text{X} \leq a) \\ &= F(b) - F(a) \end{aligned}$$

Summary of Probability Theory-III



Mathematical Expectation

A. EXPECTED VALUF

Let X be a random variable with density $f(x)$. Then the expected value of X, $E(\mathrm{X})$, is defined to be

$$E(\mathrm{X}) = \sum_x x f(x)$$

if X is discrete and

$$E(\mathrm{X}) = \int_{-\infty}^{\infty} x f(x)\, dx$$

if X is continuous. The expected value of a function g of a random variable X is defined as

$$E[g(\mathrm{X})] = \sum_x g(x) \cdot f(x)$$

if X is discrete and

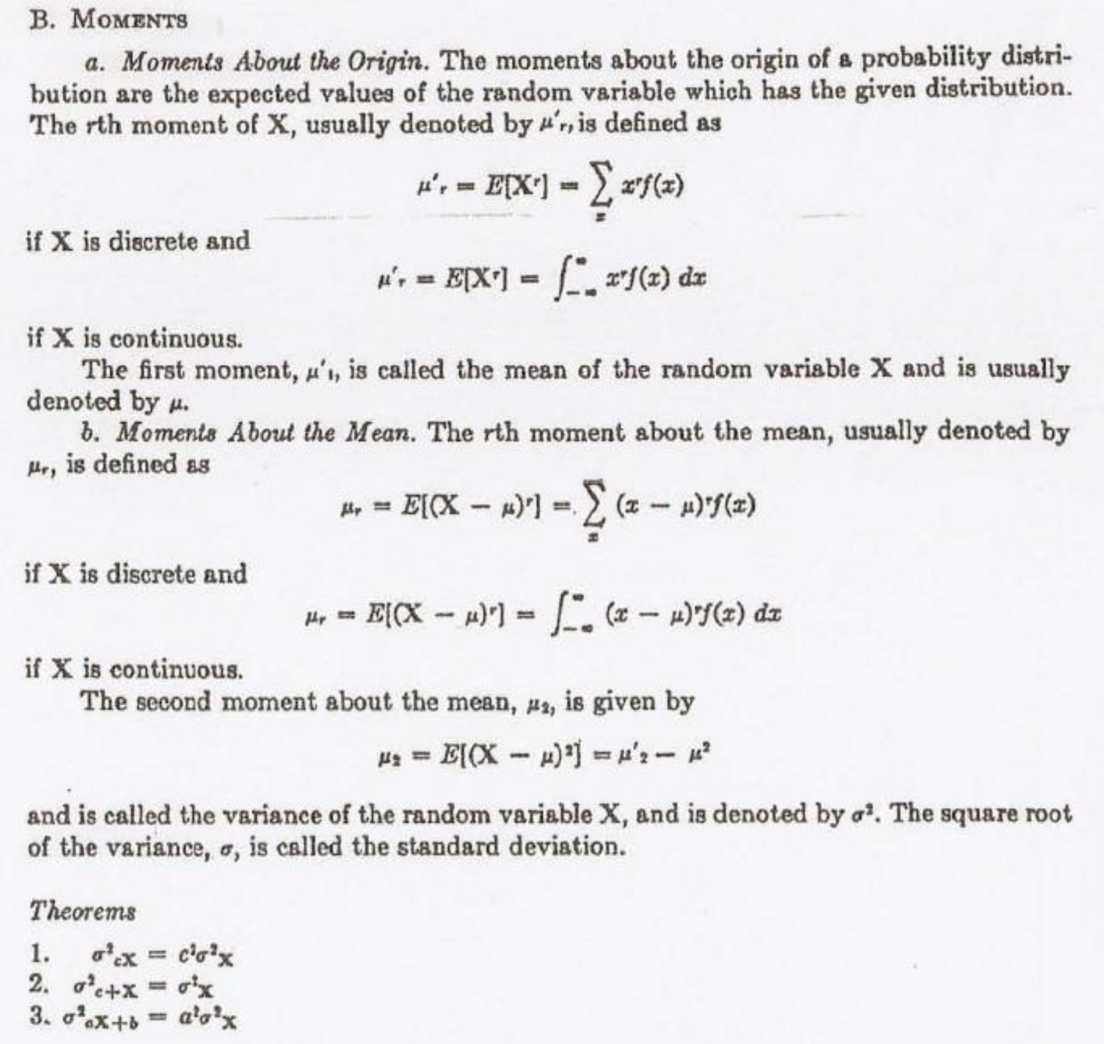
$$E[g(\mathrm{X})] = \int_{-\infty}^{\infty} g(x) \cdot f(x)\, dx$$

if X is continuous.

Theorems

1. $E[a\mathrm{X} + b\mathrm{Y}] = aE(\mathrm{X}) + bE(\mathrm{Y})$
2. $E[\mathrm{X} \cdot \mathrm{Y}] = E(\mathrm{X}) \cdot E(\mathrm{Y})$ if X and Y are statistically independent.

Summary of Probability Theory-IV



B. Moments

a. Moments About the Origin. The moments about the origin of a probability distribution are the expected values of the random variable which has the given distribution. The rth moment of X, usually denoted by μ'_r, is defined as

$$\mu'_r = E[X^r] = \sum_x x^r f(x)$$

if X is discrete and

$$\mu'_r = E[X^r] = \int_{-\infty}^{\infty} x^r f(x)\, dx$$

if X is continuous.

The first moment, μ'_1, is called the mean of the random variable X and is usually denoted by μ.

b. Moments About the Mean. The rth moment about the mean, usually denoted by μ_r, is defined as

$$\mu_r = E[(X - \mu)^r] = \sum_x (x - \mu)^r f(x)$$

if X is discrete and

$$\mu_r = E[(X - \mu)^r] = \int_{-\infty}^{\infty} (x - \mu)^r f(x)\, dx$$

if X is continuous.

The second moment about the mean, μ_2, is given by

$$\mu_2 = E[(X - \mu)^2] = \mu'_2 - \mu^2$$

and is called the variance of the random variable X, and is denoted by σ^2. The square root of the variance, σ, is called the standard deviation.

Theorems

1. $\sigma^2_{cX} = c^2\sigma^2_X$
2. $\sigma^2_{c+X} = \sigma^2_X$
3. $\sigma^2_{aX+b} = a^2\sigma^2_X$