Intermediate Lab

PHYS 3870

Lecture 2

Defining Errors

References: Taylor Ch. 1, 2, 3

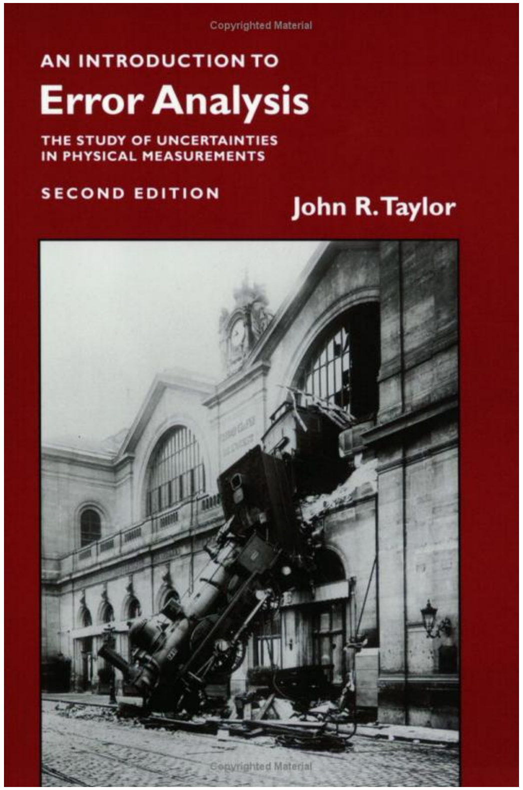
Baird Problems (Ch 5-See web site)

Also refer to [HANDOUT]

"Glossary of Important Terms in Error Analysis"

Week	Lecture	Reading	Problems in Taylor	Assignments Deadlines
1 **(8/25/14 to 8/29/14)**	Introduction Experiment Design	Baird, Ch. 5 AIP Style Manual: pp. 1-30 Taylor: Preface	Baird: 5.7,5.14, 5.18, 5.23 (problems posted on web)	
	Error Analysis Uncertainty Error Propagation	Taylor: Ch. 1 & 2 Taylor: Ch. 3	HW#1: 2.2, 2.3, 2.8, 2.9, 2.17, 2.19, 2.24, 2.26,2.28	
2	**(Memorial Day**	**No Class on Mon. 9/1)**		Problem Set 1 due 9/3/14

Problem Set #1
Due Next Week



Experimentation: An Introduction to Measurement Theory and Experiment Design

Prentice-Hall, Inc. *Englewood Cliffs, New Jersey* 1962

D. C. Baird *Associate Professor of Physics, Royal Military Colle*

PROBLEMS

In all the following problems state the variables or combination of variables that should be plotted to check the suggested variation and state how the unknown (slope, intercept, etc.) may be found.

7. The fundamental frequency of vibration of a string is given by

$$f = \frac{1}{2\ell}\sqrt{\frac{T}{m}}$$

f, ℓ, and T are measured variables. Determine m.

14. The linear expansion of a solid is described by

$$\ell = \ell_0(1 + \alpha \cdot \Delta T)$$

ℓ and ΔT are measured variables. ℓ_0 is constant but unknown. Determine α.

18. The force between electrostatic charges is described by

$$F = \frac{1}{4\pi\varepsilon_0}\frac{q_1 q_2}{r^2}$$

F and r are measured variables. q_1 and q_2 are fixed and known. How do you check the form of the function?

23. The wavelengths of the lines in the Balmer series of the hydrogen spectrum are given by

$$\frac{1}{\lambda} = R\left(\frac{1}{4} - \frac{1}{n^2}\right)$$

λ and n are measured variables. Determine R.

Useful Handout on Web

Intermediate Laboratory - PHYX 3870-3880

Glossary of Important Terms in Data And Error Analysis

Measurement Theory

Precision - A measure of the reproducibility of a measurement. If an experiment has small random errors, it is said to have high precision.

Accuracy - A measure of the validity of a measurement. If an experiment has small systematic errors, it is said to have high accuracy.

Discrepancy - The difference between two measured values of the same quantity.

Uncertainty - The outer limits of confidence within which a given measurement "almost certainly" lies. It is important to specify what criteria are used to determine the confidence limits.

average value of the quantity.

Propagation of Errors - A method of determining the error inherent in a derived quantity from the errors of the measured quantities used to determine the derived quantity.

Significant Figures - A notation convention for writing the value of measured quantities. In general, the measured quantity should have only as many significant figures as warranted by its absolute uncertainty.

Rounding - A method of truncating numbers, particularly useful in the context of significant figures. By standard convention, numbers to be rounded should be truncated for trailing numbers less than 5, rounded up for trailing numbers over 5, and rounded to the nearest even number for a trailing 5.

What is Science?

The scientific method goes further in:

- Developing a description (model) of the system behavior based on observation
- Generalizing this description (model) to other behavior and other systems
- That is to say, the scientific method is experimentation and modeling intertwined
- It is the scientific method that distinguishes science from other forms of endeavor

Scientific Method:

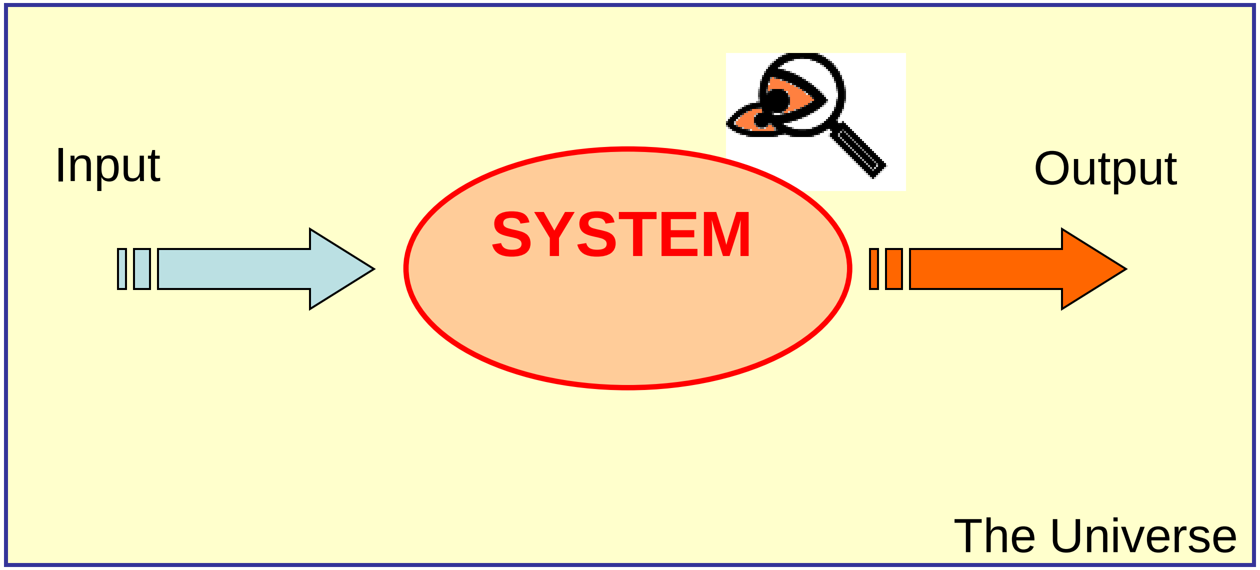
Leads to ***new discoveries*** → how scientific progress is made!

Careful measurements, Experiments

Models, Empirical Laws, Generalization

Hypothesis, Theory

Uncertainties in Observations



- **Observations characterize the system to within the uncertainty of the measurements**
- **Uncertainties can arise from:**
  - **Limitations of instrumentation or measurement methods**
  - **Statistical fluctuations of the system**
  - **Inherent uncertainties of the system**

    - Quantum fluctuations

    - Non-deterministic processes (e.g., chaos):

      - There are systems where uncertainties dominate and preclude models predicting the outcome

      - We will not (intentionally) deal with this type of system.

What is a Model?

Models of the physical world

1. A model:
   a) Describes the system
   b) Proposes how input variables interact with the system to modify output variables
2. Models versus systems
   a) A system is real. Information about the system can be known incontrovertibly.
   b) Models are not real.
      (1) Models are mankind's descriptions of reality
      (2) Models can never be fact (period), though they can be very good descriptions of how real systems behave.
      (3) Neither Newton's Law's, nor Special Realitivity, nor Einstein's Equations for General Relativity, nor TOE (Theory of Everything) are the final answer; Nature is!

What is a Model?

Models of the physical world

3. Modeling simple systems versus modeling complex systems

    a) Modeling simple systems is easier, but often insufficient

    b) This brings up the point in the *art of experimentation*; it is prudent to use the simplest model possible to get the desired level of predictions from your model.

        i. When are Newton’s Laws, or Special Relativity, or General Relativity sufficient?

        ii. Should we worry about the Quantum nature of a system or is a classical approach sufficient?}

    c) Learning how to do modeling and experimentation is easier on simple systems.

    d) Hence, we do experiments on pendula, not Pentium processors.

Designing "Good" Experiments

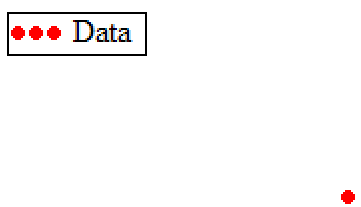
What is "Good"? One that:

- Gives "good" unambiguous results
- Gets a "good" grade

An Exercise in Experimental Design

An Example of Experimental Design

Consider a simple data set, the output for one measurement.



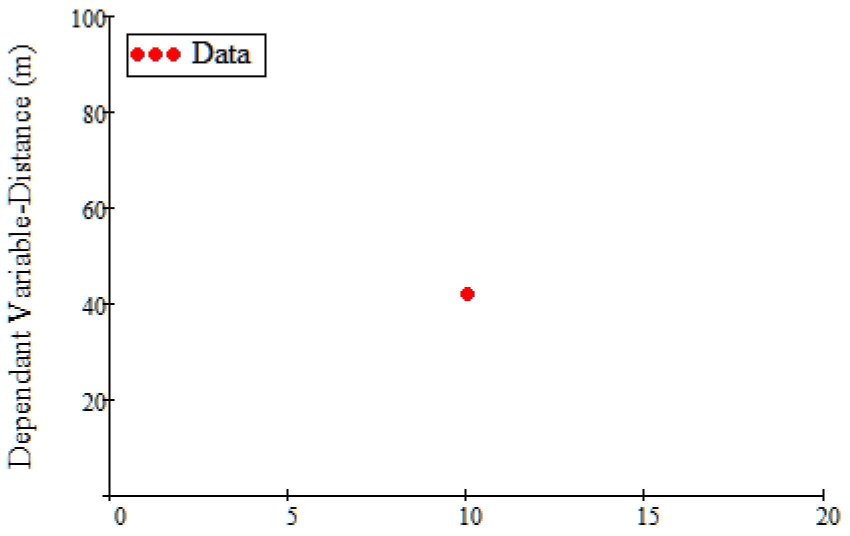
What is the best model for the data now?

How can we improve things?

Lecture 2-Exp Design.xmcd

An Exercise in Experimental Design

Add units for the outout data point. The answer is "42 (m)".



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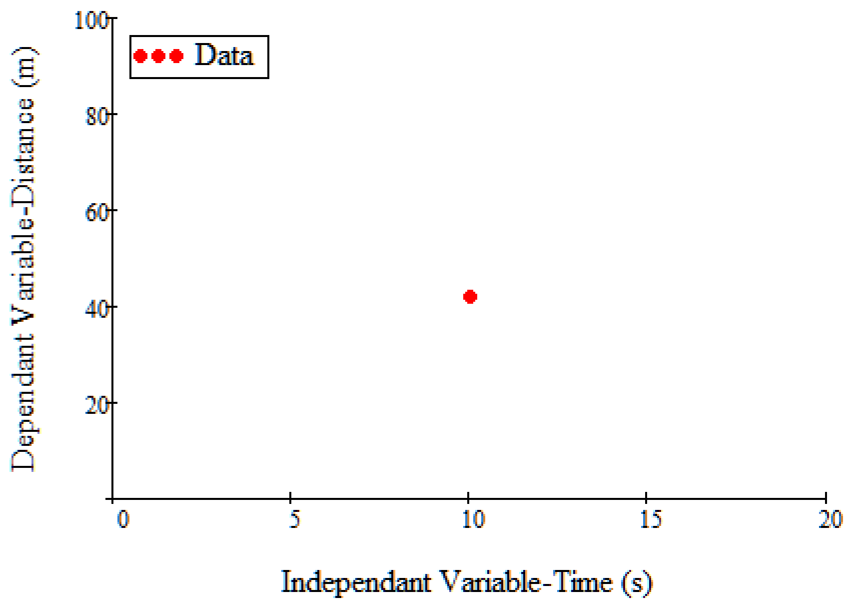
Lecture 2-Exp Design.xmcd

An Exercise in Experimental Design

Add the input value of the data point (with units).



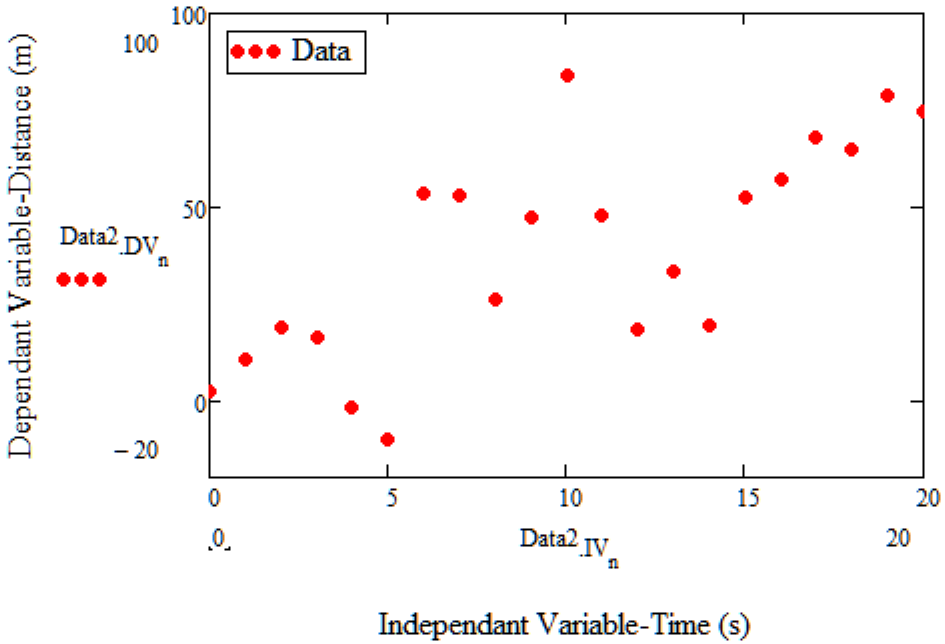
What is the best model for the data now?

How can we improve things?

Lecture 2-Exp Design.xmcd

An Exercise in Experimental Design

Add more observations, based on more pokes (inputs), to the data set.



What is the best model for the data now?

Perhaps a linear model?

Try more or less noisy data sets.

Lecture 2-Exp Design.xmcd

An Exercise in Experimental Design

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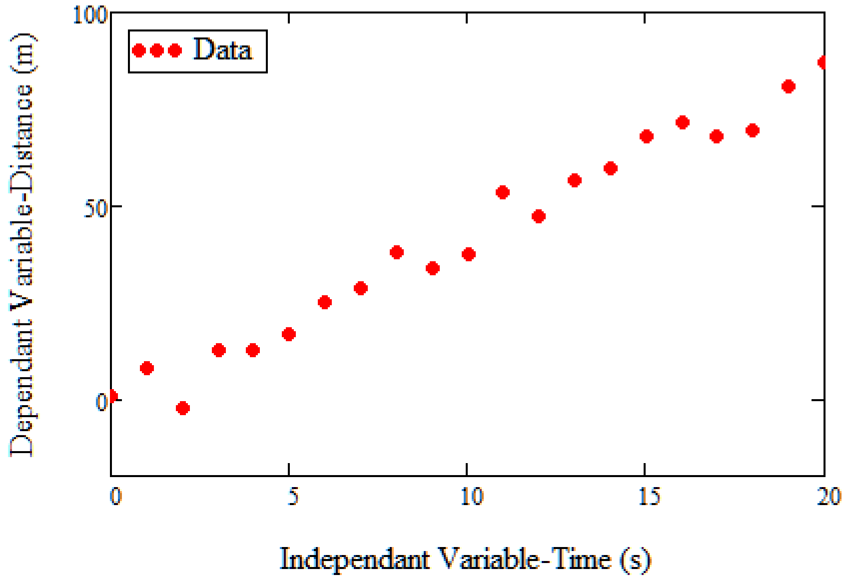

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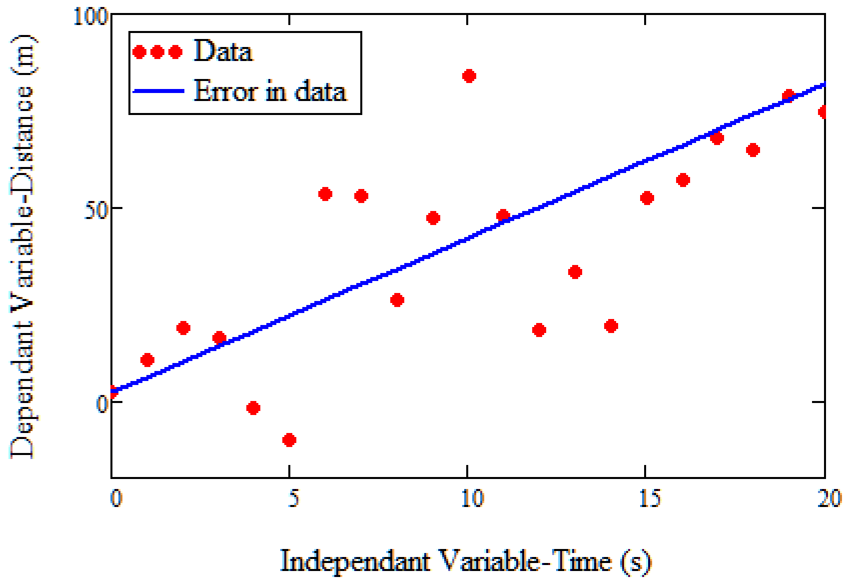

What is the best model for the data now?

Perhaps a linear model?

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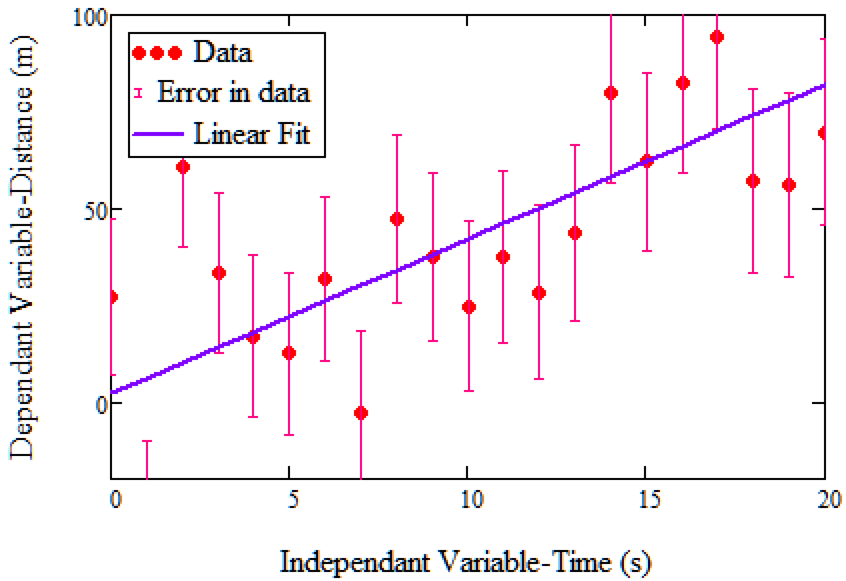
Lecture 2-Exp Design.xmcd

An Exercise in Experimental Design

Try a linear model.



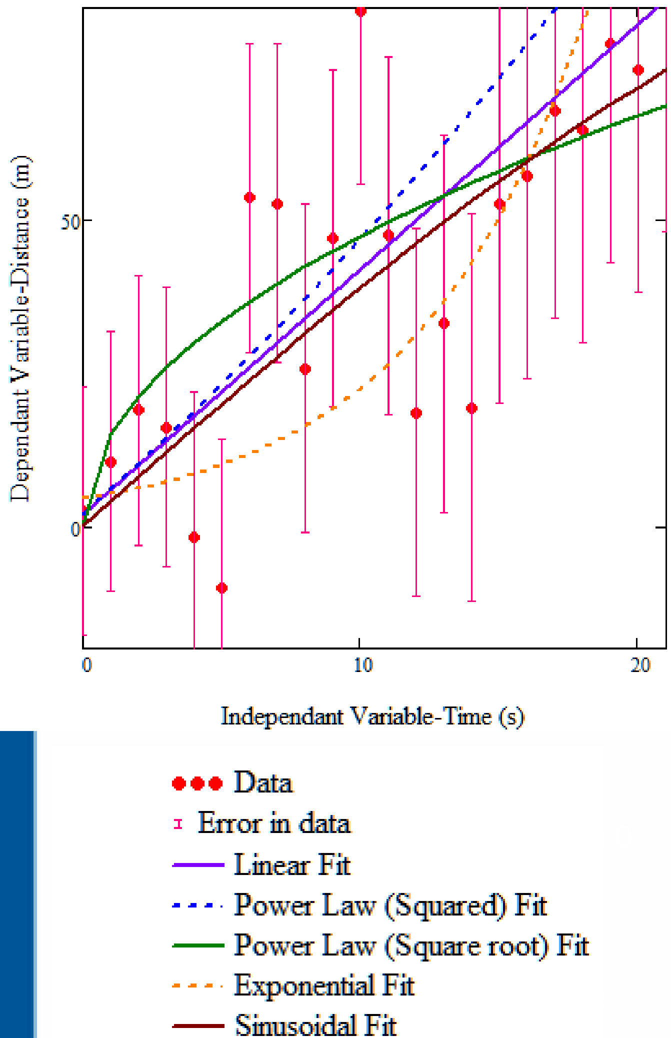
How valid is a linear model?

Perhaps add error bars to the data?

Consider adjusting the error bars.

Lecture 2-Exp Design.xmcd

An Exercise in Experimental Design



Consider other models.

How valid are the other model?

Consider adjusting the error bars.

Consider extending the width of the data set.

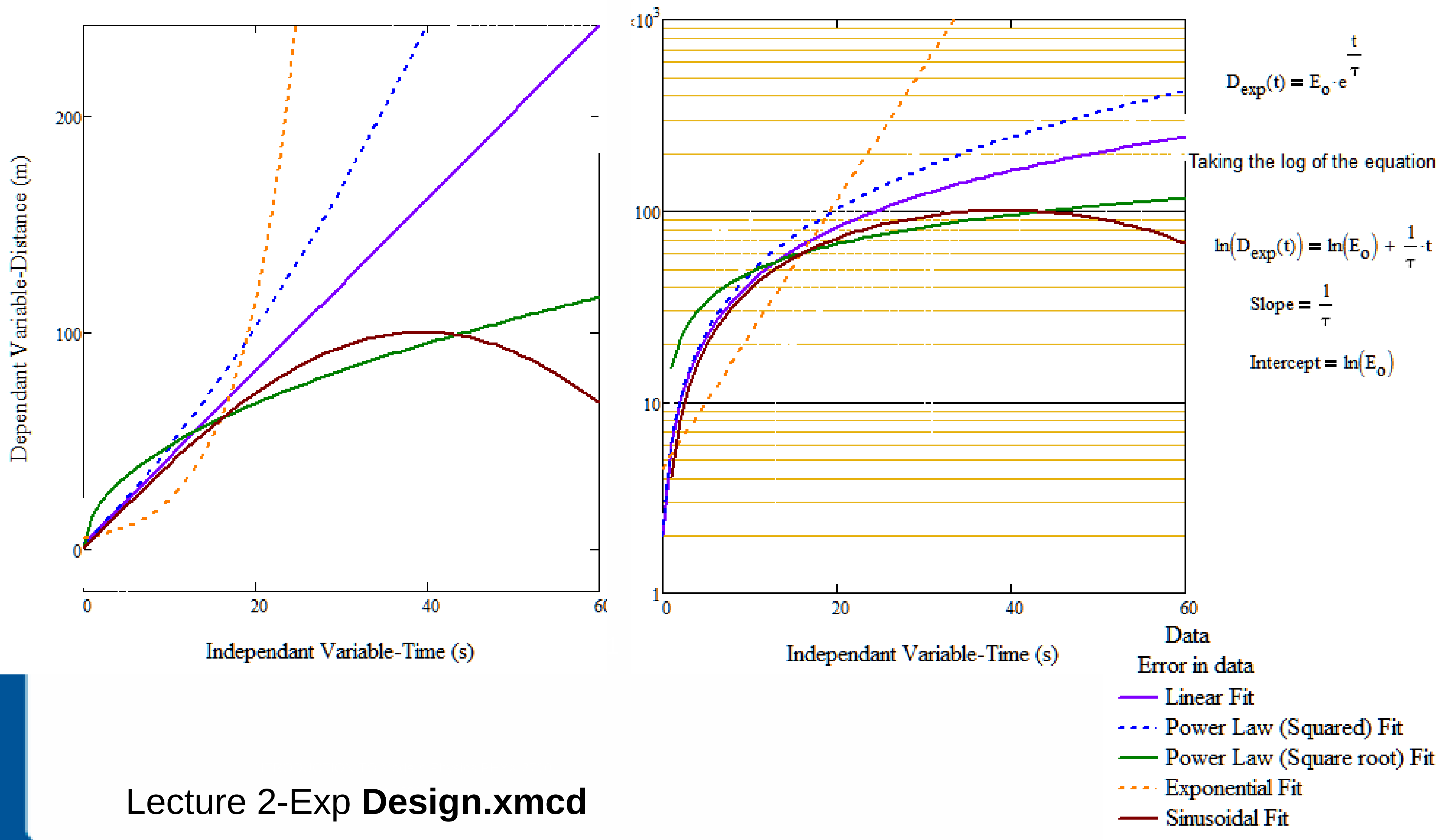
Consider "linearizing" the fitting functions.

Is the linear fit the "best" model?

Lecture 2-Exp Design.xmcd

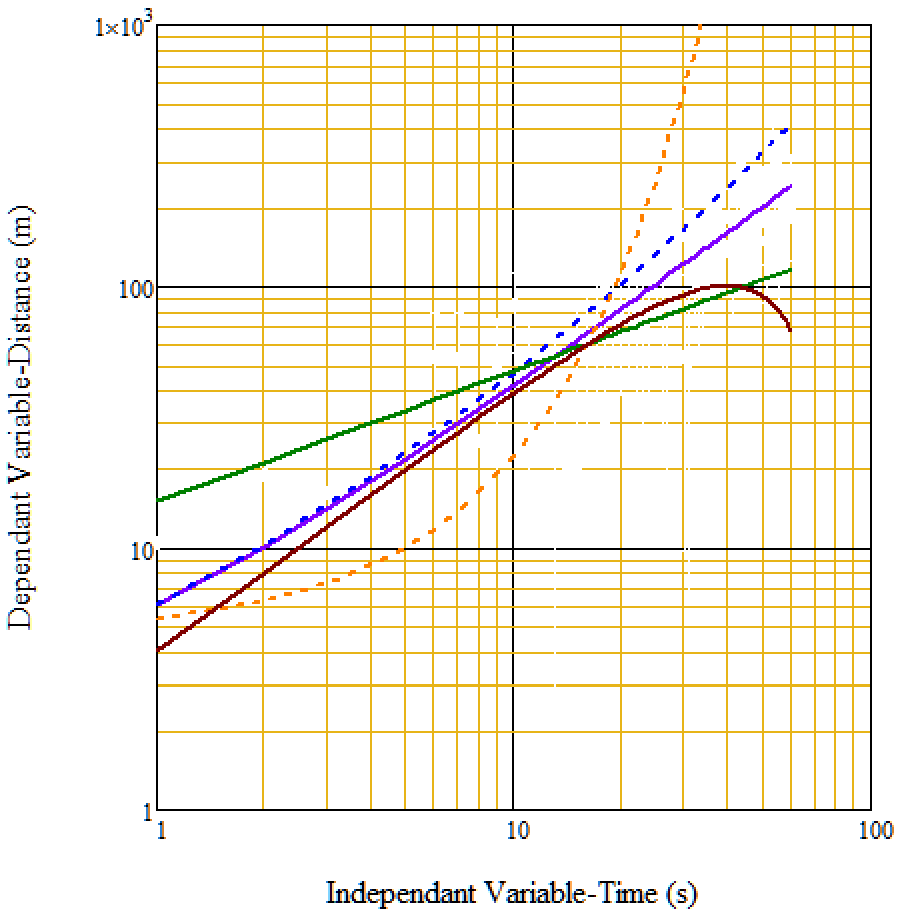
Linear Model:	$D_{linear}(t) := V_o \cdot t + D_o$	$V_o := 4 \cdot \frac{m}{s}$	$D_o := 2 \cdot m$
Square Model:	$D_{square}(t) := a_o \cdot t^2 + V_o \cdot t + D_o$	$a_o := 0.05 \cdot \frac{m}{s^2}$	
Square Root Model:	$D_{sqroot}(t) := C_1 \cdot t^{\frac{1}{2}} + C_o$	$C_1 := 15 \cdot \frac{m}{s^{0.5}}$	$C_o := 0 \cdot m$
Exponential Model:	$D_{exp}(t) := E_o \cdot e^{\frac{t}{\tau}}$	$E_o := 4.5 \cdot m$	$\tau := 6.2 \cdot sec$
Sinusoidal Model:	$D_{sin}(t) := S_1 \cdot \sin(\omega \cdot t + \psi_o)$	$S_1 := 100 \cdot m$	$\omega := \frac{1}{25} \cdot s^{-1}$ $\psi_o := 0$

An Exercise in Experimental Design



Lecture 2-Exp **Design.xmcd**

An Exercise in Experimental Design



$$D_{linear}(t) = V_o \cdot t + D_o$$

Taking the log of the equation

$$\ln\left(D_{linear}(t) - D_o\right) = \ln\left(V_o\right) + 1 \cdot t$$

Slope = 1

Intercept = $\ln\left(V_o\right)$

$$D_{sqroot}(t) = C_1 \cdot t^{\frac{1}{2}} + C_o$$

Taking the log of the equation

$$\ln\left(D_{sqroot}(t) - C_o\right) = \ln\left(C_1\right) + \frac{1}{2} \cdot t$$

Slope = $\frac{1}{2}$

Intercept = $\ln\left(C_1\right)$

Which models are now linear?

What about the square model?

Data
Error in data
Linear Fit
Power Law (Squared) Fit
Power Law (Square root) Fit
Exponential Fit
Sinusoidal Fit

Lecture 2-Exp **Design.xmcd**

Intermediate Lab

PHYS 3870

Comparing Measurements to Models

Qualitatively

Intermediate Lab

PHYS 3870

Dimensional Analysis

Units and Dimensions

Units **- An arbitrary set of measurement standards used to compare physical quantities. Common systems of units include the meter-kilogram-second (MKS or SI) system, the centimeter-gram-second (CGS) system, and the foot-pound-second (English) system.**
Fundamental SI units are s, m, kg, A, K, mole, and Cd.

Dimensions **- The fundamental quantities used to express physical quantities independent of the system of units used.**
The basic dimensions are length (L), time (T), mass (M), and electric current (A), temperature (T), amount (N), and luminous intensity (I_γ).

Dimensional Analysis

Dimensional Analysis - **The use of dimensions of physical quantities to verify calculations and formulas.**

Example:

Newton's Second Law states that F - ma.

Dimensional analysis shows that $F \overset{D}{=} MLT^{-2}$ represents the dimensions of force.

From Hooke's Law, F=-kx what are the dimensions of k?

$k=F/x \overset{D}{=} MT^{-2}$

Note the symbol $\overset{D}{=}$ represents the equality of dimensions.

Dimensional analysis is not capable of completely determining an unknown functional relationship, but it can delimit the possibilities and, in some cases it can give the complete relationship to within a constant of proportionality.

An Aside: Mathcad is great at handling units and dimensional analysis.

Units and Dimensions--A Whole New World!

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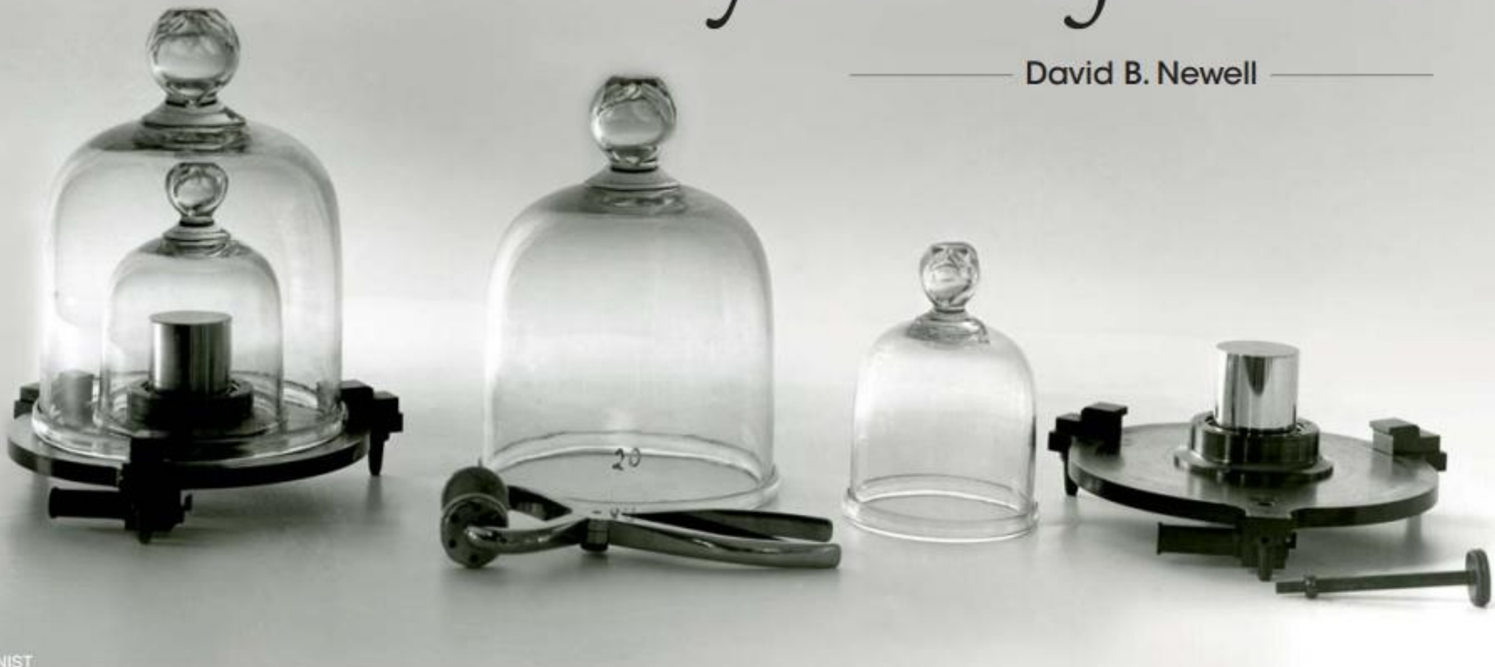
Fundamental Constants **- The combination of exact (defined) fundamental constants used to express physical quantities independent of the system of units used. The basic fundamental constants are $\Delta\nu(^{133}Cs)_{hfs}$, c, h, e, k_B, N_A, and K_{cd}.**

A more fundamental International System of Units

David B. Newell



The universally accepted method of expressing physical measurements for world commerce, industry, and science is about to get a facelift, thanks to our improved knowledge of fundamental constants.

D.B. Newell, *Physics Today,* 67(7), 35 (2014).

Units and Dimensions--A Whole New World!

Figure 1. Evolution of the SI. A brief timeline of the history of the International System of Units since John Wilkins's 1668 essay is scaled to a meter bar. The photograph shows a marble meter standard in Paris, dating from the 18th century. (Photo courtesy of LPLT\Wikimedia Commons.)

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Fundamental SI units are s, m, kg, A, K, mole, and Cd.

Units and Dimensions--A Whole New World!

Table 1. Present SI base quantities, base units, and definitions

Base quantity	Base unit	Definition
Time	second	The second is the duration of 9 192 631 770 periods of the radiation corresponding to the transition between the two hyperfine levels of the ground state of the cesium-133 atom.
Length	meter	The meter is the length of the path traveled by light in vacuum during a time interval of 1/299 792 458 of a second.
Mass	kilogram	The kilogram is the unit of mass; it is equal to the mass of the international prototype of the kilogram.
Electric current	ampere	The ampere is that constant current which, if maintained in two straight parallel conductors of infinite length, of negligible circular cross section, and placed 1 meter apart in vacuum, would produce between these conductors a force equal to 2×10^{-7} newton per meter of length.
Thermodynamic temperature	kelvin	The kelvin, unit of thermodynamic temperature, is the fraction 1/273.16 of the thermodynamic temperature of the triple point of water.
Amount of substance	mole	The mole is the amount of substance of a system which contains as many elementary entities as there are atoms in 0.012 kilogram of carbon-12; the elementary entities must be specified and may be atoms, molecules, ions, electrons, other particles, or specified groups of such particles.
Luminous intensity	candela	The candela is the luminous intensity, in a given direction, of a source that emits monochromatic radiation of frequency 540×10^{12} hertz and that has a radiant intensity in that direction of 1/683 watt per steradian.

Table 2. New SI base quantities, defining constants, and definitions

Base quantity	Defining constant	Definition
Frequency	$\Delta\nu(^{133}Cs)_{hfs}$	The unperturbed ground-state hyperfine splitting frequency of the cesium-133 atom $\Delta\nu(^{133}Cs)_{hfs}$ is exactly 9 192 631 770 hertz.
Velocity	c	The speed of light in vacuum c is exactly 299 792 458 meter per second.
Action	h	The Planck constant h is exactly $6.626X \times 10^{-34}$ joule second.
Electric charge	e	The elementary charge e is exactly $1.602X \times 10^{-19}$ coulomb.
Heat capacity	k	The Boltzmann constant k is exactly $1.380X \times 10^{-23}$ joule per kelvin.
Amount of substance	N_A	The Avogadro constant N_A is exactly $6.022X \times 10^{23}$ reciprocal mole.
Luminous intensity	K_{cd}	The luminous efficacy K_{cd} of monochromatic radiation of frequency 540×10^{12} hertz is exactly 683 lumen per watt.

The symbol X in the numerical values indicates additional digits to be set upon redefinition of the SI. The term "defining constant" is used in the broader sense to include invariants of nature such as the hyperfine splitting frequency of the cesium-133 atom and the luminous efficacy.

Intermediate Lab

PHYS 3870

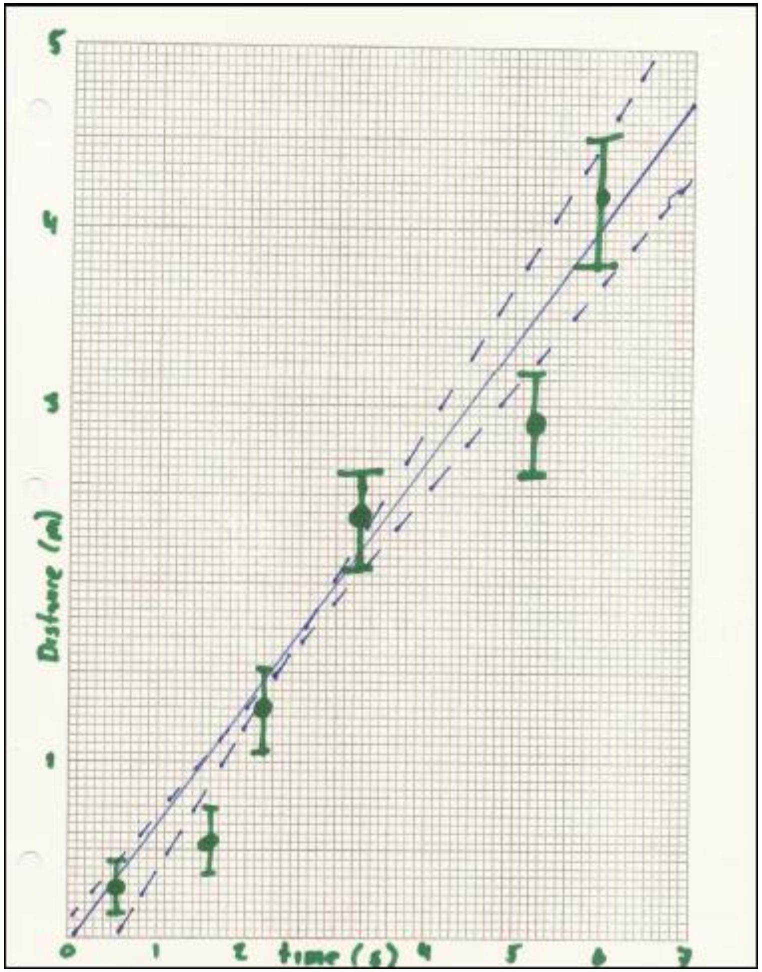
Graphical Analysis

Graphical Analysis

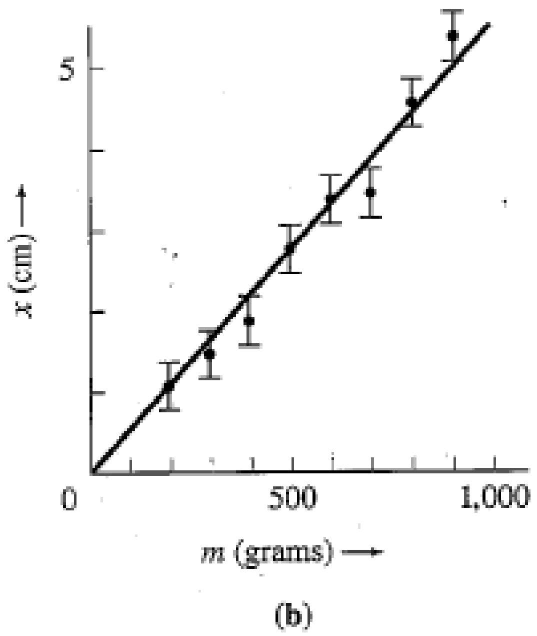
An “old School” approach to linear fits.

• Rough plot of data
• Estimate of uncertainties with error bars
• A “best” linear fit with a straight edge
• Estimates of uncertainties in slope and intercept from the error bars

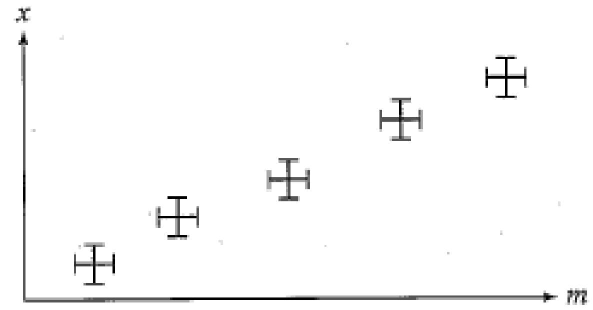
This is a great practice to get into as you are developing an experiment!



Is it Linear?

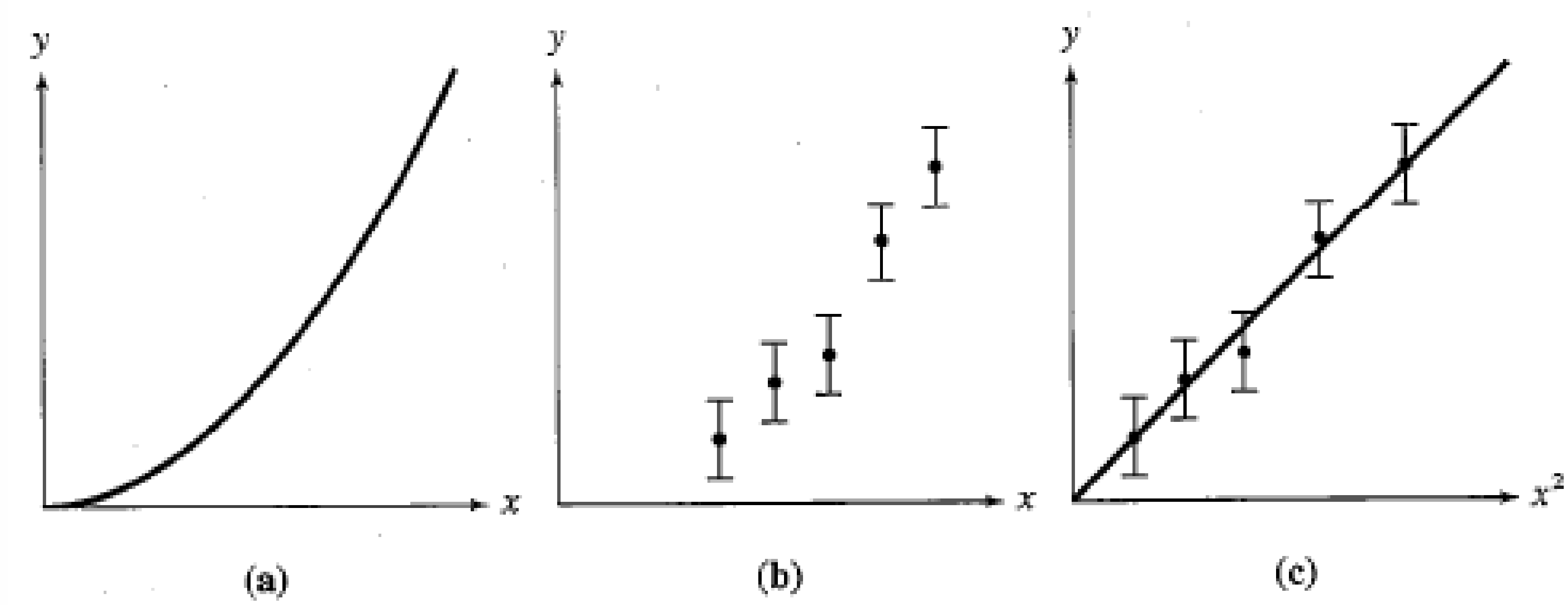


- A simple model is a linear model
- You know it when you see it (qualitatively)
- Tested with a straight edge
- Error bar are a first step in gauging the "goodness of fit"



Adding 2D error bars is sometimes helpful.

Making It Linear or Linearization



- A simple trick for many models is to linearize the model in the independent variable.
- Refer to Baird Ch.5 and the associated homework problems.

Linearizing Equations (1)

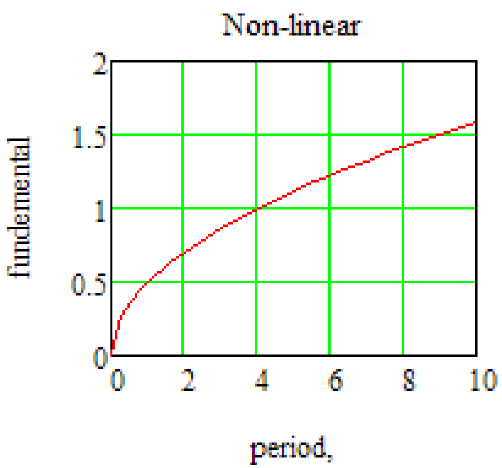
Problem 5.7

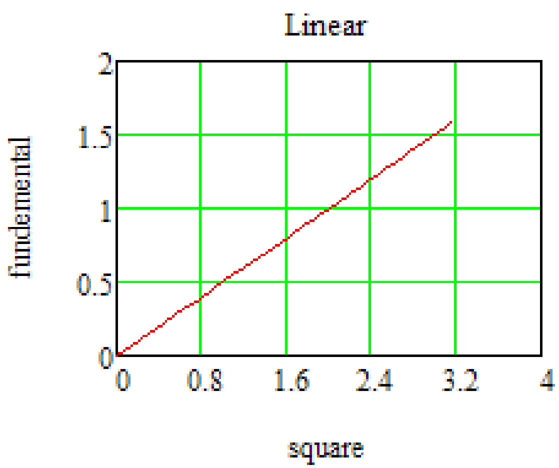
The functional form of f(L,T) is: $f(L,T) := \left(\frac{1}{2 \cdot L \cdot \sqrt{mass}}\right) \cdot \sqrt{T}$

Determine a value for mass.

By plotting the measured value the fundemental frequency, f, as a function of the square root of the period, T, one obtains a linear relationship with slope equal to $\left(\frac{1}{2 \cdot L \cdot \sqrt{mass}}\right)$. Solving for the mass, mass:

$$slope = \left(\frac{1}{2 \cdot L \cdot \sqrt{mass}}\right) \quad ==> \quad mass = (2 \cdot L \cdot slope)^2$$





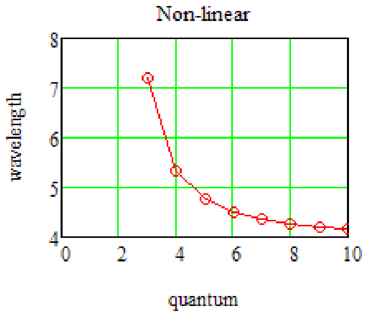
Linearizing Equations (2)

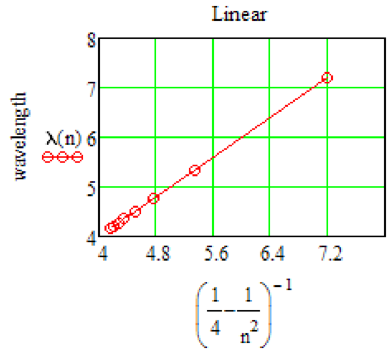
Problem 5.23

The functional form of $\lambda(n)$ is: $\lambda(n) := \left[R\cdot\left(\frac{1}{4} - \frac{1}{n^2}\right)\right]^{-1}$

Determine a value for the Rydberg constant, R.

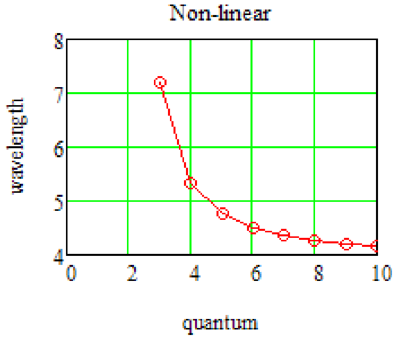
By plotting the measured value of the wavelength, λ, as a function of $\left(\frac{1}{4} - \frac{1}{n^2}\right)^{-1}$ one obtains a linear relationship with slope equal to R.

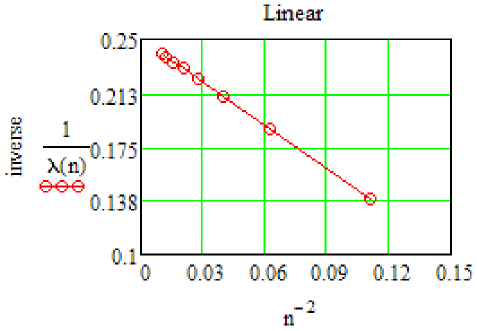




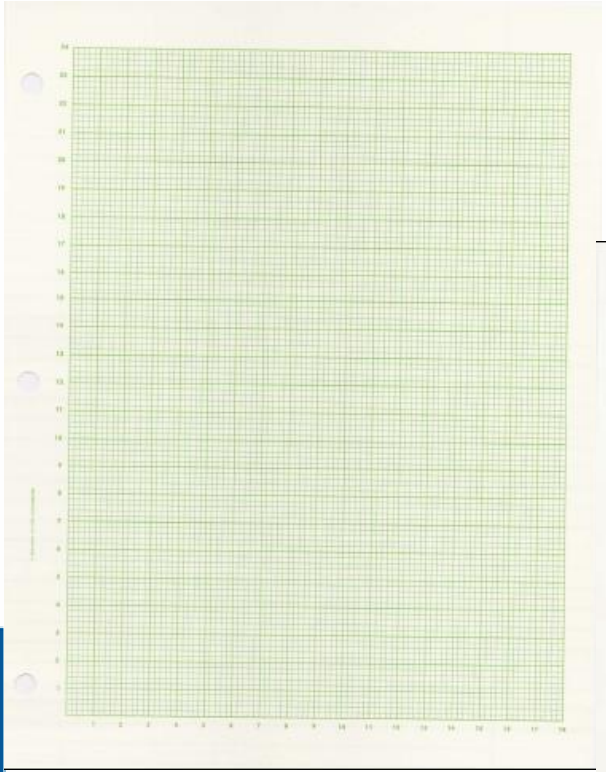
Alternately, by plotting one over the measured value of the wavelength, 1/λ, as a function of one over the square of the quantum index n, n^{-2}, one obtains a linear relationship with slope equal to $\left(\frac{-1}{R}\right)$. Solving for the mass, mass:

$$\text{slope} = \left(\frac{-1}{R}\right) \quad ==> \quad R = (-\text{slope})^{-1}$$





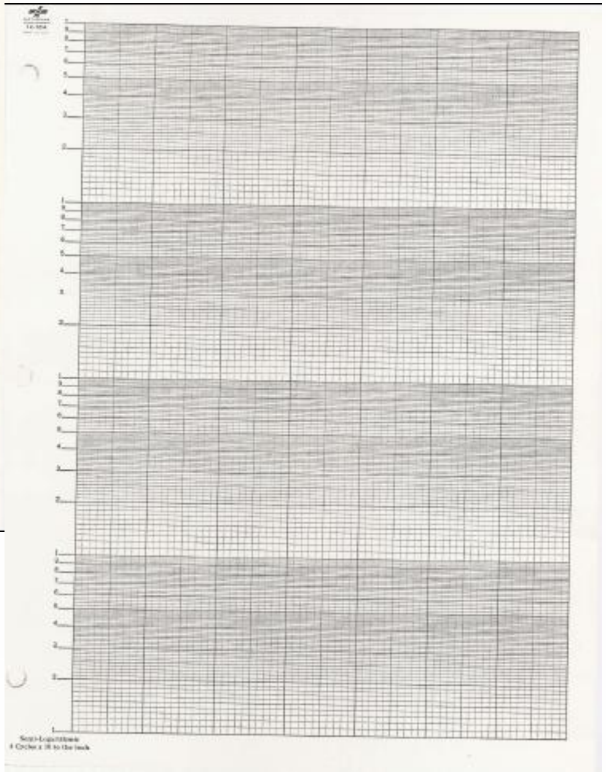
Special Graph Paper



Linear

“Old School” graph paper is still a useful tool, especially for reality checking during the experimental design process.

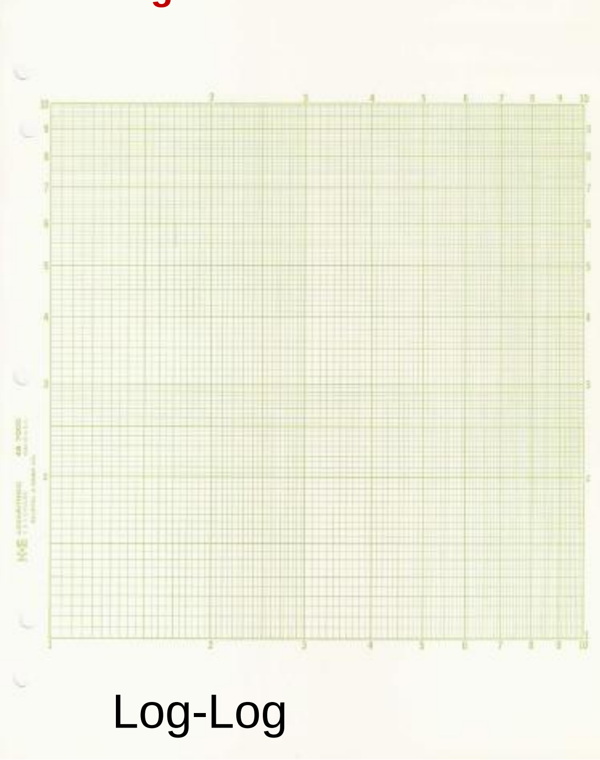
Semi-log paper tests for exponential models.



Semilog

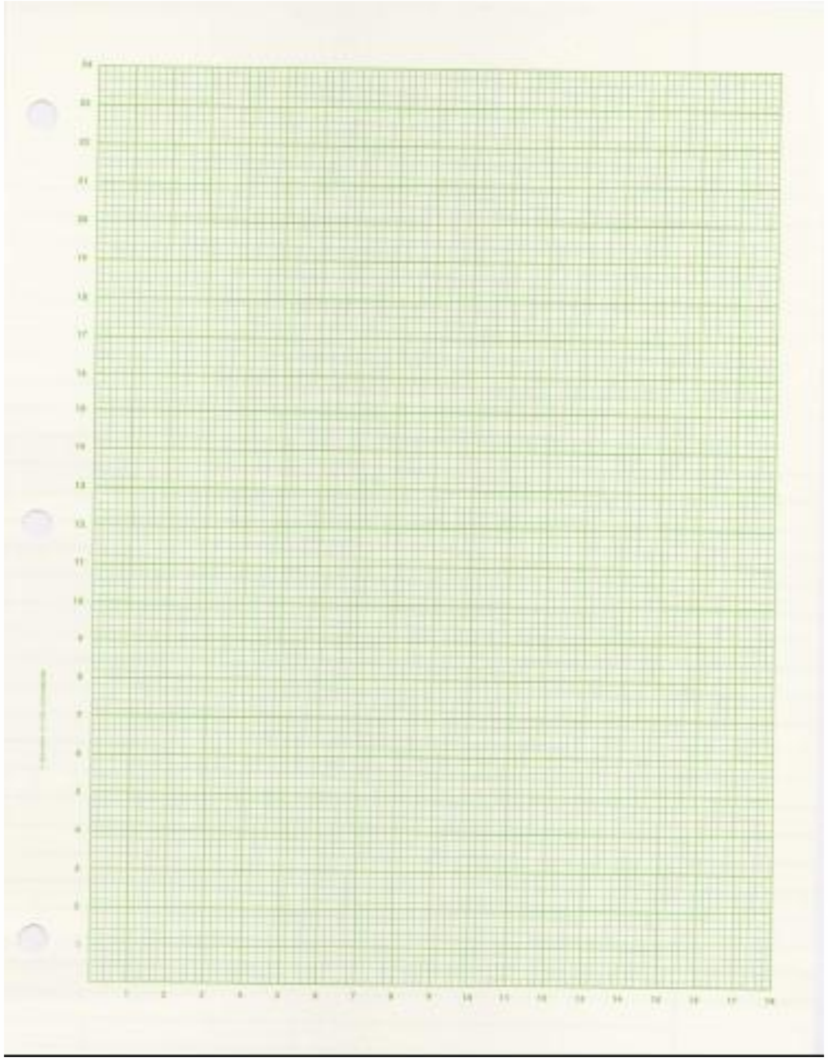
Log-log paper tests for power law models.

Both Semi-log and log -log paper are handy for displaying details of data spread over many orders of magnitude.

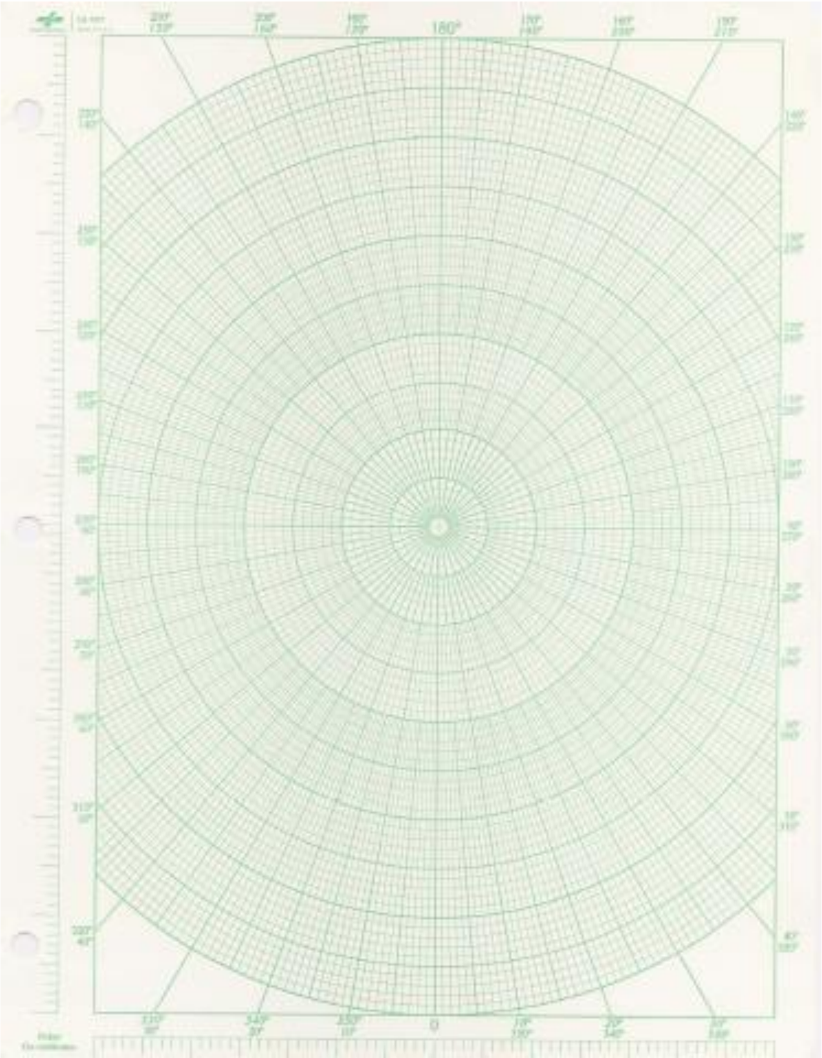


Log-Log

Special Graph Paper



Linear



Polar

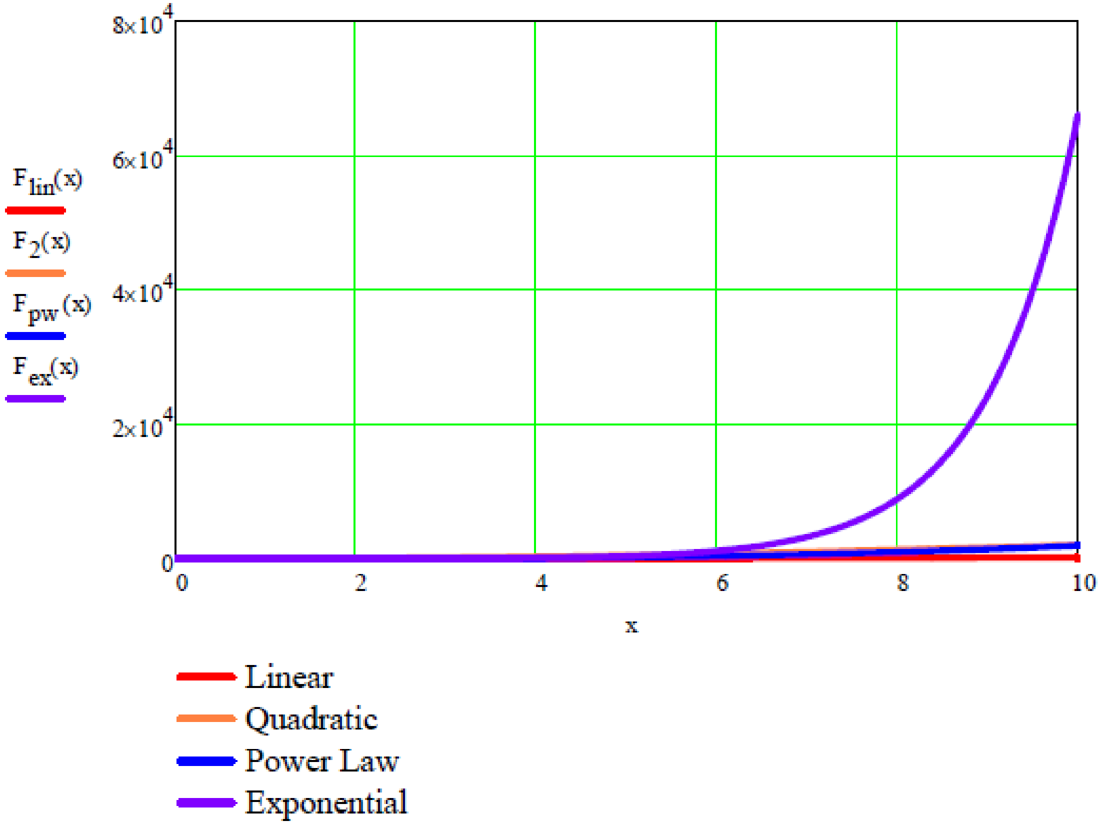
Linearizing Equations and "Magic Graph Paper"

Linear $F_{lin}(x) := A + B \cdot x$ $A := 10$ $B := 20$

Quadratic $F_2(x) := C_0 + C_1 \cdot x + C_2 \cdot x^2$ $C_0 := 5$ $C_1 := 10$ $C_2 := 20$

Power Law $F_{pw}(x) := D \cdot x^n$ $D := 2$ $n := 3$

Exponential $F_{ex}(x) := \alpha \cdot e^{\beta \cdot x}$ $\alpha := 3$ $\beta := 1$



Some Common Models

Model	Equation	Differential Eq.	Parameters
Constant	$y(x) = C$	$dy/dx\,(x) = 0$	C is constant value
Linear	$y(x) = B \cdot x + C$	$dy/dx = B$	B is slope C is constant value
Quadratic	$y(t) = A \cdot xt^2 + B \cdot t + C$	$dy/dt = 2A \cdot t + B$	A is "acceleration" B is "velocity" C is "initial position"
Power Law	$y(tx) = D \cdot x^n$	$dy/dx = D \cdot n \cdot x^{n-1}$	D is scale parameter n is power

Model	Equation	Differential Eq.	Parameters
Exponential Growth	$y(t) = P_o e^{kt}$	$dy/dx\,(x) = kt$	P_o is initial value at t=0 k>0 is growth constant (τ ≡ 1/k is time constant) (doubling time is ln(2)/k))
Exponential Decay	$y(t) = P_o e^{-\lambda t}$	$dy/dx\,(x) = -\lambda t$	P_o is initial value at t=0 λ>0 is decay constant (half life is ln(2)/ λ))
Learning Curve	$y(t) = M_o(1 - e^{kt})$	$dy/dx = k(M - y)$ y(0)=0	
Logistic Growth Curve	$y(t) = {M_o}/{(1 + e^{-M_o kt})}$	$dy/dx = ky(M - y)$	

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Comparing Measurements to Models

Errors as a Quantitative Tool

Intermediate Lab

PHYS 3870

Defining Errors

What Is Error?

- **The term "error" does not mean mistake in science.**
- **Rather, it means the inevitable uncertainty related to an observation or measurement of any physical quantity.**
- **It is a best guess at the range of values of subsequent measurements.**

Example: $x = 1.0 \pm 0.1$ m

This is shorthand for "the best estimate of x is 1.0 m. Subsequent measurements of x will 'almost certainly' lie between 0.9 m and 1.1 m

Why Are Errors Important?

http://www.math.nyu.edu/~crorres/Archimedes/Crown/CrownIntro.html

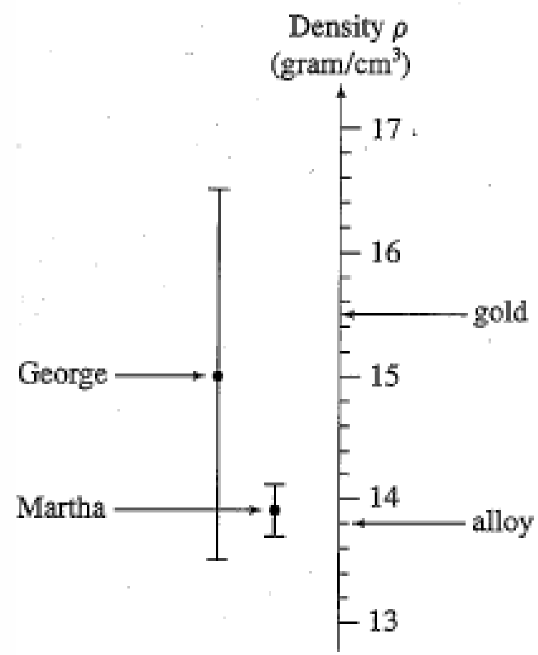
$$\rho_{gold} = 15.5 \text{ gram/cm}^3$$

and

$$\rho_{alloy} = 13.8 \text{ gram/cm}^3.$$

If we can measure the density of the crown, we should be able (as Archimedes suggested) to decide whether the crown is really gold by comparing ρ with the known densities ρ_{gold} and ρ_{alloy}.

Suppose we summon two experts in the measurement of density. The first expert, George, might make a quick measurement of ρ and report that his best estimate for ρ is 15 and that it almost certainly lies between 13.5 and 16.5 gram/cm^3. Our second expert, Martha, might take a little longer and then report a best estimate of 13.9 and a probable range from 13.7 to 14.1 gram/cm^3. The findings of our two experts are summarized in Figure 1.1.



A Timely Example of Errors in Measurements

What time is it now?

And the “best” answer is:

Data	**Reduced Data**	**Best Value**	**Discrepancy (Deviations)**	**Error**
		Average ____ + ___ sec		Range of discrepancies Average RMS (Standard) Deviation (_____ ± _____) sec For a set of N measurements (_____ ± _____) sec

or (______ ± ______) sec [absolute error]

or (______ sec ± ___%) [relative (fractional) error]

This is the best value and estimated uncertainty for a set of N measurements of the time.

What time is it now?

A Timely Example of Errors in Measurements

Intermediate Laboratory – PHX 3870

Lecture Two

Error Analysis

Uncertainties

Analysis with Mathcad sheet?

Enter the data.

Enter reduced data: $N := 10$ $n := 0..(N-1)$ $\Delta t_n :=$

N is number of data points:

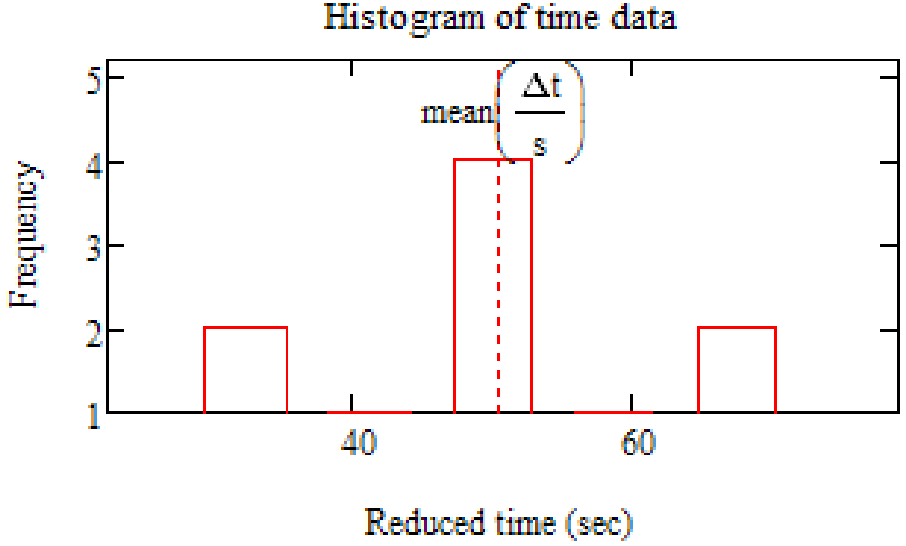
35
47
66
72
46
28
45
53
62
52

A Timely Example of Errors in Measurements

Analysis with Mathcad sheet?

Find the "best guess":

Method 1: Center of histogram



Method 2: Average (mean) Value

Longhand : $\frac{1}{N} \cdot \sum_n \Delta t_n = 50.6\,\mathrm{s}$

Shorthand : $\mathrm{mean}(\Delta t) = 50.6\,\mathrm{s}$

A Timely Example of Errors in Measurements

Analysis with Mathcad sheet?

What is the uncertainty?.

What is the uncertainty?

Calculate deviations from the mean $Dev := \Delta t - mean(\Delta t)$

n =	Δt_n = (s)	Dev_n = (s)
0.0	35.0	-15.6
1.0	47.0	-3.6
2.0	66.0	15.4
3.0	72.0	21.4
4.0	46.0	-4.6
5.0	28.0	-22.6
6.0	45.0	-5.6
7.0	53.0	2.4
8.0	62.0	11.4
9.0	52.0	1.4

Method 1: Range of deviations

$mean(\Delta t) = 50.6\,s$

+ $max(Dev) = 21.4\,s$

- $min(Dev) = -22.6\,s$

Method 2: Standard (RMS) Deviation

$$\sqrt{\frac{\sum_n (Dev_n)^2}{N-1}} = 13.5\,s$$

$Stdev(\Delta t) = 13.5\,s$

Method 3: Standard Deviation of the Mean

$$\sqrt{\frac{\sum_n (Dev_n)^2}{(N-1)\cdot N}} = 4.3\,s$$

$$\frac{Stdev(\Delta t)}{\sqrt{N}} = 4.3\,s$$

A Timely Example of Errors in Measurements

What time is it now?

And the “best” answer is:

(53 ± 9) sec
[absolute error]

or

(53 sec ± 20%)
[relative (fractional) error]

This is the best value and estimated uncertainty for a set of N measurements of the time.

Data	**Reduced Data**	**Best Value**	**Discrepancy (Deviations)**	**Error**
9:10:35	9:10 + 35 sec	Average	-18 sec	Range of discrepancies
9:10:47	9:10 + 47 sec		- 6 sec	
9:11:06	9:10 + 66 sec	9:10 + 53 sec	+13 sec	
9:45:05	9:10 + 2710 sec		+2710 sec	
9:11:12	9:10 + 12 sec		+19 sec	Average RMS (Standard) Deviation
9:10:36	9:10 + 36 sec		+17 sec	
			+46	(_____ ± _____) sec
			+28 sec	
			+45 sec	For a set of N measurements
			+53 sec	
			+62 sec	
			+52 sec	(_____ ± _____) sec

Summary of Stating Errors in Measurements

The standard format to report the best guess and the limits within which you expect 68% of subsequent (single) measurements of t to fall within is:

1. Absolute error: ($\langle t \rangle \pm \sigma_t$) sec
2. Relative (fractional) Error: $\langle t \rangle$ sec $\pm$ ($\sigma_t/\langle t \rangle$)%

It can be shown that (see Taylor Sec. 5.4) the standard deviation σ_t is a reasonable estimate of the uncertainty.

In fact, for normal (Gaussian or purely random) data, it can be shown that

3. 68% of measurements of t will fall within $\langle t \rangle \pm \sigma_t$
4. 95% of measurements of t will fall within $\langle t \rangle \pm 2\sigma_t$
5. 98% of measurements of t will fall within $\langle t \rangle \pm 3\sigma_t$
6. 99.99994% (all but 0.6 ppm) of measurements of t will fall within $\langle t \rangle \pm 5\sigma_t$ (this is the high energy physics gold standard)
7. this is referred to as the confidence limit
8. If a confidence limit is not stated it usually means ONE standard deviation or 68% confidence limit

Significant Figures (see Taylor Sec. 2.2 and 2.8)

9. There is no need to write down unnecessary (unmeaningful) digits in your answer
10. Basic idea: write down only the digitis you know something about
11. Uncertainty dictates number of sig fig displayed in best guess
12. Errors are usually stated to within 1 sig fig.
13. See sheet for rules of thumb for sig figs
14. This is a pet peeve of mine

Correct

1.23±0.02 m (±2%)

Incorrect

1.23~~4567~~±0.02~~4~~ m

Intermediate Lab

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Accuracy and Systematic Errors

Random and Systematic Errors

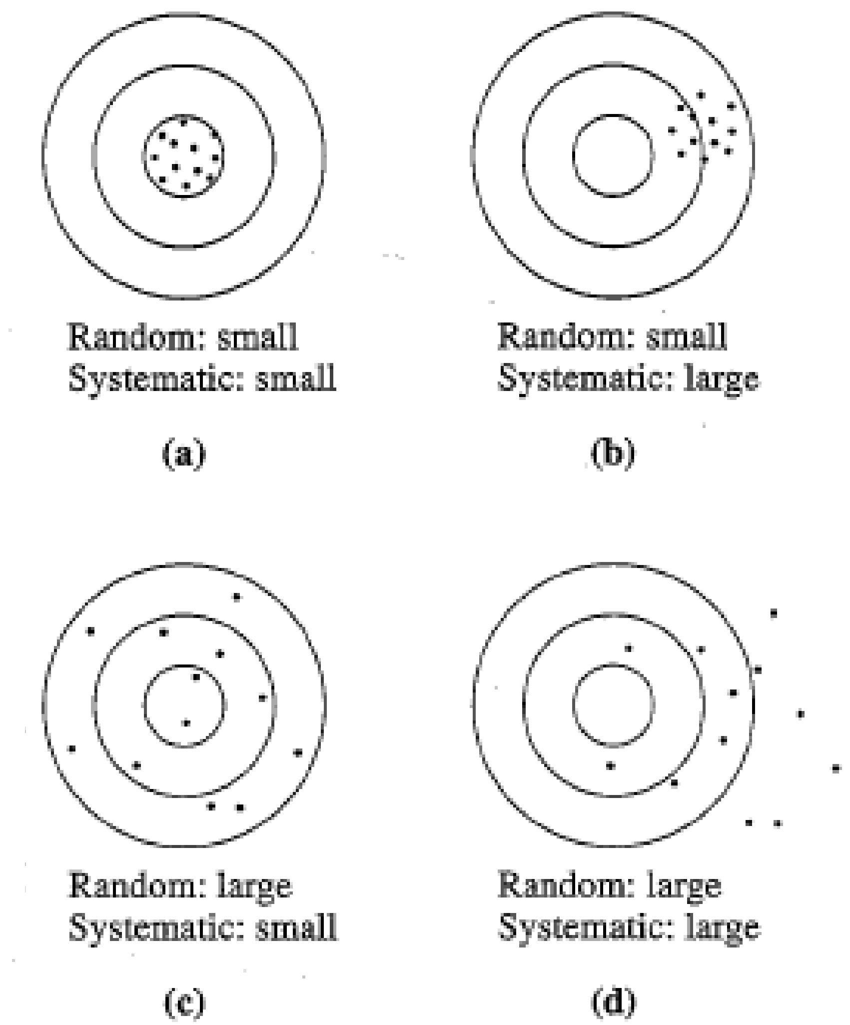
Precision is defined as a measure of the **reproducibility** of a measurement

Such errors are called random (statistical) errors.
If an experiment has small random error, it is said to have high precision.

Accuracy is a measure of the **validity** of a measurement.

If an experiment has small systematic error, it is said to have high accuracy.

Accuracy and Systematic Error

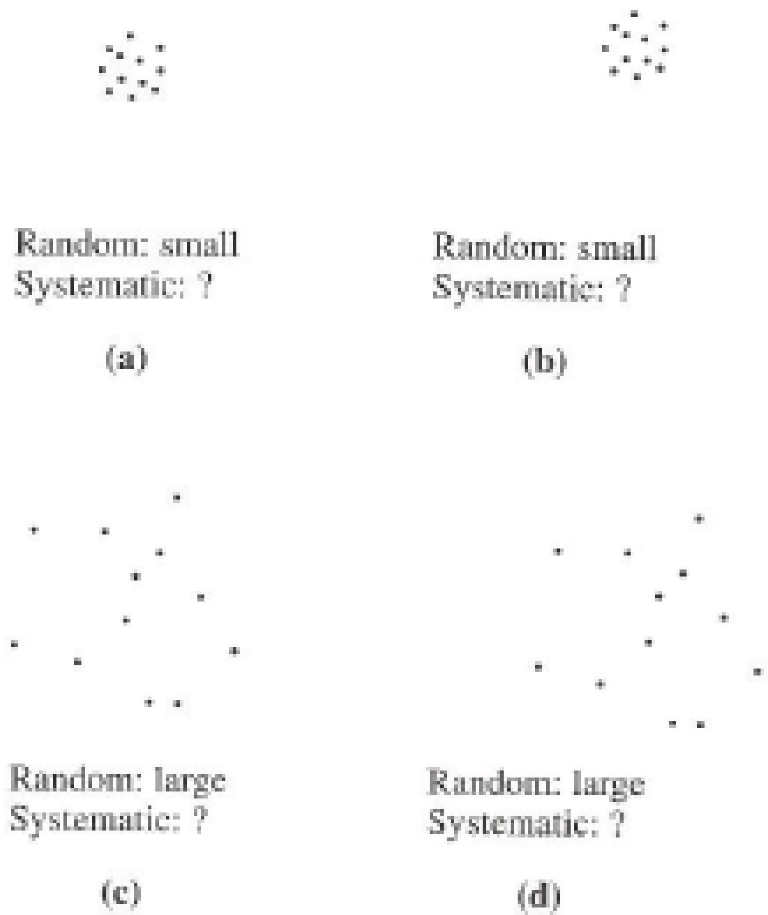


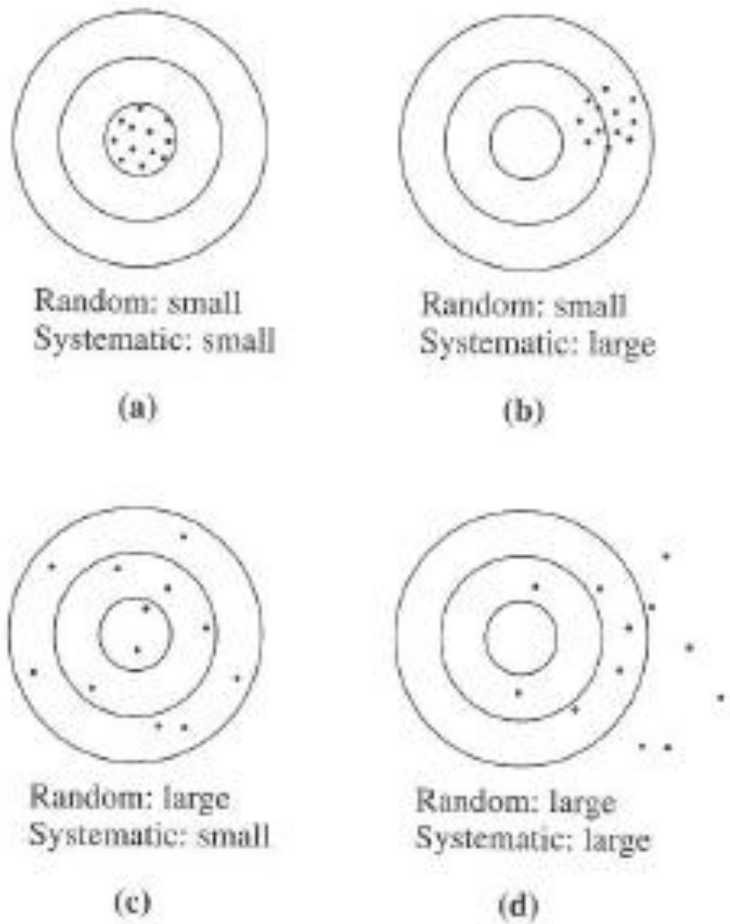
Consider 4 "dart experiments"

Which experiments are precise (have good reproducibility or low random error)?

Which experiments are accurate (are close to "true" result or have low systematic error)?

Precision and Accuracy





Accuracy and Systematic Error

A. Another (and different) question is how **accurately** do we know <t>, that is, how close is <t> to the **"true" value**

B. To be able to determine accuracy, we must know the "true" value or have access to an accurate measurement, e.g.,

   1. from my watch (calibrated before class via the NIST link) we know t to ±1 sec.
   2. from our computer via the NIST link, we know t to ±0.003 sec
   3. from the NIS atomic clock, we can know t to $\pm 1\text{x}10^{-12}$ sec
   4. from proposed NIST clocks, we will be able to know t to $\pm 1\text{x}10^{-18}$ s (see Scientific American September 2002, special issue on time keeping)

C. Using your uncalibrated watches lead to **systematic errors**, which affect the accuracy of a measurement

   1. Systematic errors – Errors which are characterized by their deterministic nature
   2. Another systematic error is the reaction time in responding to my verbal cue to read the time.

D. Repeated measurements can reduce random errors, but does not usually reduce systematic errors

E. Note systematic errors are not the same as **illegitimate errors** (blunders), e.g., writing down the time wrong is a blunder

A Timely Example of Errors in Measurements

What time is it?

nistime-32bit.exe

Log on to the National Institute of Standards and Technology (NIST) web site.

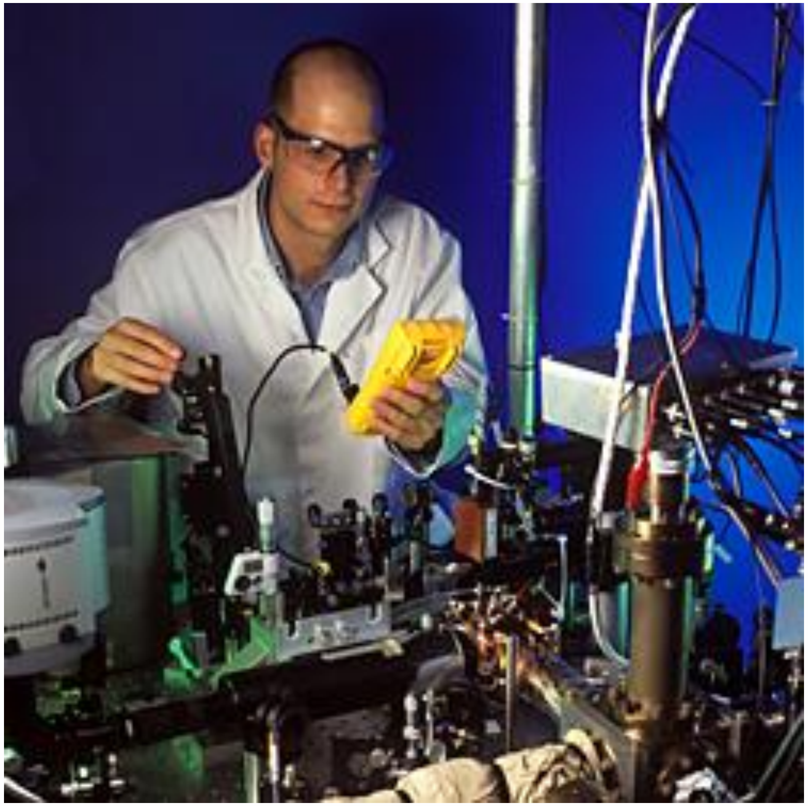
GOOD: A low resolution time stamp to 0.3 s http://nist.time.gov/

BETTER: “The NIST servers listen for a NTP request on port 123, and respond by sending a udp/ip data packet in the NTP format. The data packet includes a 64-bit (1 part in 2 10^{19}) timestamp containing the time in UTC seconds since Jan. 1, 1900 with a resolution of 200 ps. http://www.boulder.nist.gov/timefreq/service/its.htm

BEST: If that isn’t good enough, try these details on current research, with proposed precision of $\pm 1\text{x}10^{-18}$ s (±1 as!!!) (also see Sc. American, 2002 special issue on time keeping). http://www.boulder.nist.gov/timefreq/

A Timely Example of Errors in Measurements

What time is it?



EVEN BETTER: NIST's Second "Quantum Logic Clock" is World's Most Precise Clock

NIST scientists have built a second "quantum logic clock," using quantum information processing techniques on a single ion of aluminum to make a clock that would not gain or lose more than one second in about 3.7 billion years.
For more information, please see http://www.nist.gov/pml/div688/logicclock_020410.cfm .

Intermediate Lab

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Precision and Random (Statistical) Errors

Precision of Measurements

Our statement of the best value and uncertainty is: ($\langle t \rangle \pm \sigma_t$) sec

At the 68% confidence level for N measurements

1. Note the precision of our measurement is reflected in the estimated error which state what values we would expect to get if we repeated the measurement
2. Precision is defined as a measure of the reproducibility of a measurement
3. Such errors are called random (statistical) errors

Precision and Random (Statistical) Error

Precision of an instrument is (typically) determined by the finest increment of the measuring device. Sources of estimates of precision in direct measurements

1. Spread of repeated measurements (Taylor p.47)
2. Scales
   a) Digital are ±0.5 of LSD (Taylor p. 47)
   b) Analog are ±fraction of smallest division on instrument (Taylor p. 46)
   c) Verniers are ± smallest increment on vernier scale
3. Problem of definition (Taylor p. 46 and Fig. 3.1)
4. Square root of counts for timing (Taylor p. 48)
   a) $(\nu \pm \sqrt{\nu})$ = avg number of events in time T
   b) counting events that occur at random, but at a definite rate, e.g., decay of radioactive isotopes, spontaneous emission

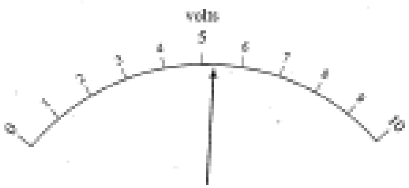
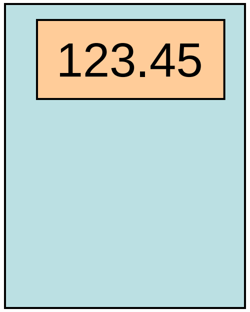


Figure 1.1. A reading on a voltmeter.

Figure 1.2. Measuring a length with a ruler.



1
2
3
...

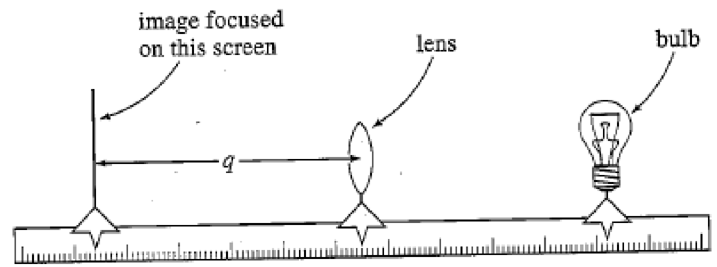
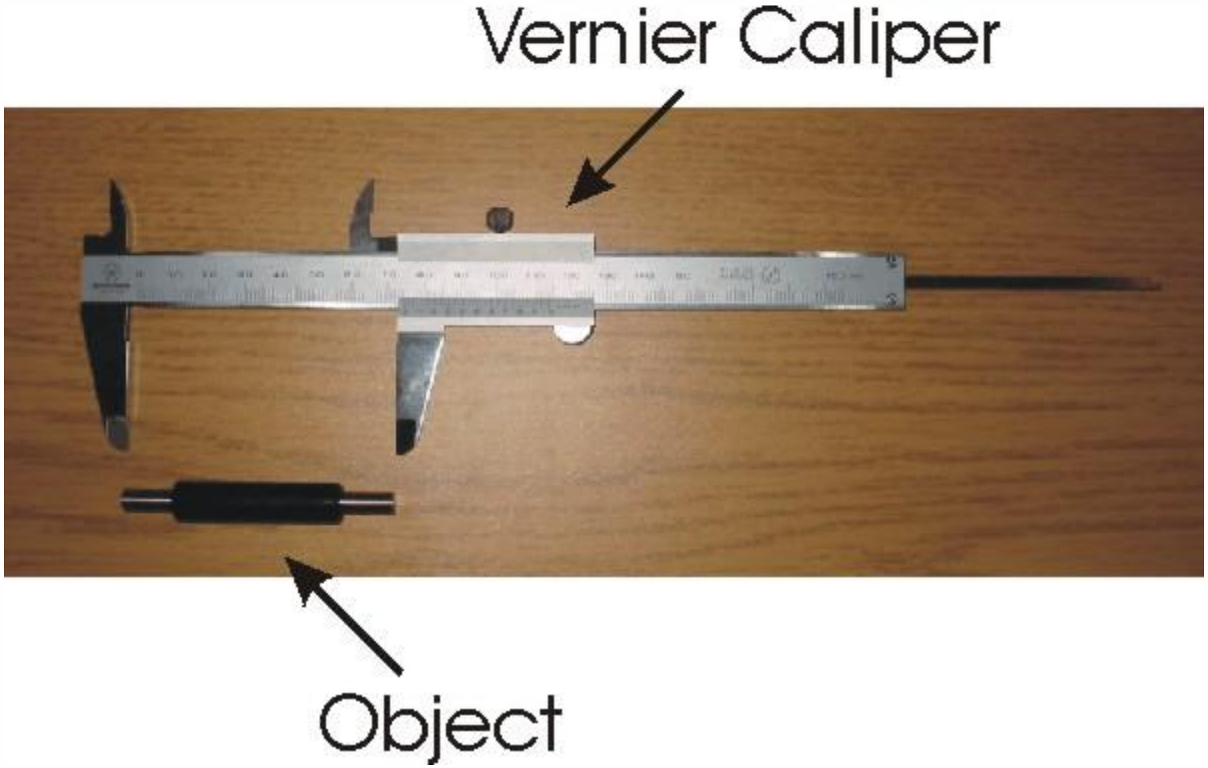
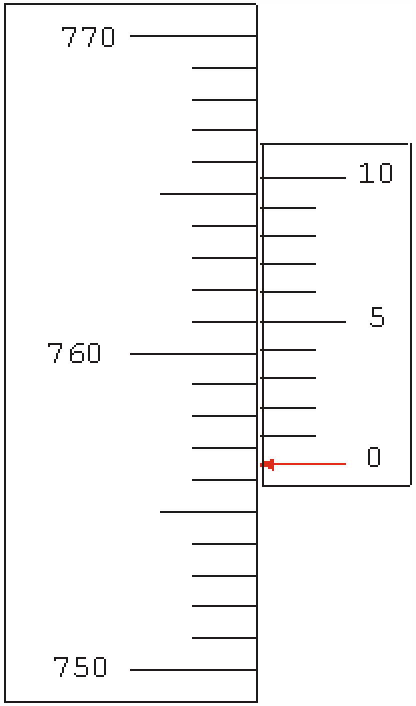


Figure 3.1. An image of the light bulb on the right is focused by the lens onto the screen at the left.

Reading a Vernier Scale

A *vernier scale* provides a way to gain added precision from an instrument scale





Intermediate Lab

PHYS 3870

The Ruler Exercise

The Ruler Exercise

A. Using the rulers provided, measure the width of the table.
B. Record, on the board, your "best" value and an estimate of the random error (the precision of your measurement)
C. Based on all the class' data, determine the best value and uncertainty of this length
D. Record this on the board
E. Based on the values on the board and comparison of the measuring devices discussed in class, discuss in class:

1. the precision of the measurements
2. the accuracy of the measurements
3. the sources of random errors in this exercise
4. the sources of systematic errors in this exercise
5. the illegitimate errors in this exercise

The Ruler Exercise

- Measure the width of the conference table.
- Record you measured value and associated error on the white board.

Name	Ruler Type	Measured Value	Uncertantity.

The Ruler Exercise

Enter reduced data: $N := 13$ $n := 0..(N-1)$

N is number of data points:

$\Delta t_n :=$

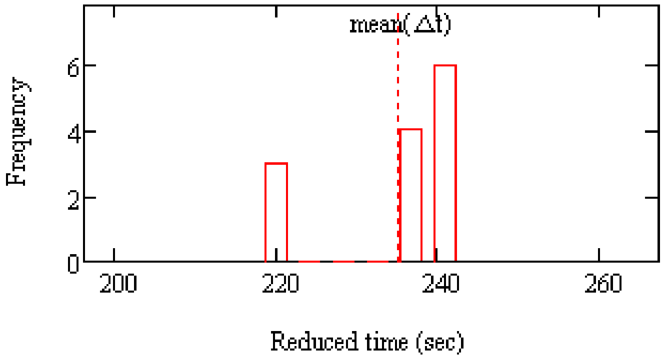
92.5·in	A
94·in	
94.625·in	
241 cm	B
237·cm	
240·cm	C
242·cm	
243·cm	
241.5·cm	
222·cm	D
236.4·cm	
218.5·cm	
221·cm	

$$\Delta t := \frac{\Delta t}{cm}$$

Find the "best guess":

Method 1: Center of histogram

Histogram of time data



Method 2: Average (mean) Value

Longhand : $\frac{1}{N} \cdot \sum_n \Delta t_n = 235.1$

Shorthand : $\text{mean}(\Delta t) = 235.1$

The Ruler Exercise

What is the uncertainty?

Minimum value: $\min(\Delta t) = 218.5$

Maimum value: $\max(\Delta t) = 243.0$

Calculate deviations from the mean $Dev := \Delta t - mean(\Delta t)$

$n =$	$\Delta t_n =$	$Dev_n =$
0.0	234.9	-0.2
1.0	238.8	3.6
2.0	240.3	5.2
3.0	241.0	5.9
4.0	237.0	1.9
5.0	240.0	4.9
6.0	242.0	6.9
7.0	243.0	7.9
8.0	241.5	6.4
9.0	222.0	-13.1
10.0	236.4	1.3
11.0	218.5	-16.6
12.0	221.0	-14.1

Method 1: Range of deviations

$mean(\Delta t) = 235.1$

\+ $\max(Dev) = 7.9$

\- $\min(Dev) = -16.6$

Method 2: Standard (RMS) Deviation

$$\sqrt{\frac{\sum_n (Dev_n)^2}{N-1}} = 8.7$$

$Stdev(\Delta t) = 8.7$

Method 3: Standard Deviation of the Mean

$$\sqrt{\frac{\sum_n (Dev_n)^2}{(N-1)\cdot N}} = 2.4$$

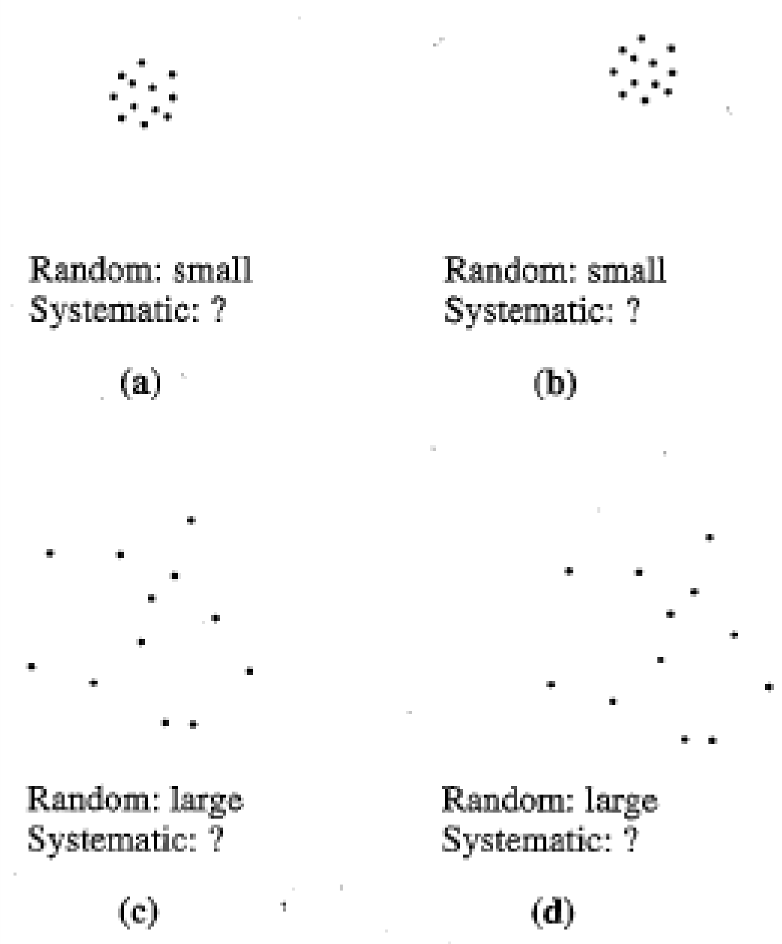
$$\frac{Stdev(\Delta t)}{\sqrt{N}} = 2.4$$

Precision, Accuracy and Systematic Errors

Precision is defined as a measure of the **reproducibility** of a measurement

Such errors are called **random (statistical) errors**. If an experiment has small random error, it is said to have high precision.

Accuracy is a measure of the **validity** of a measurement.
If an experiment has small **systematic error**, it is said to have high accuracy.



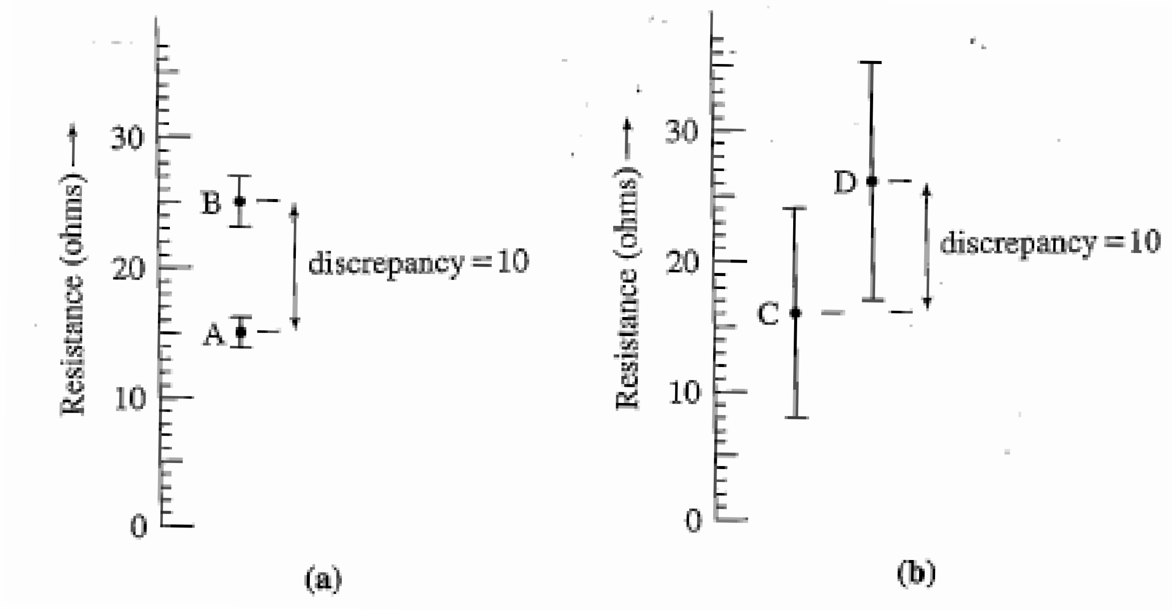
Intermediate Lab

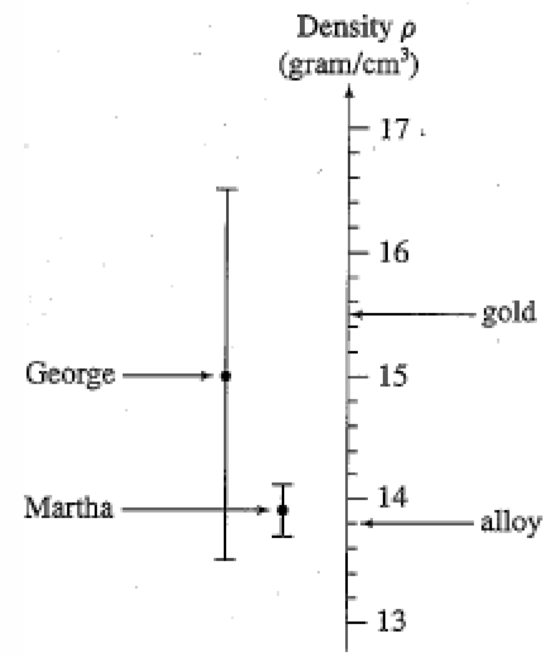
PHYS 3870

Comparing Measurements to Models

Quantitatively

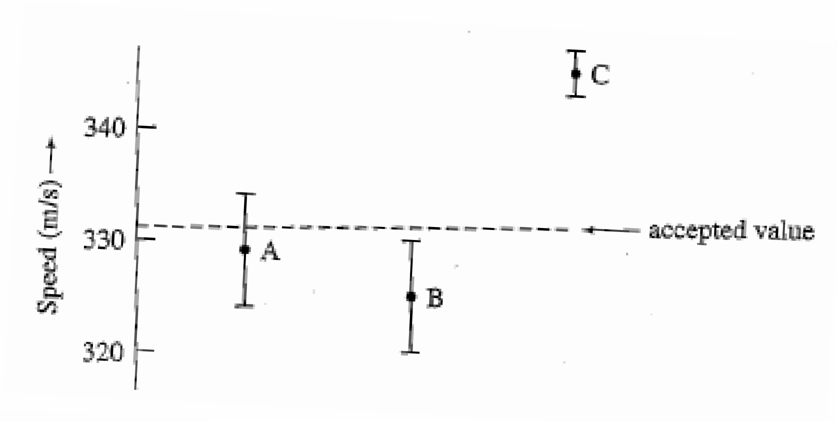
Comparison with Other Data

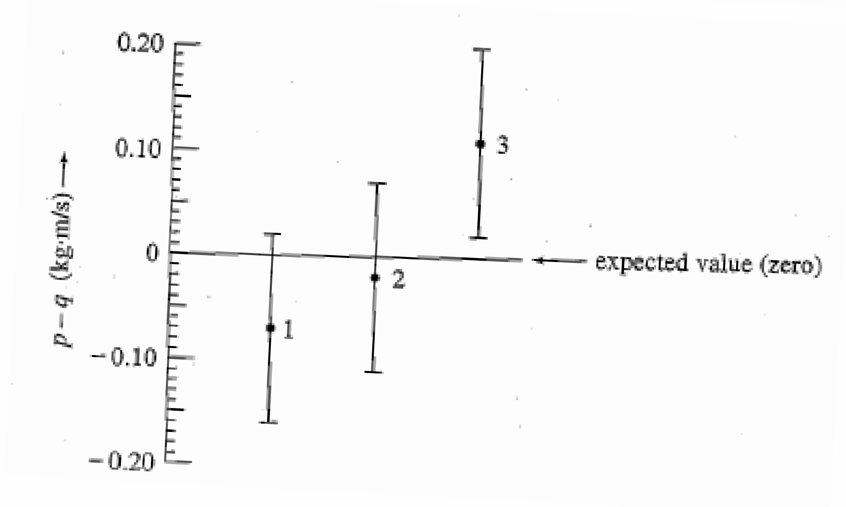


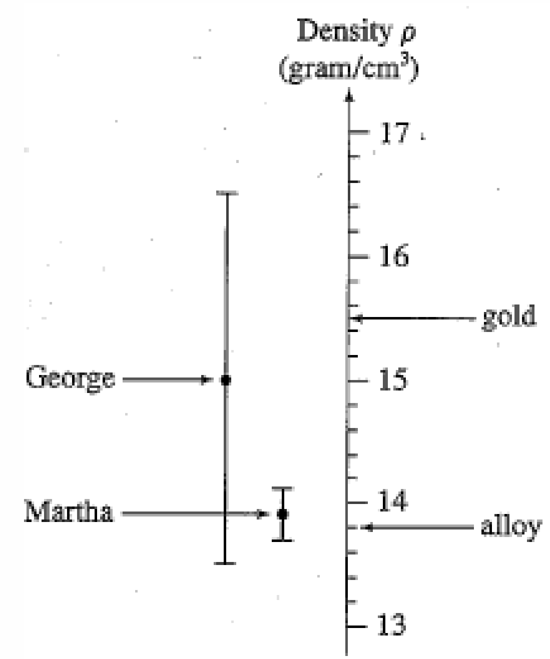


Is there agreement? (With what?)
Are these comparisons of precision or accuracy?

Direct Comparison with Standard







Is there agreement? (With what?)
Are these comparisons of precision or accuracy?

Intermediate Lab

PHYS 3870

Using Errors to Quantitatively Test Models

Basic Approach [Baird, Ch 4.1].

a. Know data and uncertainties (presumably)
b. Use this to identify system, inputs and outputs
c. Now develop a model
d. Then test model by comparison with data (first qualitatively, then quantitatively)

Testing a Model?

- Steps in a Scientific Investigation [Baird, Ch. 5-3]
    - Clearly Identify:
        - The problem or question or interaction to be addressed.
        - The system to study and its boundaries.
        - The significant variables in observation—key is to set up experiment with isolated input and output variable(s)
    - Develop a model of the system—key is to quantitatively describe interaction of inputs with system (see below).
    - Test the model through experimentation—key to designing experiment is whether data will allow quantitative evaluation of model for given input variable(s) and output variable(s) [see Baird, Ch. 5 on Experimental Design]
    - Evaluate the model as a description of the system—key is to know how good is "good enough" and how to test this quantitatively [see Baird Ch. 6 on Experiment Evaluation]
    - Refine the model to cover:
        - More precise measurements
        - More general conditions
- Basic approach to develop and evaluate the usefulness of a model [Baird, Ch 4.1].
    - Know data and uncertainties (presumably)
    - Use this to identify system, inputs and outputs
    - Now develop a model
    - Then test model by comparison with data (first qualitatively, then quantitatively)

Summary of Stating Errors in Measurements

The standard format to report the best guess and the limits within which you expect 68% of subsequent (single) measurements of t to fall within is:

1. Absolute error: $(\langle t \rangle \pm \sigma_t)$ sec
2. Relative (fractional) Error: $\langle t \rangle$ sec $\pm$ $(\sigma_t / \langle t \rangle)\%$

It can be shown that (see Taylor Sec. 5.4) the standard deviation σ_t is a reasonable estimate of the uncertainty.

In fact, for normal (Gaussian or purely random) data, it can be shown that

3. 68% of measurements of t will fall within $\langle t \rangle \pm \sigma_t$
4. 95% of measurements of t will fall within $\langle t \rangle \pm 2\sigma_t$
5. 98% of measurements of t will fall within $\langle t \rangle \pm 3\sigma_t$
6. 99.99994% (all but 0.6 ppm) of measurements of t will fall within $\langle t \rangle \pm 5\sigma_t$ (this is the high energy physics gold standard)
7. this is referred to as the confidence limit
8. If a confidence limit is not stated it usually means ONE standard deviation or 68% confidence limit

Significant Figures (see Taylor Sec. 2.2 and 2.8)

9. There is no need to write down unnecessary (unmeaningful) digits in your answer
10. Basic idea: write down only the digitis you know something about
11. Uncertainty dictates number of sig fig displayed in best guess
12. Errors are usually stated to within 1 sig fig.
13. See sheet for rules of thumb for sig figs
14. This is a pet peeve of mine

Correct

1.23±0.02 m (±2%)

Incorrect

1.23~~4567~~±0.02~~4~~ m

Quantifying Precision and Random (Statistical) Errors

The "best" value for a group of measurements of the same quantity is the

Average

What is an estimate of the random error?

Deviations

A. If the average is the the best guess, then DEVIATIONS (or discrepancies) from best guess are an estimate of error

B. One estimate of error is the range of deviations.

Standard Deviation

A better guess is the average deviation $t_{dev} = \frac{1}{N}\sum_{i=1}^{N}(t_i - \bar{t})$

…but one needs to calculate the average of the absolute value of deviations ….$\overline{t_{dev}} = \frac{1}{N}\sum_{i=1}^{N}|t_i - \bar{t}|$ (called the mean deviation) to avoid effects of positive and negative deviations canceling

…but it is easier to calculate (positive) square root of average of the square of the deviation

$$\sigma = \sqrt{\frac{1}{N}\sum_{i=1}^{N}(t_i - \bar{t})^2}$$

this is the (population) standard deviation

,,,actually, in most cases encountered in physics you should use N-1 not N (see Taylor Sec. 4.2) …$\sigma = \sqrt{\frac{1}{(N-1)}\sum_{i=1}^{N}(t_i - \bar{t})^2}$

this is the rms (root mean squared deviation or (sample) standard deviation

It can be shown that (see Taylor Sec. 5.4) σ_t is a reasonable estimate of the uncertainty. In fact, for normal (Gaussian or purely random) data, it can be shown that

(1) 68% of measurements of t will fall within <t> ± σ_t
(2) 95% of measurements of t will fall within <t> ± $2\sigma_t$
(3) 98% of measurements of t will fall within <t> ± $3\sigma_t$
(4) this is referred to as the confidence limit

Summary: the standard format to report the best guess and the limits within which you expect 68% of subsequent (single) measurements of t to fall within is <t> ± σ_t

Standard Deviation of the Mean

If we were to measure t again N times (not just once), we would be even more likely to find that the second average of N points would be close to <t>.

The standard error or standard deviation of the mean is given by…

$$\sigma_{SDOM} = \frac{\sigma_{SD}}{\sqrt{N}} = \sqrt{\frac{1}{N\,(N-1)}\,\sum_{i=1}^{N}(t_i - \bar{t})^2}$$

This is the limits within which you expect the average of N addition measurements to fall within at the 68% confidence limit

Errors in Models—Error Propagation

Define error propagation [Taylor, p. 45]

A method for determining the error inherent in a derived quantity from the errors in the measured quantities used to determine the derived quantity.

That is, the errors associated with a mathematical model of a dependant variable in terms of independent variables.

Recall previous discussions [Taylor, p. 28-29]

I. Absolute error: $(\langle t \rangle \pm \sigma_t)$ sec

II. Relative (fractional) Error: $\langle t \rangle$ sec $\pm$ $(\sigma_t / \langle t \rangle)\%$

III. Percentage uncertainty: fractional error in % units

Specific Rules for Error Propogation

Refer to **[Taylor, sec. 3.2]** for specific rules of error propagation:

1. Addition and Subtraction **[Taylor, p. 49]**
   For $q_{best}=x_{best}\pm y_{best}$ the error is $\delta q \approx \delta x+\delta y$
   Follows from $q_{best}\pm\delta q =(x_{best}\pm \delta x) \pm(y_{best} \pm\delta y)= (x_{best}\pm y_{best}) \pm(\delta x \pm\delta y)$

2. Multiplication and Division **[Taylor, p. 51]**
   For $q_{best}=x_{best} * y_{best}$ the error is $(\delta q/ q_{best}) \approx (\delta x/x_{best})+(\delta y/y_{best})$

3. Multiplication by a constant (exact number) **[Taylor, p. 54]**
   For $q_{best}= B(x_{best})$ the error is $(\delta q/ q_{best}) \approx |B| (\delta x/x_{best})$
   Follows from 2 by setting $\delta B/B=0$

4. Exponentiation (powers) **[Taylor, p. 56]**
   For $q_{best}= (x_{best})^n$ the error is $(\delta q/ q_{best}) \approx n (\delta x/x_{best})$
   Follows from 2 by setting $(\delta x/x_{best})=(\delta y/y_{best})$

Independent Uncertainties

Independent Uncertainties see **[Taylor, Secs. 3.3 and 3.4]**

A. Method A: General estimate of uncertainty (worst case):
  1. for addition and subtraction absolute errors add
  2. for multiplication and division fractional errors add

B. Method B: When original uncertainties are independent and random:
  1. for addition and subtraction absolute errors add in quadrature
     $\delta q = \sqrt{[(\delta x)^2 + (\delta y)^2]}$
  2. for multiplication and division fractional errors add in quadrature
     $\delta q/x = \sqrt{[(\delta x/x)^2 + (\delta y/y)^2]}$
  3. Note: these errors are less than rule A
  4. Given this, we must define
     a) Random
     b) Independent
     c) Addition in quadrature
  5. Note: proof of statement B is left to Ch. 5

Specific Rules for Independent Error Propagation

Easy to see why A leads to an overestimate:

1. Consider:
   a) $x \pm \delta x$ (to within 50% confidence limit)
   b) $y \pm \delta y$ (to within 50% confidence limit)
2. Thus,
   a) there is only a 25% chance for $x_{measured} > x \pm \delta x$
   b) there is only a 25% chance for $y_{measured} > y \pm \delta y$
3. Then there is only a (25%)·(25%) = 6% chance that
   $q_{calc} = x_{measured} + y_{measured} > x + y + \delta x + \delta y$

4. Thus, if x and y are:
   a) Independent (determining x does not affect measured y)
   b) Random (equally likely for $+\delta x$ as $-\delta x$)
   Then method A overestimates error

Independent (Random) Uncertaities and Gaussian Distributions

For Gaussian distribution of measured values which describe quantities with random uncertainties, it can be shown that (the dreaded ICBST), errors add in quadrature [see Taylor, Ch. 5]

$\delta q \neq \delta x + \delta y$

But, $\delta q = \sqrt{[(\delta x)^2 + (\delta y)^2]}$

1. This is proved in [Taylor, Ch. 5]
2. ICBST [Taylor, Ch. 9] Method A provides an upper bound on the possible errors

General Formula for Error Propagation

General formula for error propagation see [Taylor, Secs. 3.5 and 3.9]

Uncertainty as a function of one variable [Taylor, Sec. 3.5]

1. Consider a graphical method of estimating error
   a) Consider an arbitaray function q(x)
   b) Plot q(x) vs. x.
   c) On the graph, label:
      (1) $q_{best} = q(x_{best})$
      (2) $q_{hi} = q(x_{best} + \delta x)$
      (3) $q_{low} = q(x_{best} - \delta x)$
   d) Making a linear approximation:

$$q_{hi} = q_{best} + slope \cdot \delta x = q_{best} + \left(\frac{\partial q}{\partial x}\right) \cdot \delta x$$

$$q_{low} = q_{best} - slope \cdot \delta x = q_{best} - \left(\frac{\partial q}{\partial x}\right) \cdot \delta x$$

   e) Therefore:

$$\delta q = \left|\frac{\partial q}{\partial x}\right| \cdot \delta x$$

      Note the absolute value.

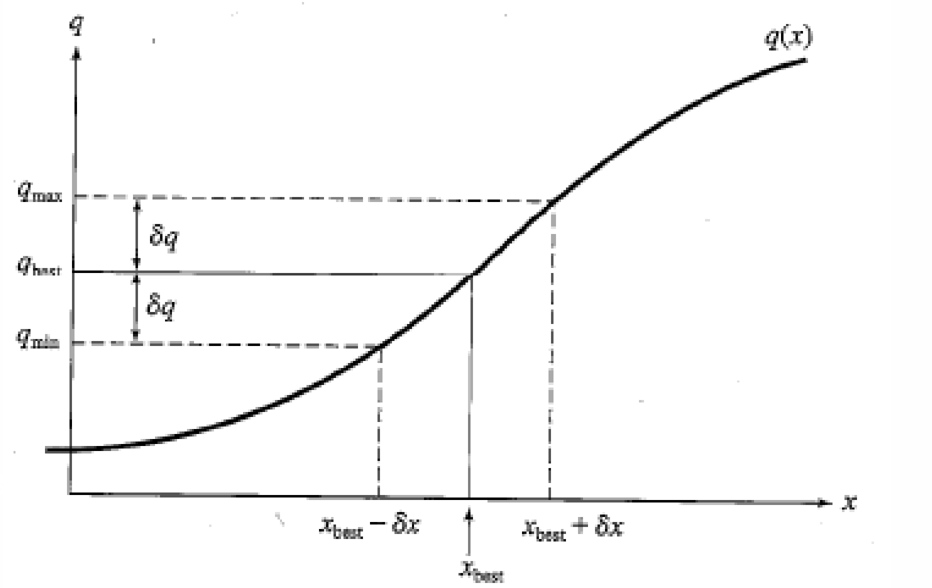


Figure 3.3. Graph of $q(x)$ vs x. If x is measured as $x_{best} \pm \delta x$, then the best estimate for $q(x)$ is $q_{best} = q(x_{best})$. The largest and smallest probable values of $q(x)$ correspond to the values $x_{best} \pm \delta x$ of x.

General Formula for Error Propagation

General formula for uncertainty of a function of one variable

$$\delta q = \left|\frac{\partial q}{\partial x}\right| \cdot \delta x$$ [Taylor, Eq. 3.23]

Can you now derive for specific rules of error propagation:

1. Addition and Subtraction **[Taylor, p. 49]**
2. Multiplication and Division **[Taylor, p. 51]**
3. Multiplication by a constant (exact number) **[Taylor, p. 54]**
4. Exponentiation (powers) **[Taylor, p. 56]**

A more complicated example: Bragg's Law

$$\lambda(d,\theta) = d \cdot \sin(\theta)$$

$$\frac{\delta\lambda}{\lambda} = \frac{\delta d}{d} + \frac{\delta[\sin(\theta)]}{\sin(\theta)}$$

$$\frac{\delta\lambda}{\lambda} = \frac{\delta d}{d} + \frac{\cos(\theta)}{\sin(\theta)} \cdot \delta\theta \, ; \quad \ldots \delta[\sin(\theta)] = {}^{d}\!/_{d\theta}[\sin(\theta)] \cdot \delta\theta = \cos(\theta) \cdot \delta\theta$$

$$\frac{\delta\lambda}{\lambda} = \frac{\delta d}{d} + \cot(\theta) \cdot \delta\theta$$

General Formula for Multiple Variables

Uncertainty of a function of multiple variables [Taylor, Sec. 3.11]

1. It can easily (no, really) be shown that (see Taylor Sec. 3.11) for a function of several variables

$$\delta q(x, y, z, \ldots) = \left|\frac{\partial q}{\partial x}\right| \cdot \delta x + \left|\frac{\partial q}{\partial y}\right| \cdot \delta y + \left|\frac{\partial q}{\partial z}\right| \cdot \delta z + \ldots$$ [Taylor, Eq. 3.47]

2. More correctly, it can be shown that (see Taylor Sec. 3.11) for a function of several variables

$$\delta q(x, y, z, \ldots) \le \left|\frac{\partial q}{\partial x}\right| \cdot \delta x + \left|\frac{\partial q}{\partial y}\right| \cdot \delta y + \left|\frac{\partial q}{\partial z}\right| \cdot \delta z + \ldots$$ [Taylor, Eq. 3.47]

where the equals sign represents an upper bound, as discussed above.

3. For a function of several *independent and random* variables

$$\delta q(x, y, z, \ldots) = \sqrt{\left(\frac{\partial q}{\partial x} \cdot \delta x\right)^2 + \left(\frac{\partial q}{\partial y} \cdot \delta y\right)^2 + \left(\frac{\partial q}{\partial z} \cdot \delta z\right)^2 + \ldots}$$ [Taylor, Eq. 3.48]

Again, the proof is left for Ch. 5.

A Complex Example for Multiple Variables