Problem 10.5

Spherical coordinates

(a) Show $[\hat{r}, \hat{p}_r] = i\hbar$

$$\hat{p}_r = -i\hbar \frac{1}{r}\frac{\partial}{\partial r} r$$

so $$[\hat{r}, \hat{p}_r]\psi = +\frac{\hbar}{i}\left[r\frac{1}{r}\frac{\partial}{\partial r}\cdot\psi - \frac{1}{r}\frac{\partial}{\partial r}(r\psi)\right]$$

$$= \frac{\hbar}{i}\left[\frac{\partial\psi}{\partial r} - \frac{\partial\psi}{\partial r} - \psi\right]$$

so $$[\hat{r}, \hat{p}_r] = -\frac{\hbar}{i}$$

(b) What properties of φ and ψ insure that $\langle\varphi|\hat{p}_r\psi\rangle = \langle\hat{p}_r\varphi|\psi\rangle$?

$$\int \varphi^* \frac{\hbar}{+i}\frac{1}{r}\frac{d}{dr}(r\psi)\, d\tau \stackrel{?}{=} \int \frac{\hbar}{-i}\frac{1}{r}\frac{d}{dr}(r\varphi^*)\,\psi\, d\tau \quad \text{where } d\tau = r^2 d\Omega\, dr$$

This simplifies to

$$\int \varphi^* \frac{1}{r}\frac{d(r\psi)}{dr} r^2 dr = -\int \frac{1}{r}\frac{d}{dr}(r\varphi^*)\,\psi\, r^2 dr$$

so $$\int r\varphi^* \frac{d}{dr}(r\psi)\, dr = -\int \frac{d}{dr}(r\varphi^*)(r\psi)\, dr$$

let $u = r\varphi$ and $v = r\psi$ so integration by parts gives

$$\int u\,dv = uv - \int v\,du \quad \text{corresponds to the above. if } uv\Big|_0^\infty = 0$$

$$uv\Big|_0^\infty = r^2\varphi\psi\Big|_0^\infty$$

This will be zero at $r=0$ if $\varphi\psi \to \infty$ slower than r^2 at $r\to 0$

We also require $\varphi\psi \to 0$ faster than $1/r^2$ as $r\to\infty$,

Prob 10.7

Show $\hat{T} = -\frac{\hbar^2}{2m}\nabla^2$ is Hermitian

Hermitian $\langle f|\hat{T}g\rangle = \langle \hat{T}f|g\rangle$ neglect $-\frac{\hbar^2}{2m}$

or $\int f^*\nabla^2 g - \int g\nabla^2 f^* \stackrel{?}{=} 0$ By greens theorem

$$= \int_S (f\vec{\nabla}g - g\vec{\nabla}f)\cdot d\vec{s}$$

Because the functions must go to zero $r \to \infty$ in order that $\int f^* f < \infty$ as in ψ_2 the surface terms vanish.

Prob 10.9

$\psi(r,t) = j_2(kr)\, e^{-i\omega t}$ $\hbar\omega = \hbar^2k^2/2m$ get $\vec{\nabla}\cdot\vec{j}$

$\vec{\nabla}\cdot\vec{J} = \frac{\hbar}{2mi}(\psi^*\nabla^2\psi - \psi\nabla^2\psi^*)$

$\vec{J} = \frac{\hbar}{2mi}(\psi^*\nabla\psi - \psi\nabla\psi^*) = 0$

use $\nabla\cdot J = -\frac{\partial\rho}{\partial t}$

$\Rightarrow \nabla^2\psi = e^{-i\omega t}\nabla^2 j_2(kr)$ Answer not in book

so ans = 0

Prob 10.10

$\psi(r,0) = 4\pi i\left[\frac{\sin kr}{(kr)^2} - \frac{\cos kr}{kr}\right]\left(3Y_1^0(\theta,\varphi) + 5Y_1^{-1}(\theta,\varphi)\right)/\sqrt{34}$

(a) $\psi(r,t) = \exp\left(-\frac{it\hat{H}}{\hbar}\right)\psi(r,0)$

$\varphi_{klm} = j_l(kr)\, Y_l^m(\theta,\varphi)$ $\frac{\sin kr}{(kr)^2} - \frac{\cos kr}{kr} = j_1(kr)$

so $\psi(r,0) = 4\pi i\, j_1(kr)\left(3Y_1^0(\theta,\varphi) + 5Y_1^{-1}(\theta,\varphi)\right)/\sqrt{34}$ so $l = 1$

this is a degenerate eigenstate of the energy therefore

$\psi(r,t) = \psi(r,0)\, e^{-iEt/\hbar}$ where $E = \hbar^2k^2/2m$

(b) $\langle E\rangle = \hbar^2k^2/2m$

(c) $L^2 = l(l+1)\hbar^2$ so L^2 will yield only $2\hbar^2$

L_z will yield 0; $\frac{P(0)}{P(\text{total})} = \left(\frac{9}{34}\right)$ and $-1\hbar$ $\frac{P(-1)}{P(\text{total})} = \frac{25}{34}$

(d) $t=0$ measurement of L^2 yields $2\hbar^2$ $\psi(\vec{r},t)$ is same as before

(e) $t=0$ " " L_z finds $-\hbar$

then $\psi(r,t) = 4\pi i\, j_1(kr)\left[\frac{\sin kr}{(kr)^2} - \frac{\cos kr}{kr}\right]Y_1^{-1}(\theta,\varphi)\, e^{-iEt/\hbar}$

Problem 10.18 was 17

$\vec{F} = \frac{e}{c}\vec{v}\times\vec{B}$

(a) show $\frac{1}{2}mv^2 = \text{const.}$ with $\vec{B} = (0, 0, B)$

~~$F = \frac{d}{dt}(m\vec{v})$~~

$T = \int F\,dt = \int$ $\qquad F = -\nabla U$

~~$\frac{1}{2}mv^2 = m\int v\,dv$~~ ~~$\frac{d(mv)}{dt} = \frac{e}{c}\vec{v}\times\vec{B}$~~ ~~$mv = \int \frac{e}{c} v\times B\,dt$~~

~~so $\frac{1}{2}mv^2 = \int\int \frac{e}{c}(v\times B)\cdot dv\,dt$~~

$dT = d\left(\frac{1}{2}mv^2\right) = m\,v\cdot dv$

$B = B\hat{k}$ $\qquad v\times B = \begin{vmatrix} i & j & k \\ v_x & v_y & v_z \\ 0 & 0 & B \end{vmatrix} = (\hat{i}\,v_y - v_x\hat{j})\,B$

$\frac{d(mv)}{dt} = \frac{eB}{c}(\hat{i}\,v_y - \hat{j}\,v_x)$ $\qquad d(mv) = \frac{eB}{c}(\hat{i}\,v_y - j\,v_x)\,dt$

$dT = m\vec{v}\cdot\left[\frac{eB}{c}(\hat{i}\,v_y - j\,v_x)\right]dt = \frac{meB}{c}(v_x v_y - v_y v_x)\,dt = 0$

so T constant.

(b) Show $p_z = mv_z = \text{const.}$

$\frac{d(mv_z)}{dt}$ from above $= 0$. $\therefore$ $p_z = \text{const.}$

(c) Since $\vec{F}\perp\vec{v}$ and $v^2 = \text{const}$ motion is circle and $|F| = \frac{mv^2}{r}$ where r = radius of curvature so $\frac{mv^2}{r} = \frac{evB}{c}$ $\qquad r = \frac{mcv}{eB}$

Period $T = \frac{2\pi r}{v} = \frac{2\pi mcv}{eBv}$ so freq $= \frac{eB}{2\pi mc}$ and ang freq $\frac{eB}{mc}$.

(d) The center of the circle depends upon initial conditions. It could just as well be at (0,0) although for the case selected in section 10.4 that wouldn't be true.

Problem 10.19 was 18

(a) $x_0 = \hat{x} + \frac{cp_y}{eB}$ $\qquad \hat{H} = \left[\frac{1}{2m}\left(p_x + \frac{eyB}{c}\right)^2 + \frac{p_y^2}{2m} + \frac{p_z^2}{2m}\right]$

Consider $\left[\hat{x}_0, \hat{H}\right] = \left[x_0, \frac{1}{2m}\left(p_x + \frac{eyB}{c}\right)^2\right] + \left[x_0, \frac{p_y^2}{2m} + \frac{p_z^2}{2m}\right]$

now $\left[x_0, \frac{p_y^2}{2m} + \frac{p_z^2}{2m}\right] = 0$ as is evident because there is no y or z coordinates and no x momentum.

now $\frac{1}{2m}\left[x_0, \left(p_x + \frac{eyB}{c}\right)^2\right] = \frac{1}{2m}\left(p_x + \frac{eyB}{c}\right)\left[x_0, p_x + \frac{eyB}{c}\right] + \frac{1}{2m}\left[x_0, p_x + \frac{eyB}{c}\right]\left(p_x + \frac{eyB}{c}\right)$

because $[AB, C] = A[B, C] + [A, C]B$

$* = \left[x_0, p_x + \frac{eyB}{c}\right] = \left[x + \frac{cp_y}{eB}, p_x + \frac{eyB}{c}\right] = \left[x, p_x\right] + 0 + 0 + \left[\frac{cp_y}{eB} + \frac{eyB}{c}\right]$

$= [x, p_x] + [p_y, y] = i\hbar - i\hbar = 0$

$\therefore \left[x_0, \hat{H}\right] = 0$

(b) Consider $\left[\hat{x}_0, \hat{y}_0\right]$ where $\hat{y}_0 = -\frac{cp_x}{eB}$

$\left[x_0, y_0\right] = \left[x + \frac{cp_y}{eB}, -\frac{cp_x}{eB}\right] = \left[x, -\frac{cp_x}{eB}\right] + 0$

$= -\frac{c}{eB} i\hbar \neq 0$

Problem 10.19 was 18

$$\hat{X}_0 = \hat{x} + \frac{c\hat{p}_y}{eB}$$

Show $[\hat{X}_0, H] = 0$ where $\hat{H} = \left[\frac{1}{2m}\left(p_x + \frac{eyB}{c}\right)^2 + \frac{p_y^2}{2m} + \frac{p_z^2}{2m}\right]$

$$\left[X_0, H\right] = \left[\left(\hat{x} + \frac{c\hat{p}_y}{eB}\right), \left[\frac{1}{2m}\left(p_x + \frac{eyB}{c}\right)^2 + \frac{p_y^2}{2m} + \frac{p_z^2}{2m}\right]\right]$$

$$= \frac{1}{2m}\left[\hat{x}, \left(p_x + \frac{eyB}{c}\right)^2\right] + \left[\hat{x}, \frac{p_y^2}{2m} + \frac{p_z^2}{2m}\right] + \left[\frac{c\hat{p}_y}{eB}, \frac{1}{2m}\left(p_x + \frac{eyB}{c}\right)^2\right] + \left[\frac{cp_y}{eB}, \frac{p_y^2}{2m} + \frac{p_z^2}{2m}\right]$$

the second + fourth commutators are clearly zero

Sorry —
I copied this over,
with a better derivation,
on another sheet &
can't find it now.
WTE

Problem 10.23

$\vec{r} = \vec{r}_1 - \vec{r}_2$ $\qquad \vec{P} = \vec{p}_1 + \vec{p}_2$

$\vec{p} = \dfrac{m_2\vec{p}_1 - m_1\vec{p}_2}{m_1+m_2}$ $\qquad \vec{R} = \dfrac{m_1\vec{r}_1 + m_2\vec{r}_2}{m_1+m_2}$

Show

(a) $\dfrac{p_1^2}{2m_1} + \dfrac{p_2^2}{2m_2} = \dfrac{p^2}{2\mu} + \dfrac{P^2}{2M}$ (Kinetic Energy)

$\dfrac{P^2}{2M} + \dfrac{p^2}{2\mu} = \dfrac{(\vec{p}_1+\vec{p}_2)^2}{2M} + \dfrac{(m_2\vec{p}_1 - m_1\vec{p}_2)^2}{(m_1+m_2)^2 \cdot 2\mu}$ $\qquad M = m_1+m_2$ $\qquad \mu = \dfrac{m_1m_2}{m_1+m_2}$

$= \dfrac{p_1^2 + p_2^2 + 2\vec{p}_1\cdot\vec{p}_2}{2(m_1+m_2)} + \dfrac{(m_2^2\vec{p}_1^2 - 2m_1m_2\vec{p}_1\cdot\vec{p}_2 + m_1^2p_2^2)}{2m_1m_2(m_1+m_2)}$

$= \dfrac{1}{2(m_1+m_2)}\left[p_1^2\left(1+\dfrac{m_2}{m_1}\right) + p_2^2\left(1+\dfrac{m_1}{m_2}\right) + 2\vec{p}_1\cdot\vec{p}_2(1-1)\right]$

$= \dfrac{1}{2(m_1+m_2)}\left[p_1^2\left(\dfrac{m_1+m_2}{m_1}\right) + p_2^2\left(\dfrac{m_2+m_1}{m_2}\right)\right]$ ✓

(b) Show $m_1\vec{r}_1^2 + m_2\vec{r}_2^2 = \mu r^2 + MR^2$ error in book not true corrected in next edition

(c) $\vec{p}_1\cdot\vec{r}_1 + \vec{p}_2\cdot\vec{r}_2 = \vec{p}\cdot\vec{r} + \vec{P}\cdot\vec{R}$

$\vec{p}\cdot\vec{r} + \vec{P}\cdot\vec{R} = \dfrac{m_2\vec{p}_1 - m_1\vec{p}_2}{m_1+m_2}\cdot\vec{r}_1 - \vec{r}_2 + (\vec{p}_1+\vec{p}_2)\cdot\left(\dfrac{m_1\vec{r}_1 + m_2\vec{r}_2}{m_1+m_2}\right)$

$= \dfrac{1}{m_1+m_2}\left[m_2\vec{p}_1\cdot\vec{r}_1 - m_1\vec{p}_2\cdot\vec{r}_1 - m_2\vec{p}_1\cdot\vec{r}_2 + m_1\vec{p}_2\cdot\vec{r}_2 + m_1\vec{p}_1\cdot\vec{r}_1 + m_2\vec{p}_1\cdot\vec{r}_2 + m_1\vec{p}_2\cdot\vec{r}_1 + m_2\vec{p}_2\cdot\vec{r}_2\right]$

$= \dfrac{1}{m_1+m_2}\left(\vec{p}_1\cdot\vec{r}_1(m_2+m_1) + \vec{p}_2\cdot\vec{r}_2(m_1+m_2)\right) = \vec{p}_1\cdot\vec{r}_1 + \vec{p}_2\cdot\vec{r}_2$

(d) $\vec{L}_1 + \vec{L}_2 = \vec{L} + \vec{L}_{cm}$

Problem 10.54

Show that u_{10}^2 has maximum at $r = a_0$

$$u_{10} = \frac{2r}{a_0^{3/2}} e^{-r/a_0}$$

$$u_{10}^2 = \frac{4r^2}{a_0^{3/2}} e^{-2r/a_0}$$

$$\frac{d}{dr}(u_{10}^2) = 0 = \frac{4}{a_0^{3/2}} \left[2re^{-r/a_0} + r^2\left(-\frac{1}{a_0}\right) e^{-r/a_0} \right]$$

$$2r - \frac{2r^2}{a_0} = 0 \qquad r = a_0$$

Problem 10.38

$$\psi(r,0) = \frac{4}{(2a_0)^{3/2}}\left[e^{-r/a_0} + A\frac{r}{a_0}e^{-r/2a_0}\left(-2Y_1^1 + Y_1^{-1} + \sqrt{7}\,Y_1^0\right)\right]$$

$$Y_0^0 = \frac{1}{\sqrt{4\pi}} \quad (p\ 333)$$

$$e^{-r/a_0} = \frac{a_0^{3/2}}{2Y_0^0}\varphi_{100} = \frac{a_0^{3/2}\sqrt{4\pi}}{2}\varphi_{100} = \sqrt{\pi}\,a_0^{3/2}\varphi_{100}$$

$$\frac{r}{a_0}e^{-r/2a_0}Y_1^1 = \sqrt{3}(2a_0)^{3/2}\varphi_{211}$$

$$\frac{r}{a_0}e^{-r/2a_0}Y_1^{-1} = \sqrt{3}(2a_0)^{3/2}\varphi_{21-1} \qquad \frac{r}{a_0}e^{-r/2a_0}Y_1^0 = \sqrt{3}(2a_0)^{3/2}\varphi_{210}$$

$$\psi(r,0) = \frac{4}{(2a_0)^{3/2}}\left[a_0^{3/2}\sqrt{\pi}\,\varphi_{100} + A\sqrt{3}(2a_0)^{3/2}\left(-2\varphi_{211} + \varphi_{21-1} + \sqrt{7}\,\varphi_{210}\right)\right]$$

$$= 4\left[\sqrt{\pi}\,\varphi_{100} + A\sqrt{3}\left(-2\varphi_{211} + \varphi_{21-1} + \sqrt{7}\,\varphi_{210}\right)\right]$$

Normalization is such that $\int \varphi^*\varphi\, d\Omega\, r^2 dr = \int_0^\infty \varphi^*\varphi\, 4\pi r^2 dr = 1$

so, for example $\int \varphi_{100}^2\, 4\pi r^2 dr = 1 \qquad \int_0 \frac{4\pi r^2}{\pi a_0^3} e^{-r/a_0}\, dr$

(d) $P_r(r) = 4\pi r^2 |\psi(r,0)|^2$

(e) $\psi(r,t) = \psi(r,0)\, e^{-iE_n t/\hbar}$ each term has $e^{-iE_n t/\hbar}$ in front

Problem 10.38 $\psi(r,0) = \frac{4}{(2a_0)^{3/2}}\left[\frac{e^{-r/a_0}}{\sqrt{4\pi}} + A\frac{r}{a_0}e^{-r/2a_0}\left(-\sqrt{2}\,Y_1^1 + Y_1^{-1} + \sqrt{2}\,Y_1^0\right)\right]$

(a) Determine A.

If $\psi(r,0) = a\varphi_{100} + b\varphi_{211} + c\varphi_{21-1} + d\varphi_{210}$ then

normalization requires $a^2+b^2+c^2+d^2 = 1$ if φ_{nlm} normalized.

From Book. $\varphi_{100} = \frac{2}{a_0^{3/2}}e^{-r/a_0}Y_0^0$ $\quad Y_0^0 = \frac{1}{\sqrt{4\pi}}$

$$\left.\begin{matrix}\varphi_{210} = \\ \varphi_{211} = \\ \varphi_{21-1} =\end{matrix}\right\} \frac{1}{\sqrt{3}(2a_0)^{3/2}}\frac{r}{a_0}e^{-r/2a_0}\begin{cases}Y_1^0\\ Y_1^1\\ Y_1^{-1}\end{cases}$$

so $a = \frac{4}{(2a_0)^{3/2}\sqrt{4\pi}}\frac{a_0^{3/2}}{2}\sqrt{4\pi} = \frac{1}{\sqrt{2}}$

$b\varphi_{210} = \frac{b}{\sqrt{3}(2a_0)^{3/2}}\frac{r}{a_0}e^{-r/2a_0}Y_1^0 = \frac{4}{(2a_0)^{3/2}}A\frac{r}{a_0}e^{-r/2a_0}\sqrt{2}\,Y_1^0$

$b = 4A\sqrt{21}$ $\quad c = 4A$ $\quad d = 4A(-2^1)$

Using $a^2+b^2+c^2+d^2 = 1$ where $a^2 \equiv a^*a$ etc.

$\frac{1}{2} + 16A^2(21+1+1) = 1$ $\quad 16A^2 \cdot 23 = \frac{1}{2}$ $\quad A^2 = \frac{1}{2\cdot 16\cdot 23}$

$A = \frac{1}{4\sqrt{46}}$

(b) Probability that measurement of L^2 yields $\ell(\ell+1)\hbar^2$

for $\ell = 0$ it is $a^2 = \frac{1}{2}$

$\ell = 1$ it is $b^2+c^2+d^2 = \frac{1}{2}$

$\ell \neq 0,1$ it is zero.

this is the last thing I'll do

(c)

Problem 10.38 revised

$\varphi_{100} = \frac{2}{a_0^{3/2}} e^{-r/a_0} Y_0^0$

$\varphi_{210} = \frac{1}{\sqrt{3}(2a_0)^{3/2}} \frac{r}{a_0} e^{-r/2a_0} Y_1^0$

$\varphi_{211} = Y_1^1$

$\varphi_{21-1} = Y_1^{-1}$

Let $\psi(r,0) = a\varphi_{100} + b\varphi_{210} + c\varphi_{211} + d\varphi_{21-1}$

(a) To get normalization. The condition is

$$\int \psi^* \psi \, r^2 dr d\Omega = 1$$

Because φ's are already normalized this leads to

$$a^2 + b^2 + c^2 + d^2 = 1$$

(b) Prob that measurement of L^2 yields $l(l+1)\hbar^2$.

For $l = 0$ it is a^2

$l = 1$ it is $a^2 + b^2 + c^2$

$l \neq 0, 1$ it is 0.

(c) Find $P_r(r)$, probability of being in spherical shell of thickness dr

$$\rho_r(r) = \left(\int_{all\,\Omega} \rho \, r^2 d\Omega\right) dr = \int \psi^* \psi \, r^2 d\Omega \, dr$$

The integrals of the Y_n^l terms over $d\Omega$ take out the cross terms, $\int Y_n^l Y_{n'}^{l'} d\Omega = c\delta_{nn'}\delta_{ll'}$

so $P_r(r) = r^2 \left[\frac{4a^2}{a_0^3} e^{-2r/a_0} + \frac{1}{3(2a_0)^3} \frac{r^2}{a_0^2} e^{-r/a_0} (b^2 + c^2 + d^2)\right] dr$

(d) $\frac{dP_r(r)}{dr} = 0 = \frac{2r}{a_0^3}\left[\frac{4a^2}{a_0} e^{-2r/a_0} + \frac{1}{24a_0^2} \frac{r^2}{a_0^2} e^{-r/a_0}(b^2 + c^2 + d^2)\right]$

$+ \frac{r^2}{a_0^3}\left[4a^2\left(-\frac{2}{a_0}\right)e^{-2r/a_0} + \frac{1}{24}a_0^2\left(-\frac{1}{a_0}\right)e^{-r/a_0}(b^2 + c^2 + d^2)\right]$

(d) solve this transcendental equation

(e) $\psi(\vec{r},t) = a\varphi_{100} e^{-iE_1 t/\hbar} + b e^{-iE_2 t/\hbar}(b\varphi_{210} + c\varphi_{211} + d\varphi_{21-1})$

obtained from $\psi(r,t) = e^{-iHt/\hbar}\psi(r,0)$ $\quad H\varphi_n = E_n\varphi_n$

(f) Measure of L_z finds $\hbar$ so

$\psi_{t=0+} = \varphi_{211}$ and $\psi(r,t) = \varphi_{211} e^{-iE_1 t/\hbar}$

(g) Measure of L_z finds 0 so

$\psi_{t=0+} = k(a\varphi_{100} + b\varphi_{210})$ normalize with k. a and b are old values

$\psi(r,t) = k(a e^{-iE_1 t/\hbar}\varphi_{100} + b e^{-iE_2 t/\hbar}\varphi_{210})$

(i) Lowest energy measured is $n = 1$.

Problem 10.~~49~~ 50

Show that the most probable values of r for $l = n-1$ states of hydrogen are

$$\tilde{r} = n^2 a_0$$

These satisfy $\frac{d}{dr}(u_{nl})^2 = 0$.

use the eq. ~~just prior to~~ 10.~~112~~ 126

$$u_{nl}(\rho) = e^{-\rho/2}\rho^{l+1}\sum_{i=0}^{n-l-1} c_i \rho^i \qquad \rho = 2K_n r$$

we can use $\frac{d}{d\rho}$ instead of $\frac{d}{dr}$ because the result $=0$ and $\frac{d}{d\rho} = \frac{1}{2K_n}\frac{d}{dr}$.

First, since we want the maximum angular momentum states where $l = n-1$

$$\sum_{i=0}^{n-l-1} c_i \rho^i = \sum_{i=0}^{n-1-(n-1)} = \sum_{i=0}^{0} c_i \rho^i = c_0 \rho^0 = 1 \text{ so.}$$

$$u_{nl} = e^{-\rho/2}\rho^{l+1} \quad \text{and}$$

$$u_{nl}^2 = e^{-\rho}\rho^{2(l+1)}$$

$$\frac{d}{d\rho}(u_{nl}^2) = -1e^{-\rho}\rho^{2(l+1)} + \underline{2(l+1)e^{-\rho}\rho^{2(l+1)-1} = e^{-\rho}\rho^{2(l+1)}}\left(-1+2(l+1)\rho^{-1}\right)$$

$$= 0 \quad \text{so}$$

$$1 = \frac{2(l+1)}{\rho} \quad \text{or} \quad \rho = 2(l+1) \quad \therefore \quad 2K_n r = 2(l+1)$$

$$K_n = \frac{Z}{a_0 n} \quad \text{so} \quad \tilde{r} = \frac{l+1}{K_n} = \frac{(l+1)}{Z/a_0 n} = \frac{a_0 n(l+1)}{Z}$$

but we assumed $n = l+1$ so

$$\tilde{r} = a_0 n^2 / Z$$

Problem 10.5D

Allowed transitions from 5D states in Hydrogen.

$\Delta l = \pm 1 \quad \Delta m = 0, \pm 1 \quad \Delta n = 1, 2 \cdots$

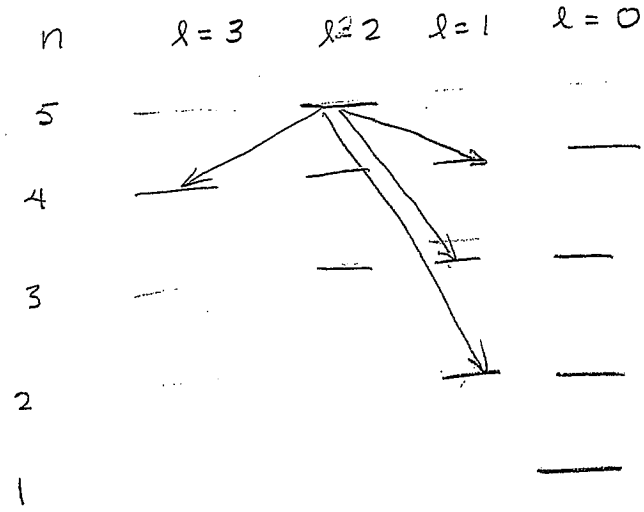
start: $n = 5, \; l = 2$

finish: $n = 4 \quad l = 3, \cancel{2}, 1, \cancel{0}$

$3 \quad l = \cancel{2}, 1, \cancel{0}$

$2 \quad l = 1, \cancel{0}$

$1 \quad l = \cancel{0}$



Problem ~~10.5~~ 10.52

show $|\vec{r}_{nn'}|^2 = |\langle n|\vec{r}|n'\rangle|^2 = \frac{1}{2}\{|\langle n|x+iy|n'\rangle|^2 + |\langle n|x-iy|n'\rangle|^2 + |\langle n|z|n'\rangle|^2$

$\vec{r}_{nn'} = \langle n|x\hat{i}+y\hat{j}+z\hat{k}|n'\rangle$

$\vec{r}_{nn}^{\,2} = |\langle n|x|n'\rangle|^2 + |\langle n|y|n'\rangle|^2 + |\langle n|z|n'\rangle|^2$

expand $\frac{1}{2}\{|\langle n|x+iy|n'\rangle|^2 + |\langle n|x-iy|n'\rangle|^2$

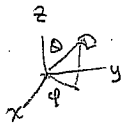
$= \frac{1}{2}\{\langle n|x+iy|n'\rangle\langle n|x+iy|n'\rangle^* + \langle n|x-iy|n'\rangle\langle n|x-iy|n'\rangle^*\}$

$= \frac{1}{2}\{\langle n|x|n'\rangle\langle n|x|n'\rangle^* + (i)(i)^*\langle n|y|n'\rangle\langle n|y|n'\rangle^*$
$+ (-i)(-i)^*\langle n|y|n'\rangle\langle n|y|n'\rangle^* + \langle n|x|n'\rangle\langle n|x|n'\rangle^*$
$+ i\langle n|x|n'\rangle\langle n|y|n'\rangle^* + i\langle n|y|n'\rangle\langle n|x|n'\rangle^*$
$- i\langle n|x|n'\rangle\langle n|y|n'\rangle^* - i\langle n|y|n'\rangle\langle n|x|n'\rangle^*$

$= |\langle n|x|n'\rangle|^2 + \langle n|y|n'\rangle^2$

Problem ~~10.52~~ 10.55

(a) show $\langle n|x\pm iy|n'\rangle = \int_{4\pi}[Y_\ell^m]^*[Y_{\ell'}^{m'}]\sin\theta\, e^{\pm i\varphi}\,d\Omega \int_0^\infty R_{n\ell}R_{n'\ell'}r^3dr$



$x = r\sin\theta\cos\varphi \quad y = r\sin\theta\sin\varphi \quad$ so $x \pm iy = r\sin\theta(\cos\varphi \pm i\sin\varphi)$
$= r\sin\theta\, e^{\pm i\varphi}$

$|n'\rangle = Y_{\ell'}^{m'}R_{n'\ell'} \quad d\tau = r^2\sin\theta\,dr\,d\theta\,d\varphi \quad d\Omega = r^2\sin\theta\,d\theta\,d\varphi/r^2$

$\langle n|A|n'\rangle = \int[Y_\ell^m]^* R_{n\ell}^* A\, Y_{\ell'}^{m'}R_{n'\ell'}\,d\Omega\,dr\,r^2$

so $\langle n|x\pm iy|n'\rangle = \int_{2\pi}\sin\theta[Y_\ell^m]^* Y_\ell^m\, e^{\pm i\varphi}\,d\Omega \int_0^\infty r^3 R_{n\ell}R_{n'\ell'}\,dr$

(b) Similarly, because $z = r\cos\theta$

$\langle n|z|n'\rangle = \int[Y_\ell^m]^*[Y_{\ell'}^{m'}]\cos\theta\,d\Omega \int_0^\infty R_{n\ell}R_{n'\ell'}r^3dr$

Problem ~~10.53~~ 10.54

Get Selection rules $\Delta l = \pm 1 \quad \Delta m = 0, \pm 1$

$$Y_l^m(\theta,\varphi) = \left[\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}\right]^{1/2} P_l^m(\cos\theta)\, e^{im\varphi} \qquad \text{from Table 9.1}$$

$$\int Y_l^m \left(Y_{l'}^{m'}\right)^* \sin\theta\, d\Omega\, e^{\pm i\varphi}$$

$$= \left[\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}\right]^{1/2}\left[\frac{2l'+1}{4\pi}\frac{(l'-m')!}{(l'+m')!}\right]^{1/2} \int \left(P_l^m e^{im\varphi}\right)^* P_{l'}^{m'} e^{im'\varphi}\, e^{\pm i\varphi} \sin\theta\, d\Omega$$

$d\Omega = \sin\theta\, d\theta\, d\varphi = -d\mu\, d\varphi \qquad \sin\theta\, d\Omega = (1-\mu^2)^{1/2}\, d\mu\, d\varphi$

$$= (\;)^{1/2} \int P_l^m P_{l'}^{m'} (1-\mu^2)^{1/2} d\mu \int e^{i((m'-m)\pm 1)\varphi}\, d\varphi$$

if $\Delta m = \pm 1$ $\int d\varphi$ finite

if $\Delta m \pm 1 \neq 0$ $\int e^{i(\Delta m \pm 1)\varphi} d\varphi = \left.\dfrac{e^{i(\Delta m\pm 1)\varphi}}{i(\Delta m \pm 1)}\right|_0^{2\pi} = 0$

$\int_0^{2\pi} e^0\, d\varphi = 2\pi$ nonzero if $\boxed{\Delta m = \pm 1}$

$$\cdots P_l^m P_{l'}^{m+1} (1-\mu^2)^{1/2} d\mu$$

From Table 9.3 ~~$P_{l'}^{m+1}(1-\mu^2) = (l'+m+1)P_{l'}^m + (l'-m+1)P_{l+1}^m$~~

so ~~$\int P_l^m P_{l'}^{m+1}(1-\mu^2)^{1/2} d\mu = -(l+m+1)\int P_l^m P_{l'}^m d\mu + (l'-m+1)\int P_l^m P_{l+1}^m d\mu$~~

↓ nonzero if $l = l'+1$

From Table 9.3 $(1-\mu^2)^{1/2} P_l^m = \frac{1}{2l+1}\left[P_{l-1}^{m+1} - P_{l+1}^{m+1}\right]$

so $\int P_l^m P_{l'}^{m+1}(1-\mu^2)^{1/2} d\mu = \frac{1}{2l+1}\left\{\int P_{l-1}^{m+1} P_{l'}^{m+1} d\mu - \int P_{l+1}^{m+1} P_{l'}^{m+1} d\mu\right\}$

one of the $\int$'s is zero if $l' = l \pm 1$ or $\boxed{\Delta l = \pm 1}$

Now look at z component $\langle n l m | z | n' l' m' \rangle = \int [Y_l^m]^* [Y_{l'}^{m'}] \cos\theta\, d\Omega \int R_{nl} R_{n'l'} r^3 dr$

$\int [Y_l^m]^* Y_{l'}^{m'} \mu\, d\mu d\varphi$ = (constants) $\int P_l^m P_{l'}^{m'} \mu\, e^{i\Delta m\varphi}\, d\mu d\varphi$ to keep $\int e^{i\Delta m\varphi} d\varphi \neq 0$ $\boxed{\Delta m = 0}$

Then $\int P_l^m P_{l'}^m \mu\, d\mu$ Table 9.3 $\mu P_l^m = \frac{1}{2l+1}\left[(l-m+1)P_{l+1}^m + (l+m)P_{l-1}^m\right]$

$$\int P_l^m \mu P_{l'}^m d\mu = \frac{1}{2l+1}\left[(l-m+1)\int P_{l+1}^m P_{l'}^m d\mu + (l+m)\int P_{l-1}^m P_{l'}^m d\mu\right]$$

one of these is nonzero if $\boxed{l' = l \pm 1}$

Note in all $\int R_{nl} R_{n'l'} r^3 dr \neq 0$ for $n' \neq n$ + $l' \neq l$ because orthogonality is $\int R_{nl} R_{n'l'} r^2 dr$ ↓

Leighton

5-4 In Eq (3) set $\frac{1}{\Phi}\frac{d^2\Phi}{d\varphi^2} = -m^2$ That achieves the first separation and Eq. 4(c)

Then divide what is left to Eq. (3) by $\sin^2\theta$ and get

$$-\frac{\sin^2\theta}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) - \frac{2Ze^2mr}{4\pi\varepsilon_0\hbar^2} - \frac{2Emr^2}{\hbar^2}$$

$$= \frac{\sin\theta}{\sin^2\theta\,\Theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) - \frac{m^2}{\sin^2\theta}$$

This accomplished the second separation so let each side equal $\ell(\ell+1)$ which anticipates what the separation constant must look like. This gives the final two equations 4(a) and 4(b).

5-5 For $e^{\pm im\varphi}$ be single valued we must have (take + sign)

$$e^{im\varphi} = e^{im(\varphi+2\pi)} \quad \text{or dividing by } e^{im\varphi}$$

$$1 = e^{im2\pi} \quad \text{or} \quad \cos m2\pi + i\sin m2\pi = 1$$

or $\sin m2\pi = 0$ and $\cos m2\pi = 1$

This is satisfied by m = integer or 0 $\therefore m = 0, \pm1, \pm2 \cdots$

5-6 See next page

Leighton $\mu = \cos\theta$

5-6 Assume $P_\ell(\mu)$ solves Eq.(9),

$$(1-\mu^2)\frac{d^2P_\ell}{d\mu^2} - 2\mu\frac{dP_\ell}{d\mu} + \ell(\ell+1)P_\ell = 0. \qquad (9)$$

Leighton shows that $\Theta = W\sin^m\theta$ into Eq(7) leads to

$$(1-\mu^2)W'' - 2(m+1)\mu W' + [\ell(\ell+1) - m(m+1)]W = 0$$

so Eq above gives Eq.9 if $W = (d^m/d\mu^m)P_\ell$

$\therefore \Theta = W\sin^m\theta = W(1-\cos^2\theta)^{\frac{m}{2}} = W(1-\mu^2)^{m/2}$ and hence

$$P_\ell^m(\mu) = (1-\mu^2)^{m/2}(d^m/d\mu^m)P_\ell$$

5-8 This may be too much busy work to take all of the polynomials in #(14) and show they solve Eq (8), so take a sampling.

Eq.(8) $P_\ell^m(\mu) = (1-\mu^2)^{m/2}\, d^m P_\ell(\mu)/d\mu^m$

I) $P_0 = 1$ trivial

II) $P_1 = \mu$ so $P_1' = (1-\mu^2)^{1/2}\frac{d}{d\mu}(\mu) = (1-\mu^2)^{1/2}$ OK

III) $P_2(\mu) = \frac{1}{2}(3\mu^2-1)$ so $P_2' = (1-\mu^2)^{1/2}\frac{d}{d\mu}\left[\frac{1}{2}(3\mu^2-1)\right] = (1-\mu^2)^{1/2}3\mu$ OK

$$P_2^2 = (1-\mu^2)^{3/2}\frac{d^2}{d\mu^2}\left[\frac{1}{2}(3\mu^2-1)\right] = (1-\mu^2)^{3/2}\frac{d}{d\mu}(3\mu) = 3(1-\mu^2)^{3/2}$$

and so on

5-9 The eigenvalue equation for a one-dimensional particle of mass m moving in a potential is

$$-\frac{\hbar^2}{2m}\frac{d^2f(x)}{dx^2} + V(x)f = Ef \quad \text{and use } V = -\frac{Ze^2}{4\pi\varepsilon_0 r} + \frac{\ell(\ell+1)\hbar^2}{2mr^2}$$

Sub $\chi(r) = rR(r)$ into 5-3(4a) which is

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) + \frac{2m}{\hbar^2}\left[E + \frac{Ze^2}{4\pi\varepsilon_0 r} - \frac{\hbar^2\ell(\ell+1)}{2mr^2}\right]R = 0$$

$R = \chi/r \quad dR/dr = -\chi/r^2 + \chi'/r$

Eq. becomes $\frac{1}{r^2}\frac{d}{dr}\left(r^2\left(\frac{\chi'}{r} - \frac{\chi}{r^2}\right)\right) + \cdots = \frac{1}{r^2}\frac{d}{dr}(r\chi' - \chi) = \frac{1}{r^2}(\chi' + r\chi'' - \chi')$

So the Eq. becomes

$$\chi''/r = +\frac{2m}{\hbar^2}\left(E + \frac{Ze^2}{4\pi\varepsilon_0 r} - \frac{\hbar^2\ell(\ell+1)}{2mr^2}\right]\frac{\chi}{r} = 0 \quad \text{or} \quad \frac{\chi''}{r} = \frac{2m}{\hbar^2}V\frac{\chi}{r} - \frac{2mE}{\hbar^2}\chi/r$$

now multiply through by $\frac{r\hbar^2}{2m}$ and replace χ by $-\chi$ and get

$-\frac{\hbar^2}{2m}\chi'' + V\chi = E\chi$ as desired.

Leighton

5-12 Show that $\frac{d^k}{d\rho^k} L_j(\rho)$ solves Eq. (32) if L_j solves (34)

Eq. (34) $\rho \frac{d^2 L_j}{d\rho^2} + (1-\rho)\frac{dL_j}{d\rho} + jL_j = 0$

Try $k=1$ and take $\frac{d}{d\rho}$ of Eq. (34) and get

$$\frac{d^2 L_j}{d\rho^2} + \rho \frac{d^3}{d\rho^3} L_j + (1-\rho)\frac{d^2 L_j}{d\rho^2} - \frac{dL_j}{d\rho} + j\frac{dL_j}{d\rho} = 0$$

Rearranging

$$\rho \frac{d^2}{d\rho^2}\left(\frac{dL_j}{d\rho}\right) + \frac{d}{d\rho}\left(\frac{dL_j}{d\rho}\right)(2-\rho) + (j-1)\frac{dL_j}{d\rho} = 0$$

so $\rho \frac{d^2}{d\rho^2}(L_j') + \frac{d}{d\rho} L_j')(2-\rho) + (j-1)L_j' = 0$

$$= \rho \frac{d^2}{d\rho^2} L_j' + (1+1-\rho)\frac{dL_j'}{d\rho} + (j-1)L_j' = 0$$

which is Eq. (32) with $k=1$.

Do it once more to show it follows for $k=2$ and infer it holds for all k by induction.

5-13 Show that Eq. (35) solves Eq (34)

Eq. (34) $\frac{1}{\rho}\frac{d^2L_j}{d\rho^2} + (1-\rho)\frac{dL_j}{d\rho} + jL_j = 0$

Eq (35) $L_j(\rho) = e^\rho \frac{d^j}{d\rho^j}(\rho^j e^{-\rho})$

$$\frac{dL_j}{d\rho} = e^\rho \frac{d^j}{d\rho^j}(\rho^j e^{-\rho}) + e^\rho \frac{d^{j+1}}{d\rho^{j+1}}(\rho^j e^{-\rho})$$

$$= e^\rho \frac{d^j}{d\rho^j}(\rho^j e^{-\rho}) + e^\rho \frac{d^j}{d\rho^j}(-\rho^j e^{-\rho} + j\rho^{j-1}e^{-\rho})$$

$$= e^\rho \frac{d^j}{d\rho^j}(\cancel{\rho^j e^{-\rho}} - \cancel{\rho^j e^{-\rho}} + j\rho^{j-1}e^{-\rho})$$

$$\frac{d^2L_j}{d\rho^2} = e^\rho \frac{d^j}{d\rho^j}(j\rho^{j-1}e^{-\rho}) + e^\rho \frac{d^{j+1}}{d\rho^{j+1}}(j\rho^{j-1}e^{-\rho})$$

$$= e^\rho \frac{d^j}{d\rho^j}(\cancel{j\rho^{j-1}e^{-\rho}} + e^\rho \frac{d^j}{d\rho^j}(\cancel{j(j-1)\rho^{j-2}e^{-\rho}} - \cancel{j\rho^{j-1}e^{-\rho}})$$

Substitute in Eq (34)

$$\frac{1}{\rho}e^\rho \frac{d^j}{d\rho^j}[j(j-1)\rho^{j-2}e^{-\rho}] + (1-\rho)e^\rho \frac{d^j}{d\rho^j}(j\rho^{j-1}e^{-\rho}) + je^\rho \frac{d^j}{d\rho^j}(\rho^j e^{-\rho})$$

now show that this equals zero

5-14 Leighton

Show that $F(\rho) = \rho^{\ell} L_{n+\ell}^{2\ell+1}$ solves (24) with $\lambda_n = n$

$$(24)\quad \frac{d^2F}{d\rho^2} + \left(\frac{2}{\rho} - 1\right)\frac{dF}{d\rho} + \left[\frac{\lambda_n - 1}{\rho} - \frac{\ell(\ell+1)}{\rho^2}\right]F = 0$$

$$(32)\quad \rho\frac{d^2L_j^k}{d\rho^2} + (k+1-\rho)\frac{dL_j^k}{d\rho} + (j-k)L_j^k = 0$$

$$\frac{dF}{d\rho} = \ell\rho^{\ell-1}L_{n+\ell}^{2\ell+1} + \rho^{\ell}\frac{dL_{n+\ell}^{2\ell+1}}{d\rho}$$

$$\frac{d^2F}{d\rho^2} = \ell(\ell-1)\rho^{\ell-2}L_{n+\ell}^{2\ell+1} + \ell\rho^{\ell-1}\frac{dL_{n+\ell}^{2\ell+1}}{d\rho} + \ell\rho^{\ell-1}\frac{dL_{n+\ell}^{2\ell+1}}{d\rho} + \rho^{\ell}\frac{d^2L}{d\rho^2}$$

$$= \ell(\ell-1)\rho^{\ell-2}L + 2\ell\rho^{\ell-1}\frac{dL}{d\rho} + \rho^{\ell}\frac{d^2L}{d\rho^2}$$

put in (24) with $n = \lambda_n$

$$\ell(\ell-1)\rho^{\ell-2}L + 2\ell\rho^{\ell-1}\frac{dL}{d\rho} + \rho^{\ell}\frac{d^2L}{d\rho^2} + \left(\frac{2}{\rho}-1\right)\left[\ell\rho^{\ell-1}L + \rho^{\ell}\frac{dL}{d\rho}\right] + \left[\frac{n-1}{\rho} - \frac{\ell(\ell+1)}{\rho^2}\right]L\rho^{\ell} = 0$$

$$0 = \frac{d^2L}{d\rho^2}\left[\rho^{\ell}\right] + \frac{dL}{d\rho}\left[2\ell\rho^{\ell-1} + \left(\frac{2}{\rho}-1\right)\rho^{\ell}\right] + \left[\ell(\ell-1)\rho^{\ell-2} + \left(\frac{2}{\rho}-1\right)\ell\rho^{\ell-1} + (n-\ell)\rho^{\ell-1} - \ell\rho^{\ell-2}\right]L$$

$$(\ell^2 - \ell - \ell^2 - \ell + 2\ell)\rho^{\ell-2} + (n-\ell-1)\rho^{\ell-1}$$

divide by $\rho^{\ell-1}$

$$\rho\frac{d^2L}{d\rho^2} + \frac{dL}{d\rho}\left[2\ell + \left(\frac{2}{\rho}-1\right)\rho\right] + L\left[\quad + \left(\frac{2}{\rho}-1\right)\ell + n - \ell\right]$$

$$n-\ell-1+$$

$$= \rho\frac{d^2L}{d\rho^2} + \frac{dL}{d\rho}\left[2(\ell+1) - \rho\right] + (n-\ell-1)L = 0$$

agree

(32) with $j = n+\ell$ and $k = 2\ell+1$

$$\rho\frac{d^2L^{2\ell+1}}{d\rho^2} + (2\ell+1+1-\rho)\frac{dL}{d\rho} + (n+\ell-2\ell-1)L = 0$$

$$\rho\frac{d^2L}{d\rho^2} + (2(\ell+1)-\rho)\frac{dL}{d\rho} + (n-\ell-1)L = 0$$

5-18 $P = \int_{-\pi/2}^{\pi/2} u^* u \, 2\pi r \cdot r \, d\theta = 2\pi A^* A \left(\frac{2Zr}{na_0}\right)^{2n-1} e^{-2Zr/na_0'} \int_{-\pi/2}^{\pi/2} \sin^{2(n-1)}\theta \, r^2 \, d\theta$

$= C \left(\frac{2Zr}{na_0}\right)^{2n} e^{-2Zr/na_0'}$

$\frac{dR}{dr} = 0 = \frac{2Z}{na_0} 2n \left(\frac{2Zr}{na_0}\right)^{2n-1} e^{-2Zr/na_0'} - \frac{2Z}{na_0'} e^{-2Zr/na_0'} \left(\frac{2Zr}{na_0}\right)^{2n}$

$\frac{2Z}{na_0} \frac{n^2 a_0}{2Zr} = \frac{Z}{na_0'} \qquad r_{max} = \frac{n^2 a_0'}{Z}$

5-19 $P = \int_{r_m}^{\infty} 4\pi r^2 dr \left(\frac{Z}{a_0}\right)^3 \frac{1}{\pi} e^{-2Zr/a_0'} = \left\{ e^{-2Zr/a_0'} \left[-\frac{r^2 a_0'}{2Z} - \frac{2r a_0'^2}{4Z^2} - \frac{2a_0'^3}{8Z^3} \right] 4 \left(\frac{Z}{a_0}\right)^3 \right\}_{r_m}^{\infty}$

$= e^{-4} \left[2 + 1 + 1/4 \right] 4 \left(\frac{Z}{a_0}\right)^3 \left(\frac{a_0'}{Z^3}\right) = 13 e^{-4}$ where $r_m = 2a_0'/Z$

(note: $a_0' = \left(1 + \frac{m}{M}\right) a_0 \approx a_0$)