Examples of Error Analysis Calculations

Compiled: September 12, 2011 by Michael Olson

1 Collect Data

Assume we have a box with the given dimensions of,

$$L = 140\ cm, \qquad w = 95\ cm, \qquad h = 75\ cm.$$

Also, assume 8 people have measured the box as shown in the following table. The mean is also calculated and shown below the measurements.

	Length (cm)	width (cm)	height (cm)
Alice	139.3	94.7	75.6
Benjamin	141.7	93.9	74.3
Cassandra	140.2	96.2	75.8
Daniel	138.4	94.6	76.1
Erin	141.3	95.5	74.8
Fred	138.9	95.3	73.9
Georgia	140.6	96.6	75.3
Mean	140.06	95.26	75.11

2 Uncertainty of each measurement

To find the uncertainty, we need to find the difference of the measurement from the mean, and the square of that difference.

	Length (cm)	$L - \bar{L}$	$(L - \bar{L})^2$
Alice	139.3	-0.76	0.58
Benjamin	141.7	1.64	2.69
Cassandra	140.2	0.14	0.02
Daniel	138.4	-1.66	2.76
Erin	141.3	1.24	1.54
Fred	138.9	-1.16	1.35
Georgia	140.6	0.54	0.29
$\sum_{i=1}^{N}(L - \bar{L})^2$			9.22
$\frac{1}{N-1}\sum_{i=1}^{N}(L - \bar{L})^2$			1.54
$\delta L = \sqrt{\frac{1}{N-1}\sum_{i=1}^{N}(L - \bar{L})^2}$			1.24

	Width (cm)	$L - \bar{L}$	$(L - \bar{L})^2$
Alice	94.7	-0.56	0.31
Benjamin	93.9	-1.36	1.85
Cassandra	96.2	0.94	0.88
Daniel	94.6	-0.66	0.44
Erin	95.5	0.24	0.06
Fred	95.3	0.04	0.00
Georgia	96.6	1.34	1.80
$\sum_{i=1}^{N}(w - \bar{w})^2$			5.34
$\frac{1}{N-1}\sum_{i=1}^{N}(w - \bar{w})^2$			0.89
$\delta w = \sqrt{\frac{1}{N-1}\sum_{i=1}^{N}(w - \bar{w})^2}$			0.94

	Height (cm)	$L - \bar{L}$	$(L - \bar{L})^2$
Alice	75.6	0.49	0.24
Benjamin	74.3	-0.81	0.66
Cassandra	75.8	0.69	0.48
Daniel	76.1	0.99	0.98
Erin	74.8	-0.31	0.10
Fred	73.9	-1.21	1.46
Georgia	75.3	0.19	0.04
$\sum_{i=1}^{N}(h - \bar{h})^2$			3.95
$\frac{1}{N-1}\sum_{i=1}^{N}(h - \bar{h})^2$			0.66
$\delta h = \sqrt{\frac{1}{N-1}\sum_{i=1}^{N}(h - \bar{h})^2}$			0.81

But the uncertainties should only have one significant figure. So, we rewrite that them as $\delta L = 1\ cm, \delta w = 0.9\ cm, \delta h = 0.8\ cm$. These are the uncertainties for each measurement. We rewrite our initial measurements as,

	Length (cm)	width (cm)	height (cm)
Alice	139 ± 1	94.7 ± 0.9	75.6 ± 0.8
Benjamin	142 ± 1	93.9 ± 0.9	74.3 ± 0.8
Cassandra	140 ± 1	96.2 ± 0.9	75.8 ± 0.8
Daniel	138 ± 1	94.6 ± 0.9	76.1 ± 0.8
Erin	141 ± 1	95.5 ± 0.9	74.8 ± 0.8
Fred	139 ± 1	95.3 ± 0.9	73.9 ± 0.8
Georgia	141 ± 1	96.6 ± 0.9	75.3 ± 0.8

3 Uncertainty of the Mean

The mean uses a different uncertainty–the uncertainty of the mean. This is calculated as $\delta\bar{x} = \delta x/\sqrt{N}$.

$\delta L = 1$	$\delta w = 0.9$	$\delta h = 0.8$
$\delta\bar{L} = 1/\sqrt{7} = 0.5$	$\delta\bar{w} = 0.9/\sqrt{7} = 0.4$	$\delta\bar{h} = 0.8/\sqrt{7} = 0.3$

Thus, our means are written as,

	Length (cm)	width (cm)	height (cm)
Mean	140.1 ± 0.5	95.3 ± 0.4	75.1 ± 0.3

4 Perimeter

To calculate the perimeter, we need to change the general equations to something we can use:

$$\begin{aligned} q &= x_1 + x_2 + \ldots & \rightarrow \quad & P = L + L + w + w \\ \delta q &= \sqrt{\delta x_1^2 + \delta x_2^2 + \ldots} & \rightarrow \quad & \delta P = \sqrt{\delta L^2 + \delta L^2 + \delta w^2 + \delta w^2} \end{aligned}$$

Let's use the measurements made by Alice. Recall:

$L + \delta L = 139 \pm 1\ cm$, $\quad w + \delta w = 94.7 \pm 0.9\ cm$, $\quad h + \delta h = 75.6 \pm 0.8\ cm$

So, the perimeter itself is:

$$\begin{aligned} P &= L + L + w + w \\ &= 139\ cm + 139\ cm + 94.7\ cm + 94.7\ cm \\ &= 467.4\ cm \end{aligned}$$

The error would be,

$$\begin{aligned} \delta P &= \sqrt{\delta L^2 + \delta L^2 + \delta w^2 + \delta w^2} \\ &= \sqrt{1^2 + 1^2 + 0.9^2 + 0.9^2} \\ &= \sqrt{1 + 1 + 0.81 + 0.81} \\ &= \sqrt{3.62} \\ &= 1.9\ cm \approx 2\ cm \end{aligned}$$

Our perimeter is then written as,

$$\mathbf{P} \pm \delta\mathbf{P} = \mathbf{467} \pm \mathbf{2}\ \mathbf{cm}.$$

If we wanted to find the mean perimeter, then we would follow the same method, only using the uncertainty of the mean:

$\bar{L} + \delta\bar{L} = 140.1 \pm 0.5\ cm$, $\quad \bar{w} + \delta\bar{w} = 95.3 \pm 0.4\ cm$, $\quad \bar{h} + \delta\bar{h} = 75.1 \pm 0.3\ cm$

$$\begin{aligned} \bar{P} &= \bar{L} + \bar{L} + \bar{w} + \bar{w} \\ &= 140.1 + 140.1 + 95.3 + 95.3 \\ &= 470.8\ cm \end{aligned}$$

$$
\begin{aligned}
\delta\bar{P} &= \sqrt{\delta\bar{L}^2 + \delta\bar{L}^2 + \delta\bar{w}^2 + \delta\bar{w}^2} \\
&= \sqrt{0.5^2 + 0.5^2 + 0.4^2 + 0.4^2} \\
&= \sqrt{0.22 + 0.22 + 0.13 + 0.13} \\
&= \sqrt{0.70} \\
&= 0.84\ cm
\end{aligned}
$$

The average perimeter is then,

$$\mathbf{\bar{P} \pm \delta\bar{P} = 740.8 \pm 0.8\ cm}.$$

5 Area

As with the perimeter, we need to change the general equations to something we can use to find the area and its uncertainty:

$$
\begin{aligned}
q &= \frac{x_1 x_2 \dots}{y_1 y_2 \dots} && \rightarrow && A = Lw \\
\frac{\delta q}{q} &= \sqrt{\left(\frac{\delta x_1}{x_1}\right)^2 + \dots + \left(\frac{\delta y_1}{y_1}\right)^2 + \dots} && \rightarrow && \frac{\delta A}{A} = \sqrt{\left(\frac{\delta L}{L}\right)^2 + \left(\frac{\delta w}{w}\right)^2}
\end{aligned}
$$

Again, with Alice's measurements,

$$
\begin{aligned}
A &= Lw \\
&= (139\ cm)(94.7\ cm) \\
&= 13,163.3\ cm^2
\end{aligned}
$$

$$
\begin{aligned}
\frac{\delta A}{A} &= \sqrt{\left(\frac{1}{139}\right)^2 + \left(\frac{0.9}{94.7}\right)^2} \\
&= \sqrt{0.000052 + 0.000090} \\
&= \sqrt{0.00014} \\
&= 0.012 \\
\delta A &= 0.012A \\
&= (0.012)(13,163.3\ cm^2) \\
&= 156.9\ cm^2 \approx 200\ cm^2
\end{aligned}
$$

$$\mathbf{A \pm \delta A = 13,200 \pm 200\ cm^2}$$

Likewise, the mean area would be,

$$\begin{aligned}\bar{A} &= \bar{L}\bar{w} \\ &= 13,351.53\ cm^2\end{aligned}$$

$$\begin{aligned}\frac{\delta\bar{A}}{\bar{A}} &= \sqrt{\left(\frac{0.5}{140.1}\right)^2 + \left(\frac{0.4}{95.3}\right)^2} \\ &= 0.0056 \\ \delta\bar{A} &= 0.0056A \\ &= 74.34\ cm^2 \approx 70\ cm^2\end{aligned}$$

$$\mathbf{\bar{A} \pm \delta\bar{A} = 13,350 \pm 70\ cm^2}$$

6 Volume

Similarly, the Volume would be calulated using the same method as the Area, but with one more variable. The answers would be:

$$\begin{aligned}\mathbf{V \pm \delta V} &= \mathbf{1,000,000 \pm 20,000\ cm^3} \\ \mathbf{\bar{V} \pm \delta\bar{V}} &= \mathbf{1,003,000 \pm 7,000\ cm^3}\end{aligned}$$