Error Analysis Lab Outline

Compiled: August 31, 2011 by Michael Olson

1 Story of Archimedes

According to legend, Archimedes was given the task of determining if a crown was made of pure gold, or if it was some alloy. However, he could not do this without damaging the crown. As he went to take a bath, he noticed that the water level rose. This gave him an idea on how to measure the volume. He successfully found the density of the crown, and found that it was not pure gold.

Assume two students were asked to do this experiment. The density of gold and a particular alloy are

$$\rho_{gold} = 19.3\ g/cm^3,$$
$$\rho_{alloy} = 15.7\ g/cm^3.$$

If student A measured a density of 18.9 g/cm^3, he would thus conclude that the crown is gold, as that is closes to his measurement. But if student B measured a density of 15.9 g/cm^3 for the same crown, he would conclude that the crown is made of the alloy. Which is correct?

This is where error analysis becomes useful. Due to methods of measurement and calculation, errors in the experiment can become significantly large. If student A were to say he measured 18.9 g/cm^3 with a range from 15.5 g/cm^3 to 22.3 g/cm^3, then we would say he had a measurement of

$$\rho_A = 18.9 \pm 3.4\ g/cm^3.$$

Likewise, if student B measured 15.9 g/cm^3 with a range from 15.5 g/cm^3 to 16.3 g/cm^3, we would say he had a measurement of

$$\rho_B = 15.9 \pm 0.4\ g/cm^3.$$

Even though student A got a result near the density of gold, his range covers both density values. Because of the large uncertainty, his answer is *uncertain.*

Student B, on the other hand, was more *accurate*, since the uncertainty is low. This gives us more confidence in his answer.

Note: Either answer could still be correct, though ρ_A is useless, and ρ_B is precise. **You must be able to justify your error estimates.**

2 How to make a measurement and determine uncertainties

Have all students measure the length, width, and height of the tabletop, then write their responses on the board. If all answers are close, introduce a number with a large error. Then do the procedure in this section using the measurements of the length of the table.

Now, we'll work on finding the error δx of a measurement x Let us assume we have the following measurements:

$$x_1 = 75.2\ cm$$
$$x_2 = 76.6\ cm$$
$$x_3 = 77.4\ cm$$

The **mean** is found with the equation,

$$\bar{x} = \frac{1}{N}\sum_{i=1}^{N} x_i.$$

So, in our example, we would get a value of $\bar{x} = 76.4\ cm$. Now, we want to see how each answer compares to this average, using the departure: $d_i = x_i - \bar{x}$. In order to find the average departure, we would add some of these up. But this causes problems as positive and negative departures will cancel each other out. To fix this, we consider the square of the deviation:

x_i	d_i	d_i^2
$75.2\ cm$	$-1.2\ cm$	$1.4\ cm^2$
$76.6\ cm$	$0.2\ cm$	$0.04\ cm^2$
$77.4\ cm$	$1.0\ cm$	$1.0\ cm^2$

$$\bar{d}^2 = \frac{1}{N}\sum_{i=1}^{N} d_i^2 = 0.8\ cm^2$$

To regain the appropriate units, we need to take the square root. This is our uncertainty:

$$\bar{d} = \sqrt{\frac{1}{N}\sum_{i=1}^{N} d_i^2} = 0.9\ cm$$

This is knows as the *Root Mean Square*. With some more intensive mathematics that we won't cover here, we can get a more accurate value using $\frac{1}{N-1}$ instead of $\frac{1}{N}$. Replacint d_i with $x_i - \bar{x}$, we get our equation for *uncertainty*,

$$\delta x = \sqrt{\frac{1}{N-1}\sum_{i=1}^{N}(x_i - \bar{x})^2}$$

This is also called the *Standard Deviation* (σ_x). For our example, we now have an uncertainty of 1.2 cm. We now report our numbers as:

$$x_1 = 75.2 \pm 1.2\ cm$$
$$x_2 = 76.6 \pm 1.2\ cm$$
$$x_3 = 77.4 \pm 1.2\ cm$$

2.1 Standard Deviation of the Mean

When we talk of the uncertainty of the mean, we must consider the uncertainty of all the measurements. We do this with the following equation:

$$\delta\bar{x} = \frac{\delta x}{\sqrt{N}} \qquad \text{or} \qquad \sigma_{\bar{x}} = \frac{\sigma_x}{\sqrt{N}}$$

2.2 About Significant Figures

When dealing with errors, it makes no sense to have an uncertainty more precise than the measurement. So,

$$75.3 \pm 1.22846539\ cm$$

makes no sense. Here is the general rule to follow with significant figures in error analysis[1]:

- Experimental uncertainties should almost always be rounded to one (at most two) significant figure(s).
- The last significant figure in any stated answer should usually be of the same order of magnitude (in the same decimal position) as the uncertainty.

Thus, it would be better to say,

$$75.3 \pm 1.2\ cm$$

or

$$75 \pm 1\ cm$$

2.3 Fractional Uncertainty

Another way to write the uncertainty is through a fractional representation, or percentages. The fractional uncertainty is simply the error over the measurement:

$$\frac{\delta x}{x}.$$

[1] Taylor, J.R. (1997). *An Introduction to Error Analysis*, The study of uncertainties in physical measurements, Second Edition, p.15. Sausalito: University Science Books.

From this equation, the fractional uncertainty of $75.3 \pm 1.2\ cm$ would be $\frac{1.2}{75.3} = 1.6\%$. So, another way to write our measurement would be,

$$75.3\ cm \pm 1.6\%$$

At this point, show the students that even though there was one erroneous measurement, it does not become significant, due to the multiple measurements and uncertainty. Have the students do the same procedure for the width of the table. This number will be needed to discuss propagation of error.
HW: *Do the same calculations for the height of the table.*

3 Propagation of Error

At times, it is necessary to combine two measurements (addition, subtraction, multiplication, division, etc.). This section is meant to describe how to combine measurements and their uncertainties.

3.1 Addition and Subtraction

First, let us try to add two errors together. Say we have two measurements: $x_1 \pm \delta x_1$ and $x_2 \pm \delta x_2$. Also, let q be the sum of our measurements, and δq be the error. Simply enough, q is just the sum of the measurements, as expected:

$$q = x_1 + x_2.$$

To determine the error, we need to consider all possible values. In other words, we need to find a range that will cover all errors. There are four possible total errors we can have:

$$\begin{aligned} \delta q &= \delta x_1 + \delta x_2 \\ \delta q &= \delta x_1 - \delta x_2 \\ \delta q &= -\delta x_1 + \delta x_2 \\ \delta q &= -\delta x_1 - \delta x_2 \end{aligned}$$

Since the value of δx_i is positive by convention, the two extreme errors are the first and last options, or,

$$\delta q = \pm(\delta x_1 + \delta x_2).$$

However, this uncertainty overestimates δq. In order to better estimate the error, we say instead that,

$$\delta q = \pm\sqrt{(\delta x_1)^2 + (\delta x_2)^2}.$$

So, our two added numbers are now,

$$q \pm \delta q = (x_1 + x_2) \pm \sqrt{(\delta x_1)^2 + (\delta x_2)^2}.$$

Now, let's subtract the two measurements. The value of q is simply,

$$q = x_1 - x_2.$$

Again, to find the error, we find the two extremes. The possible errors are the same as we determined in addition, with extreme values at $\pm(\delta x_1 + \delta x_2)$. So, subtracting errors is thus done the same way as in addition, giving us,

$$q \pm \delta q = (x_1 - x_2) \pm \sqrt{(\delta x_1)^2 + (\delta x_2)^2}.$$

We can do this for as many measurements as needed:

$$q = x_1 + x_2 + \cdots - y_1 - y_2 - \ldots$$
$$\delta q = \sqrt{(\delta x_1)^2 + (\delta x_2)^2 + \cdots + (\delta y_1)^2 + (\delta y_2)^2 + \ldots}$$

Calculate the Perimeter of the tabletop, using the numbers calculated earlier.

3.2 Multiplication and Division

Multiplying errors is a story very different from addition. This can be seen as such: Let us assume we measured a table with length and width:

$$l = 173.4 \pm 1.2\ cm$$
$$w = 64.2 \pm 0.8\ cm$$

The area of the table is simply the product of the two,

$$A = lw$$

So, the measurement of w had a smaller error. However, looking at it with fractional uncertainties,

$$l = 173.4\ cm \pm 0.7\%$$
$$w = 64.2\ cm \pm 1.2\%$$

In other words, even though the error for l was larger, it was more insignificant in comparison to the whole length. So, if we had just multiplied the errors, it would have misrepresented the error of the length. Instead, we multiply errors using fractional uncertainties. The whole procedure is outlined as follows:

$$
\begin{aligned}
A \pm \delta A &= (l \pm \delta l)(w \pm \delta w) \\
A \pm \delta A &= lw \left(1 \pm \frac{\delta l}{l}\right) \left(1 \pm \frac{\delta w}{w}\right) \\
A \pm \delta A &= lw \left(1 \pm \frac{\delta l}{l} \pm \frac{\delta w}{w} \pm \frac{\delta l \; \delta w}{lw}\right) \\
1 \pm \frac{\delta A}{A} &\approx 1 \pm \left(\frac{\delta l}{l} + \frac{\delta w}{w}\right) \\
\frac{\delta A}{A} &\approx \frac{\delta l}{l} + \frac{\delta w}{w}
\end{aligned}
$$

This method is the same as the general case,

$$\frac{\delta q}{q} = \frac{\delta x_1}{x_1} + \frac{\delta x_2}{x_2} + \dots$$

As in the addition case, this overestimates the error. So, it is better to use,

$$\frac{\delta q}{q} = \sqrt{\left(\frac{\delta x_1}{x_1}\right)^2 + \left(\frac{\delta x_2}{x_2}\right)^2 + \dots}$$

The proof for quotients is similar, giving the same result. So, for products and quotients, we can use the following equation:

$$q = \frac{x_1 x_2 \dots}{y_1 y_2 \dots}$$

$$\frac{\delta q}{q} = \sqrt{\left(\frac{\delta x_1}{x_1}\right)^2 + \left(\frac{\delta x_2}{x_2}\right)^2 + \dots + \left(\frac{\delta y_1}{y_1}\right)^2 + \left(\frac{\delta y_2}{y_2}\right)^2 + \dots}$$

Calculate the Area of the tabletop, using the numbers calculated earlier.
HW: *Calculate the Volume of the tabletop, with its appropriate error.*

3.3 Powers

Powers work simply as multiplication of one number by itself. Thus, we can simply use the product rule stated above. The gives us:

$$q = x^n$$

$$
\begin{aligned}
\frac{\delta q}{q} &= \frac{\delta x}{x} + \frac{\delta x}{x} + \dots \\
\frac{\delta q}{q} &= |n| \left|\frac{\delta x}{x}\right|
\end{aligned}
$$

4 Discrepancies - Return to Archimedes

Now that we have seen how to determine the error of a measurement, we now have to compare our measurement (and its uncertainty) to the expected value. Recall the following from the Archimedes' experiment:

$$\begin{aligned}\rho_{gold} &= 19.3\ g/cm^3\\ \rho_{alloy} &= 15.7\ g/cm^3\\ \rho_A &= 18.9 \pm 3.4\ g/cm^3\\ \rho_B &= 15.9 \pm 0.4 g/cm^3\end{aligned}$$

If the crown is made of an alloy, then by observing these numbers, we see that both students A and B have an answer with a range that accepts this value. Thus we say that ρ_A and ρ_B are within 1 standard deviation of the correct answer.

However, if the crown is made of gold, then only student A is within 1 standard deviation of the correct answer. Since $\rho_{gold} - \rho_B = 3.4$, and $\delta\rho_B = 0.4\ g/cm^3$, we find that student B is 8.5 standard deviations (3.4/0.4) away from the correct answer.

HW: How do your measurements of the table (length, width, height, preimeter, area, volume) compare to the expected value?

5 References

Taylor, J.R. (1997). *An Introduction to Error Analysis*, The study of uncertainties in physical measurements, Second Edition. Sausalito: University Science Books.