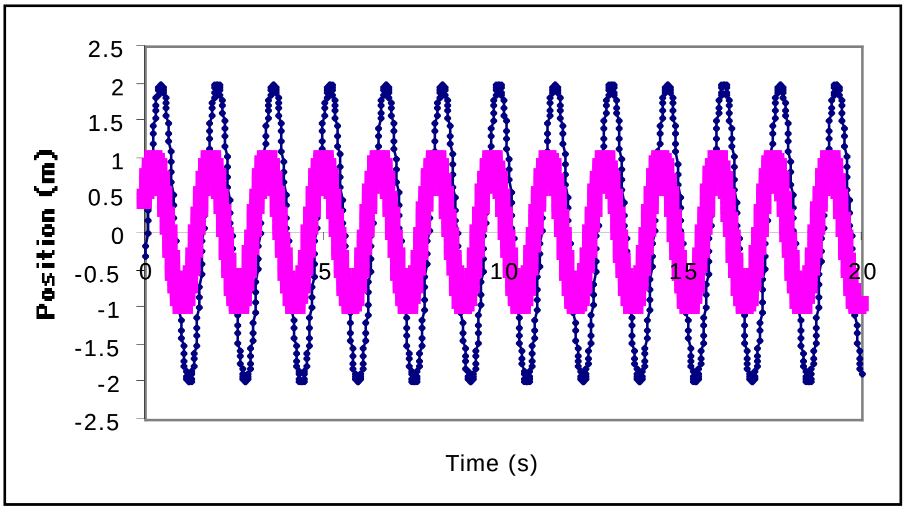
Notes on Resonance

Simple Harmonic Oscillator

- A mass on an ideal spring with **no friction** and **no external driving force**
- Equation of motion: $ma_x = -kx$
- Late time motion: $x(t) = A\sin(\omega_0 t)$ OR $x(t) = A\cos(\omega_0 t)$ OR

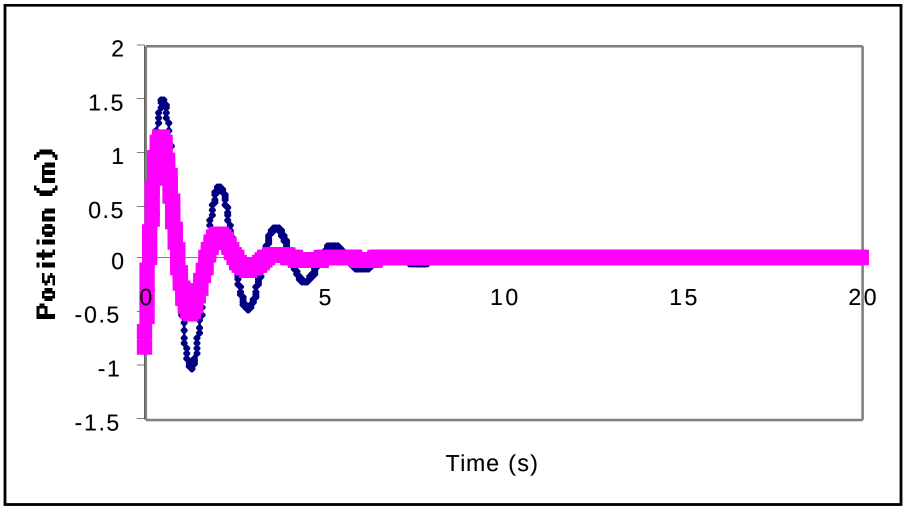
$x(t) = A(\alpha \sin(\omega_0 t) + \beta\cos(\omega_0 t))$, where $\omega_0 = \sqrt{\frac{k}{m}}$ is the so-called **natural frequency** of the oscillator

- Late time motion is the same as beginning motion; the amplitude *A* is determined completely by the initial state of motion; the more energy put in to start, the larger the amplitude of the motion; the figure below shows two simple harmonic oscillations, both with *k* = 8 N/m and *m* = 0.5 kg, but one with an initial energy of 16 J, the other with 4 J.



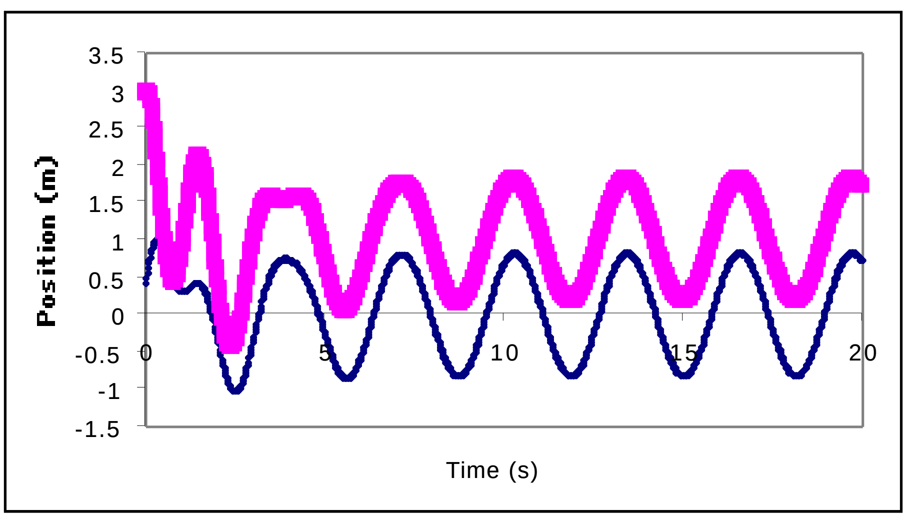
Damped Harmonic Motion

- A mass on an ideal spring **with friction**, but **no external driving force**
- Equation of motion: $ma_x = -kx + friction$; friction is often represented by a velocity dependent force, such as one might encounter for slow motion in a fluid: $friction = -bv_x$
- Late time motion: friction converts coherent mechanical energy into incoherent mechanical energy (**dissipation**); as a result a mass on a spring moving with friction always "runs down" and ultimately stops; this "dead" end state $x = 0$, $v_x = 0$ is called **an attractor of the dynamics** because **all** initial states ultimately end up there; the late time amplitude of the motion is always **zero** for a damped harmonic oscillator; the following figure shows two different damped oscillations, both with *k* = 8 N/m and *m* = 0.5 kg, but one with *b* = 0.5 Ns/m (the oscillation that lasts longer), the other with *b* = 2 Ns/m.

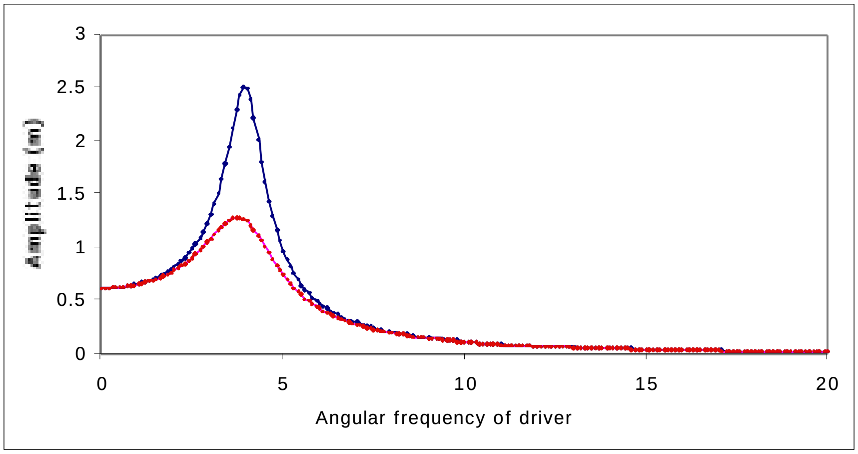


Damped, Driven Harmonic Oscillator

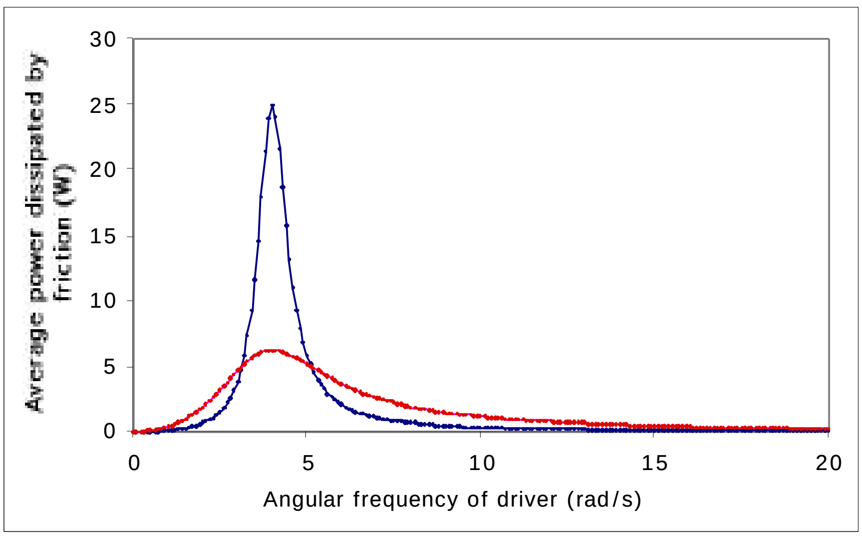
- A mass on an ideal spring **with friction**, and **with an external driving force**
- Equation of motion: $ma_x = -kx + friction + driving$; here $friction = -bv_x$ and $driving = F_0 \sin(\omega_d t)$.
- Late time motion: friction causes conversion of coherent ME into incoherent, and the driving force attempts to replace the coherent ME; at first, the oscillator will "attempt to" oscillate as if there were no external force, but, ultimately, the late time motion will become $x_{late}(t) = A_{late}\sin(\omega_d t + \varphi)$, where $\boldsymbol{A_{late}}$ **is the same no matter what the initial state of the motion is**; in other words, the **late time behavior is also an attractor**, but instead of being "dead" as in the previous case, it is "alive" and wiggling with the same frequency as the driving force; the funny quantity φ in the argument of the sin is a "phase shift" that indicates that the motion is not necessarily in phase with the driver—that is, x need not be a maximum when F_{driver} is a maximum; the following figure shows two examples of damped, driven oscillations, both with k = 8 N/m, m = 0.5 kg, b = 0.5 Ns/m, and F_0 = 5 N, but one with a starting energy of 4 J and the other with a starting energy of 16 J; the top curve is offset from the bottom so that both can be seen simultaneously; notice that after about 5 or 6 seconds both motions are identical.



- Resonance: the late time amplitude, A_{late}, is a function of the driving force amplitude, the amount of friction, and the driving frequency; the following figure shows plots of two attractor amplitudes for different driving frequencies, the difference being one has less friction (the steeper curve) that the other; the maxima of the amplitudes occur at a driving frequency that is essentially the natural frequency ω_0 of the oscillator; the phenomenon of maximum amplitude at the natural frequency of the oscillator is called **resonance**;



another manifestation of resonance is that at when the oscillator is driven at its natural frequency the rate at which energy is dissipated by friction is a maximum; that's shown in the following figure; again, the steeper graph corresponds to less friction; notice that when friction is increased the response is flatter and



broader.

BIG PICTURE: Attractors are dynamical behaviors that are robust against environmental changes; in the absence of dissipation every starting condition produces a different behavior; when dissipation is present attractors can appear; when there's no input the attractor is "dead;" when there's energy input as well as dissipation, the attractors are more interesting, resonance being the most interesting of all. **This most interesting attractor occurs when friction is dissipating energy at its maximum rate.**