Notes on deterministic chaos:

One of the most mysterious aspects of the natural world is the coexistence of order and disorder. Some things appear to be fairly predictable. These things appear to obey fairly clear, rigid rules. When you flip the light switch the lights come on (mostly). When you turn the key in car's ignition, the engine starts (mostly). The sun rises at a very precisely predictable time each morning and sets at an equally predictable time each evening. As a species, we are able to build dams and roads and bridges; we are able to cure diseases; we are able to place communication satellites into orbit; we are able to tame (much of) our environment. All of these things are predictable, obey rules, are orderly, can be controlled.

Then there are other things: the weather, the stock market, some sporting events, games of "chance," outcomes of some elections, and so on. These things are much less predictable, and certainly not controllable. We often say that they contain aspects of *randomness*, meaning no rhyme-or-reason, no predictability. Random is the opposite of deterministic. In randomness, future events are not *rigidly* determined by past events. In *deterministic* behavior, the future *is* rigidly determined by the past. Newton's Laws of Motion are an exquisite example of determinism. Once the initial state of a Newtonian system is known, the future is completely determined by the rules that relate state (position and velocity) to the forces acting.

Why is the universe filled with some stuff that's orderly, deterministic, Newtonian, and other stuff that's not? And how do we know when a system is one kind or the other? There is a growing awareness that at least some of the irregularity in nature is **not** due to random rulelessness. In the last 30 years or so, it has become apparent that some erratic and seemingly unpredictable behavior is actually deterministic. The emergence of irregularity from order is called **deterministic chaos**. (Deterministic chaos has been brought to the attention of large numbers of nonscientists through a popular book *Chaos* by James Gleick and via Jeff Goldblum's character in the movie of Michael Chrichton's *Jurassic Park*.)

Our understanding of chaos (the "deterministic" part is usually deleted, but it's important to remember that it's implied) has its origins in the study of fluids. When a real fluid (that is, one that has viscosity) flows in a pipe at low speeds, the flow is steady, smooth, and streamline. If the pipe is straight and horizontal, the streamlines at low flow rates are straight lines parallel to the pipe walls. As the flow speed is increased the streamlines begin to wiggle, at first smoothly undulatory like a sine function, then more-and-more jaggedly, as the flow rate is made larger. As the flow speed is increased even further, the streamlines begin to break up into a regular series of swirls called *vortices*. Finally, at high flow speeds, no regular patterns are observed in the flow at all; this situation is called *turbulence.* Turbulence is filled with erratic and unpredictable variations. It appears to be random. But the smooth flow at slow speeds is in exact agreement with the predictions of Newton's Laws of Motion—low speed flow is perfectly deterministic. So how does randomness creep into the behavior of the fluid just because its flow rate is increased?

Today, we strongly suspect it doesn't. We believe (although it hasn't yet been completely demonstrated) that turbulence and smoothness are really two sides of the same coin—and both are equally deterministic. Let's examine the transition from smooth behavior to chaotic behavior as the flow rate is increased a little more closely. In several different fluid systems where a similar progression from streamline to turbulent flow is observed a qualitatively identical pattern of events appears to be present. For the pipe flow example, suppose the velocity perpendicular to the pipe walls is measured in the center of the pipe. Suppose at the input end a little paddle wiggles in such a way that it tries to make little sinusoidal undulations in the streamlines near the middle of the pipe. The fluid will always get little ripples in it near the paddle but at low flow rates these ripples tend to disappear as we look downstream some distance from the paddle. In other words, at low flow rates viscosity causes the input variations to die out. *There is an* ***attractor*** *for the fluid flow that at low flow rates has zero undulations perpendicular to the net flow direction*. At higher flow rates, this zero undulation behavior becomes *unstable* and we find that the streamlines downstream now undulate periodically with the same period as that of the paddle. If the speed perpendicular to the walls is recorded every time the velocity reaches a maximum, the result is (except for measurement error) the same nonzero value, again-and-again. The attractor for the fluid flow in this case is a periodic variation perpendicular to the net flow direction.

At yet another precise and repeatable value of the flow rate, the undulation downstream suddenly becomes more complicated. The velocity perpendicular to the net flow direction still undulates, but instead of having the same amplitude every cycle the amplitude alternates every other cycle. That is, the speed measured every time the velocity reaches a maximum is one time a higher value, the next time a lower value, the next time the higher value, the next time the lower value, and so on. Instead of the value repeating every time the speed is sampled, it repeats every other time. The attractor of the fluid flow is still periodic, but the repeat time—the period of the undulation—is double what it was before. Note that, the paddle is still wiggling at its former rate. The new period of the fluid downstream is not due to changes in the way the fluid is being wiggled at the input end of the pipe.

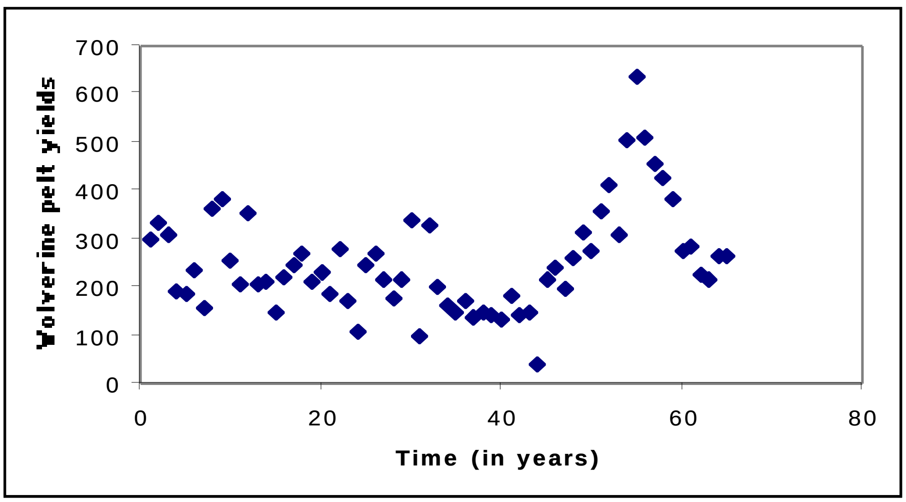
In very careful experiments of this kind, it is observed that as the flow rate is increased further, at precise and repeatable values of the flow rate the repeat time for the maximum speed doubles again, and again, and so forth. Eventually, when turbulence emerges, the maximum perpendicular speed never repeats. The motion of the fluid through the pipe ceases being smooth and the perpendicular variations of velocity become **aperiodic** (never repeat). In this condition, the fluid flow is said to be **chaotic**. (Incidentally, the reverse of this process is how blood pressure measurements are made. A cuff that can be pressurized is wrapped around a major artery and the flow is temporarily cut off. Gradually the constraining force is eased up and at some point flow begins, but through a very narrow opening. Because the of requirement that Av = constant, the speed of the rushing blood through the small opening is fast, and, therefore, turbulent. The turbulent flow makes a characteristic rumbling sound that can be heard with a stethoscope. As the force is released further, the opening widens and eventually the flow becomes smooth. The applied pressure at which turbulent flow first begins is called the *systolic* pressure [that corresponds to the highest pressure the heart can deliver] and where smooth flow commences the *diastolic* pressure [that corresponds to the lowest pressure the heart can deliver].)

The complicated swirling of the fluid in turbulent flow can be analyzed statistically and when we do so we find that irrespective of how the paddle wiggles at the input end of the pipe the statistical properties of the turbulence are the same. In other words, when the fluid is flowing chaotically there is still something repeatable about the irregularity. Because of that, we say that the motion is associated with a *chaotic, or strange, attractor*.

This so-called **period doubling route to chaos** is observed in *many* different, seemingly unrelated, phenomena. Besides fluids it has been reported in: electronic circuits, the intensity fluctuations in lasers, various mechanical devices, dripping faucets, woodwind instruments, chemical reactions, the synchronized contractions of cardiac cells, the levels of armaments of antagonistic countries, and even in babies' crying. Though we now know that period doubling is not the only route to chaos, its appearance in such diverse situations implies that it is a remarkably robust and ubiquitous phenomenon. And where period doubling exists, we probably should expect that deterministic chaos is possible as well.

The key to producing chaos is to have a sufficient degree of **nonlinearity**. A **linear** mathematical model contains dependent and independent variables all raised to the power *one*. A **nonlinear** model has variables raised to other powers, or contained in complicated functions like sines and exponentials (and worse). So where does nonlinearity come into fluid dynamics? The answer is, through the acceleration in Newton's Second Law applied to a lump of fluid. There are two ways a lump of fluid can accelerate. One way is that the velocity at every point in the flow is changing in time. But, even if the flow is steady—that is, no change in time anywhere—a lump of the fluid can accelerate if it is being swept by the flow from a region of one speed to a region of another. A good example is a narrowing pipe. To conserve mass, as the pipe narrows the flow rate has to increase. Thus, a lump of fluid when it is upstream will travel more slowly than when it is downstream. The acceleration of the lump is its rate of change of velocity. Thus, if we take the velocity downstream minus that upstream and divide by the time necessary to get from upstream to downstream, we get the acceleration. The time that we divide by is roughly the distance between the up- and downstream points divided by the average velocity between up and down points. When you do the algebra you find that the acceleration due to sweeping is the product of a velocity and a change in velocity. That is, the acceleration due to sweeping essentially involves the square of velocity. The larger the flow speed the larger will be this nonlinear term. Newton's Second Law for a lump of fluid is inherently nonlinear.

The kind of order/disorder, yin and yang harbored by fluids can be studied without having to get into a lot of complicated mathematics. One very simple example of a nonlinear dynamical system (**dynamical** means a system that changes in time) that shows much of the interesting behavior that fluids do comes from population biology. Many wildlife populations are known to have regular periods and wildly fluctuating ones. The graph to the right shows the variation of wolverine pelts taken in British Columbia over a period of 65 years. The fluctuations may be attributable to several factors, but underlying population is probably one the stronger ones. The population dynamics model that we will examine is due called the *Ricker Equation*. This equation is supposed to describe one species' population size in a finite environment—a finite landscape or waterscape region. In it the variable x_k is the size of the population divided by the environment's *carrying capacity* (the population size the environment can support at optimal energy consumption). In the fluid dynamics example described above, there was a single *control parameter*—the flow rate. In the Ricker Equation there is also a single



control parameter, b, the intrinsic birth rate—the average number of births per generation per individual in the population.

The population is measured every generation, and for Ricker beasties, there is no overlap of generations: every individual lives for exactly one generation. (While not valid for many animals, this is an appropriate assumption for certain insects and fishes.) The **Ricker Equation** relates the next population size, x_{k+1}, to the previous, x_k:

$$x_{k+1} = bx_k\exp(-x_k).$$

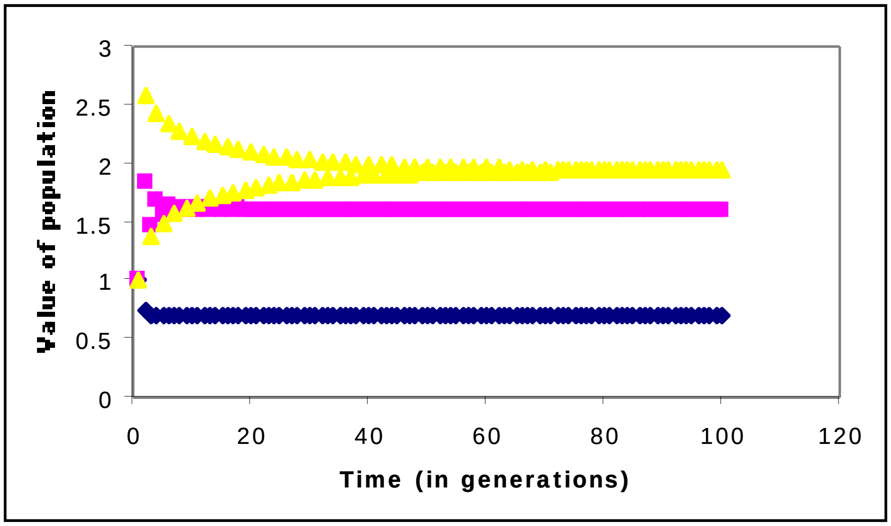
The term $\exp(-x_k)$ means the *exponential constant* e = 2.718281828... raised to the minus x_k power. (The constant e always appears wherever there is *exponential* growth or decay, as in population dynamics, chain reactions, compounding of interest, and so forth.) The values of x are numbers greater than zero. The value of b depends on the *fecundity* of the species. When x is much smaller than 1 (that is, when the size of the population is much less than the carrying capacity), $\exp(-x)$ is approximately 1. So, for small population sizes the Ricker Equation is equivalent to $x_{k+1} = bx_k$. Such an equation is linear in the x's. As long as this approximate relation holds it is easy to show that $x_k = b^k x_0$, where x_0 is some starting value. (Make sure you understand why. Start with $x_1 = bx_0$. Then $x_2 = bx_1 = b(bx_0)$, and keep going.) If $b < 1$ (less than one offspring produced per individual), the population size steadily decreases (and eventually extincts). If, on the other hand, $b > 1$ (more than 1 offspring produced per individual), the population grows by a larger amount in each generation because b^k increases as k increases *nonlinearly*. This is called *exponential growth*. If left unfettered exponential growth will lead to an enormous population. That's where the "exp" term comes in.

The bigger x the smaller is $\exp(-x)$. This term describes the effects of too many individuals competing for a finite amount of energy in the environment (finite carrying capacity). You can think of it as being the probability an offspring will survive to maturity to produce more offspring. The product $b\exp(-x)$ is therefore the effective birth rate of each individual. The number of births per individual times the probability each birth will survive to reproduce. If the population size ever becomes too large, the species pays a price in the next generation because few offspring are produced, and the population decreases. Of course, if the population ever becomes very small, exponential growth returns it to ever-larger values. This interplay between growth and collapse is akin to the opposing effects of pressure and friction in a fluid. Pressure speeds flow up, friction slows it down.

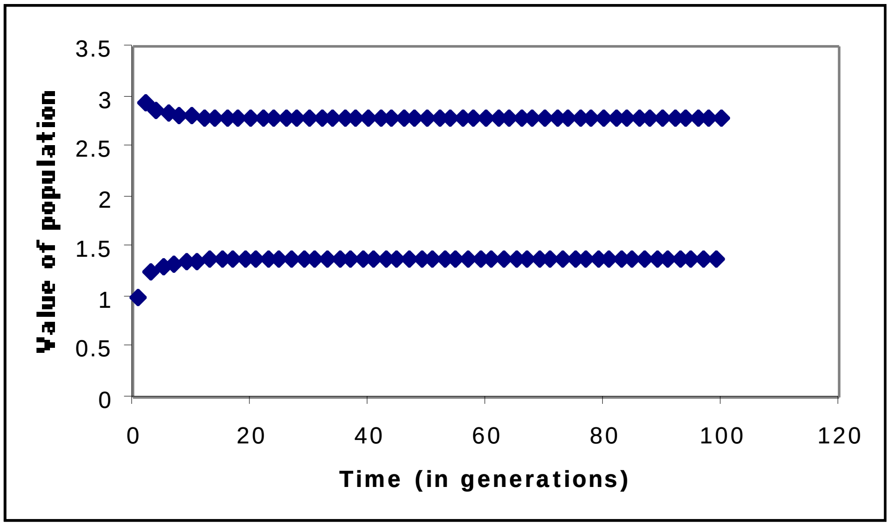
The interplay is also responsible for some very interesting results. First, note that $x = 0$ is a special value. If you put 0 in for x on the right hand side of the Ricker Equation you get 0 back out. The value 0 never changes. Such an unchanging value is called a **fixed point** of the dynamics. When $b < 1$, every starting value of x eventually evolves to 0. For $b < 1$, zero is an attractor, a **stable fixed point**. This situation is identical to the lowest speed fluid flow, where there is zero velocity perpendicular to the walls downstream. If you intentionally make little perpendicular wiggles in the fluid to try to induce perpendicular motion, they die out at low flow rates and straight line, streamline flow returns.

Zero is always a fixed point of the Ricker Equation, but when $b > 1$, it is **unstable**. That means that any slight deviation from zero will evolve away from zero, not come back. (Stability and instability can be thought of in terms of a marble and a bowl. If the bowl is open side up, the bottom of the bowl is a fixed point for a marble, because a marble placed there will stay there. It is also a stable fixed point. If you displace the marble away from the bottom a bit and let go, the marble will slide back down toward the bottom of the bowl. On the other hand, suppose the bowl is inverted, so the open side is down. Assuming the bottom of the bowl is perfectly round—a tough bowl to eat from because it will rock back and forth whenever you touch it—you can still, with *great* care, perch a marble at the very top. But now any slight nudge will send the marble careening away from the top. In this case, the top of the bowl is an *unstable* fixed point.)

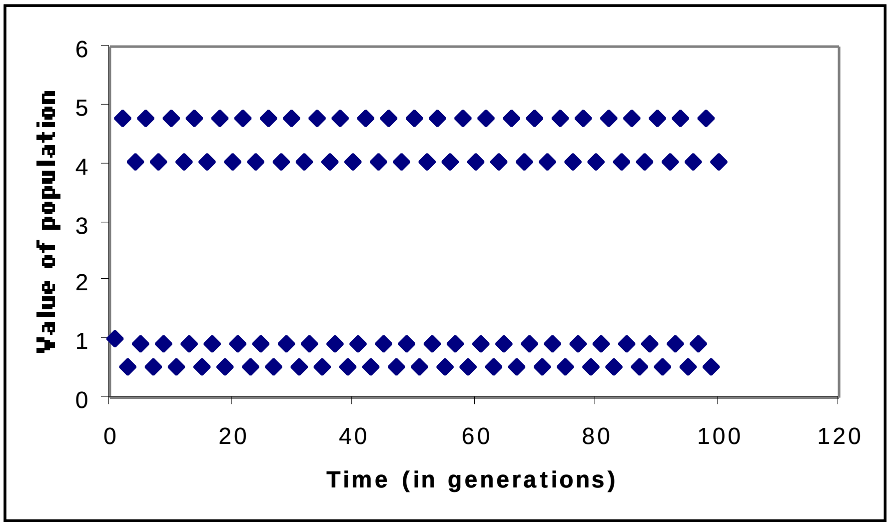
What happens when $b > 1$, if zero is not stable? The answer is that the population reaches a new stable fixed point the value of which is *not* zero. The figure to the right shows what happens for any x except zero as the starting value and for $b = 2$ (bottom), $b = 5$ (middle), and $b = 7$ (top). Eventually, x goes to a fixed value greater than zero. The fixed value reached in each case is greater for greater values of b. Again, this is just like the fluid flow described at the outset. This is analogous to the situation where, downstream, the flow has a simple periodic wiggle perpendicular to the flow direction. Every cycle the maximum perpendicular speed is the same. Note that as b increases it takes longer and longer for the dynamics to settle into the fixed-point value. That is evidence that the fixed point is getting increasingly less stable as b increases.

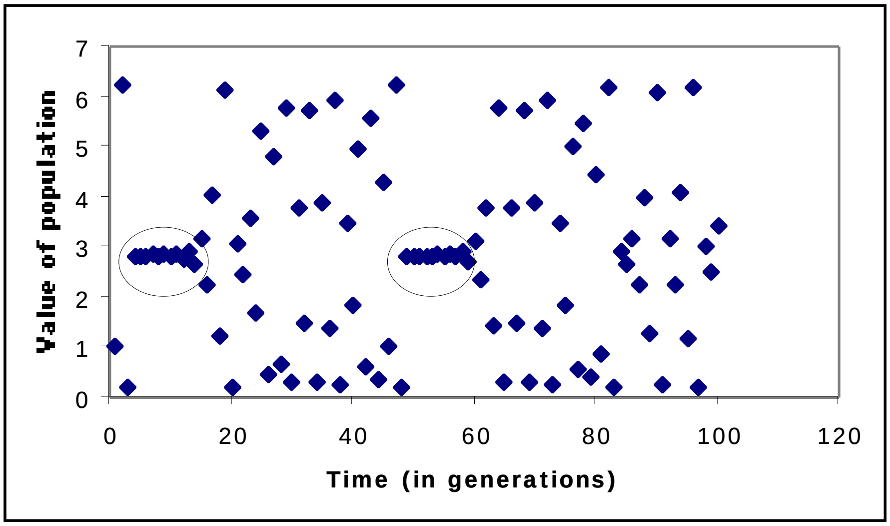


In fact, at about $b = 8$, the nonzero fixed point also becomes unstable. Instead, we observe a pattern of population values that repeats every other cycle instead of every cycle. The repeat period has doubled. This repeat-every-other-time behavior is called a **two-cycle** (repeats every two samples). For $b = 8$, the fixed point at zero and the nonzero fixed point still exist, but both are unstable. The stable behavior is instead a two-cycle. Again, this is the same as for fluid flow.



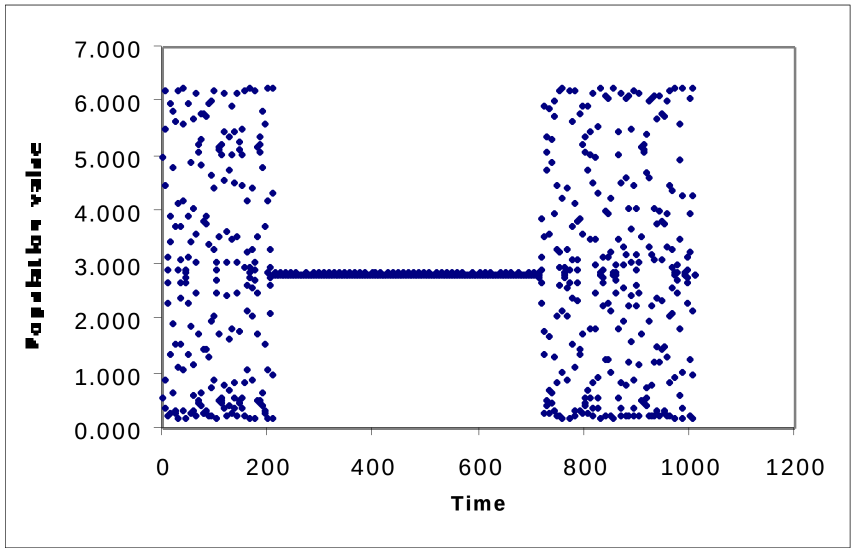
When b is made a little bit bigger than 12, the two-cycle becomes unstable and is replaced by a stable four-cycle, as in the figure to the right (recall the fluid). This sequence of period doublings occurs again-and-again as b is made larger and larger (4-cycle goes to 8-cycle to 16-cycle to 32-cycle, and so on). Eventually, the repeat time becomes so long, the population size just never repeats. The latter situation is called deterministic chaos. An example of deterministic chaos in the Ricker Equation for $b = 17$ is shown in the figure to the right. (You may think a birth rate of 17 is pretty high but many insects and fishes produce many more progeny than that.) The big picture point here is that Ricker Equation reproduces qualitatively all of the behavior seen in fluids as the flow rate is increased. Both are nonlinear dynamical systems and essentially any nonlinear dynamical system can produce chaos provided the system control parameters are in the correct range.

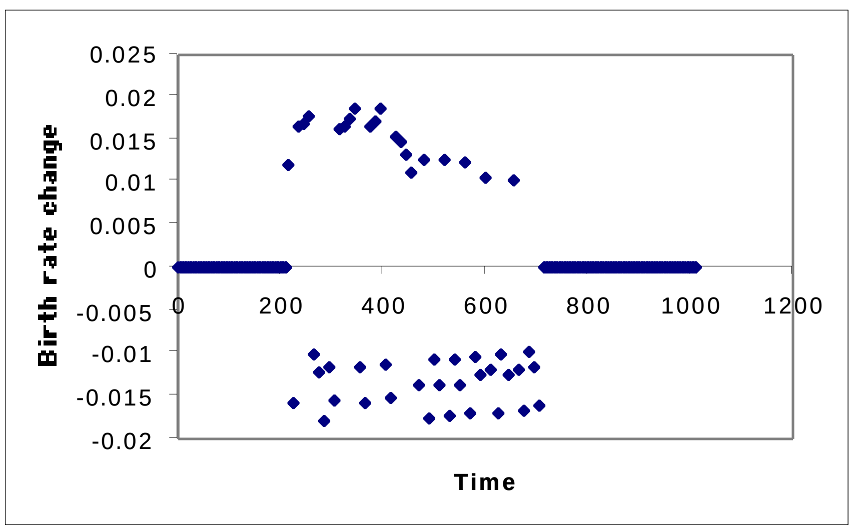




Please note the two circled stretches in this graph where the value of x seems to settle into a repeat-every-time, fixed-point pattern. The values that x takes on in each stretch are about the same. In fact, those values are almost the value of the unstable, nonzero fixed point for $b = 17$ (a value that can be calculated exactly—it's $\ln(17) = 2.833...$; *I'll give you 10 extra course points if you prove it*). This behavior is no accident. Deterministic chaos is a recurrent attempt to hit one or other periodic behavior, it's just that every attempt fails because the periodic behaviors are all unstable (you'd have to hit them infinitely precisely—any slight miss and away you go). In fact, almost immediately after the second attempt to be the unstable fixed point, the behavior attempts to be an unstable four-cycle. Do you see it?

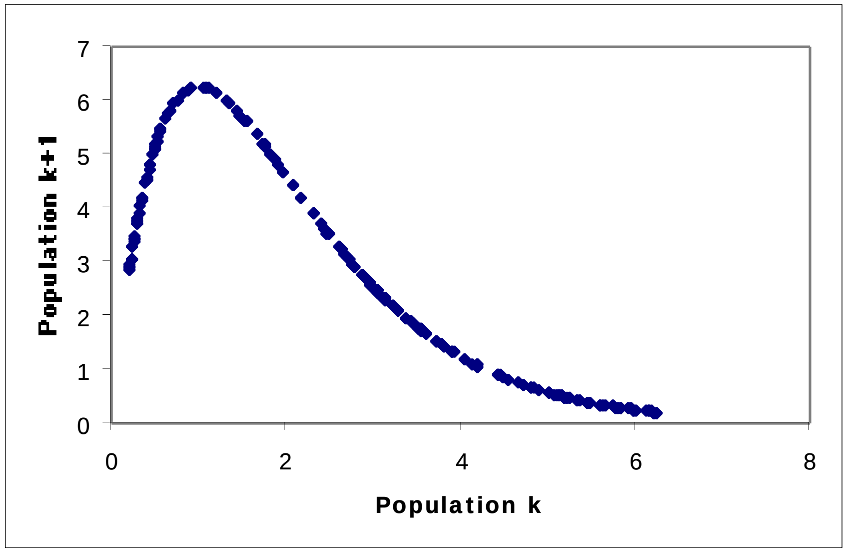
The fact that deterministic chaos is a continual, recurrent, ever-failing pursuit of periodicity allows it to be controlled by a very subtle technique. Usual engineering control is accomplished in a "brute force" manner. The usual method for control starts with a mathematical model of the system to be controlled and a target behavior. A controller is then designed based on the model. Often the control is energy intensive and not at all optimal. Chaos control is different. It exploits the failed periodicity character of chaos and is extremely efficient. It doesn't require a model of the behavior. The way it works is, you (or your computer) look for examples of recurrent unstable periodic behaviors in the output of the system (such as the two circled stretches in the previous graph). Once those are identified, you wait until the same behavior recurs (it will, though you may have to wait a bit). When that happens you change the system's control parameter ever so slightly, and only when needed, to keep the system close to the identified behavior. The details of how the parameter adjustment is actually done are fairly easy to work out, but for our purposes it is sufficient to know that little parameter jiggles are often sufficient to trap the system's behavior in a mode that it ordinarily would not like to stay in.

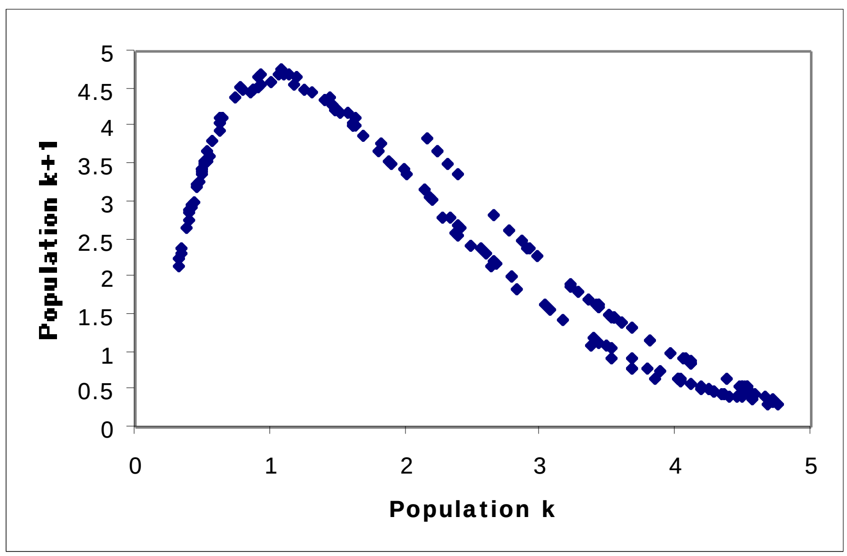




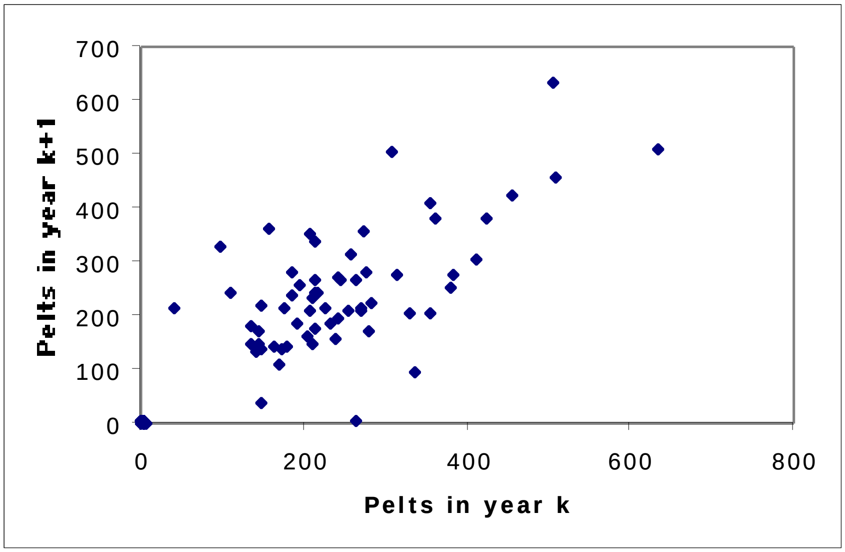
The top figure to the right shows control of the output of the Ricker Equation (with b = 17) using this method. Though *we* know the source is the Ricker Equation, the computer doesn't. Control comes on at step # 212, after an unstable fixed point value has been identified. The control stays on for 500 steps, after which it is turned off. Before the control is on the output is wildly chaotic. After the control is turned off the output is again wildly chaotic. Control is maintained to within ±5% of the fixed point value by adjusting the birth rate when necessary. The lower graph to the right shows the changes in the birth rate needed to accomplish this control. Note that the changes are few (49 changes in 500 steps), and range between ±0.02 (about ±0.1% of the unchanged birth rate of 17). Very small parameter adjustments applied at just the right time can keep the system from returning to its wildly chaotic behavior. In the usual kind of brute force control, the controller would typically always be doing something and the adjustments would typically always be much larger than shown in the graph. The downside to chaos control is that you can't control to just any behavior. You can only control to the unstable periodic behaviors that the system permits. You can't, for example, control to a fixed point value of 1 or 5 from the Ricker Equation with b = 17, because the only fixed point the Ricker Equation has with that b is about 2.83. Nonetheless, if wild chaotic swings are very undesirable (as, for example, in a wildly fibrillating heart) any control at all may be of significant practical utility.

To close this discussion on deterministic chaos, it is necessary to point out that chaos comes in two forms—simple, or **low-dimensional**, and complex, or **high-dimensional**. The meaning of low- and high-dimensional can be seen by making graphs of the next value of the output versus the previous value—a so-called **first return map**. If a first return map of output from the Ricker Equation is made for b = 17, for example, we obtain the graph to the right. The fact that we obtain a smooth, simple curve tells us that the population in generation k+1 depends only on the population in generation k. When the future depends on only one value in the past the dynamics is said to be **one-dimensional**. If the Ricker Equation is slightly modified so that $x_{k+1} = bx_k\exp(-x_k) - cx_{k-1}$, the first return map (for b = 15.9, c = 0.31) looks like the lower figure to the right. You can still see hints of the curve of the previous graph, but the new plot shows considerable smudging out. In the modified Ricker Equation, the population in generation k+1 depends on the populations in generation k and generation k-1. It is an example of **two-dimensional dynamics**.





The rule is, the higher the dimension of the dynamics the more smudged out a first return map will be. We almost always see smudged out first return maps for real data. A typical example from the real world is the pelt data displayed earlier. Its first return map is shown to the right. Although there isn't much data, pelt yields appear to be an example of high-dimensional dynamics. We infer that almost all real systems are relatively high-dimensional.



The interesting and encouraging thing about chaos is that when it episodically tries to be periodic, the dimension of the dynamics effectively becomes lower. Thus, even if the chaotic fluctuations of a real system are high-dimensional, every once in awhile, the complexity of the dynamics spontaneously reduces. When that happens, control of even a high-dimensional system can be put into effect. There are two physiological examples of this, one involving rabbit fibrillating rabbit hearts, the other electrical activity in rat brains. In both cases the intervals between events (beating of the heart, electrical spikes in the brain) make a very blobby first return map—suggesting high-dimensional chaos. Nonetheless, in both episodic simplification is observed and control using the same principles as in the Ricker example has been implemented. There is a considerable interest in seeing whether these primitive experiments can be extended to humans with positive clinical implications (stabilizing fibrillation, preventing epileptic seizures).