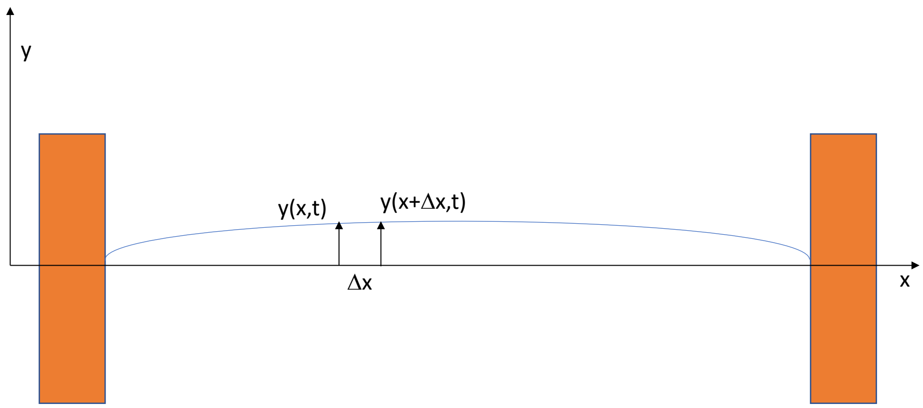

Assume y(x,t) << Δx

Let the ends of a string of length L and mass M be fixed so that the string is under tension T. Neglect gravity. Describe motions of the string.

Write the mass per unit length of the string as

$$\mu \equiv \frac{M}{L}$$

Let the undisturbed string lie along the x-axis and the displacement be in the y-direction, so that the displacement for any small disturbance may be written as a function

$$y\left(x\right)$$

We also want to describe the motion of points along the string, so this function changes in time, hence

$$y = y\left(x, t\right)$$

Consider a small element of the string of length ds. It will have mass $dm = \lambda ds$, where

$$ds = \sqrt{dx^2 + dy^2}$$

and for a given change in x, the change in y is

$$dy = \frac{\partial y}{\partial x} dx$$

We assume this is very small, so that

$$\begin{aligned} ds &= \sqrt{dx^2 + dy^2} \\ &= \sqrt{dx^2 + \left(\frac{\partial y}{\partial x}\right)^2 dx^2} \\ &= dx\sqrt{1 + \left(\frac{\partial y}{\partial x}\right)^2} \end{aligned}$$

Since $dy \ll \frac{\partial y}{\partial x} dx$, we may neglect its square, which is even smaller. Then $ds \approx dx$.

We would like to apply Newton's second law to the motion of the string. The net force on the string is the vector sum of the tensions pulling on the ends of the string. In general these act in slightly different directions, but we know they act in a direction along the string, and always have the same magnitude. A vector along the string will have components dx and dy, so the tension at the left end of our segment is parallel to (dx, dy):

$$\overrightarrow{T}(x) \parallel (dx, dy)$$

We know that the magnitude everywhere is fixed, $\left|\overrightarrow{T}\right| = T$. We may write a unit vector in the direction of $\overrightarrow{T}$ as

$$\begin{aligned}
\hat{T} &= \frac{(dx, dy)}{ds} \\
&\approx \frac{dx\left(1, \frac{\partial y}{\partial x}\right)}{dx} \\
&= \left(1, \frac{\partial y}{\partial x}\right)
\end{aligned}$$

We need to evaluate this at both x and at $x + dx$. We have:

$$\begin{aligned}
\overrightarrow{T}(x) &= -T\left(1, \frac{\partial y}{\partial x}(x)\right) \\
\overrightarrow{T}(x+dx) &= T\left(1, \frac{\partial y}{\partial x}(x+dx)\right)
\end{aligned}$$

How do we find $\frac{\partial y}{\partial x}(x + dx)$? Another Taylor series:

$$\begin{aligned}
\frac{\partial y}{\partial x}(x+dx) &= \frac{\partial y}{\partial x}(x) + \frac{\partial}{\partial x}\left(\frac{\partial y}{\partial x}(x)\right) dx \\
&= \frac{\partial y}{\partial x}(x) + \frac{\partial^2 y}{\partial x^2}(x)\, dx
\end{aligned}$$

Substituting,

$$\begin{aligned}
\overrightarrow{T}(x+dx) &= T\left(1, \frac{\partial y}{\partial x}(x) + \frac{\partial^2 y}{\partial x^2}(x)\, dx\right) \\
&= \overrightarrow{T}(x) + \left(0, \frac{\partial^2 y}{\partial x^2}(x)\right) dx
\end{aligned}$$

The net increment of force, $d\overrightarrow{F}$ is the sum of these tensions,

$$\begin{aligned}
d\overrightarrow{F} &= \overrightarrow{T}(x+dx) + \overrightarrow{T}(x) \\
&= T\left(0, \frac{\partial^2 y}{\partial x^2}(x)\, dx\right) \\
&= T\frac{\partial^2 y}{\partial x^2}(x)\, \hat{\jmath}
\end{aligned}$$

where we neglect the small difference from 1 in the denominator.

Now apply Newton's second law, which applies only in the y direction:

$$\begin{aligned}
\overrightarrow{F} &= m\overrightarrow{a} \\
d\overrightarrow{F} &= dm\left(\frac{\partial^2 y}{\partial t^2}\right)\hat{\jmath} \\
T\frac{\partial^2 y}{\partial x^2} dx &= \mu ds\left(\frac{\partial^2 y}{\partial t^2}\right)
\end{aligned}$$

and once again taking $dx \approx ds$,

$$\frac{\partial^2 y}{\partial x^2}(x) = \frac{\mu}{T}\frac{\partial^2 y}{\partial t^2}$$

For reasons that will be cleare soon, we define

$$\begin{aligned} v &\equiv \sqrt{\frac{T}{\mu}} \\ \left[\sqrt{\frac{T}{\mu}}\right] &= \sqrt{\left(\frac{kg \cdot m}{s^2}\right) / \left(\frac{kg}{m}\right)} \\ &= \sqrt{\frac{m^2}{s^2}} \\ &= \frac{m}{s} \end{aligned}$$

so that v has units of velocity. The equation of motion for the string therefore satisfies

$$-\frac{1}{v^2}\frac{\partial^2 y}{\partial t^2} + \frac{\partial^2 y}{\partial x^2} = 0$$