Solutions to the wave equation

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We have derived the 2-dimensional wave equation

$$-\frac{1}{c^2}\frac{\partial^2 q}{\partial t^2}+\frac{\partial^2 q}{\partial x^2}=0 \tag{1}$$

in two different ways as a continuum limit from Newton's second law. Just as generic potentials give rise to simple harmonic oscillation at any minimum of the potential, extended bodies almost always have wavelike solutions. We now explore some solutions to Eq.(1).

1 Standing wave solution

Recall that in solving for N coupled oscillators, we assumed that the displacements of the N masses varied sinusoidally and that the time dependence was oscillatory, $e^{i\Omega t}$. Let's try the same sort of solution for the wave equation. Let

$$q\left(x,t\right)=A\sin\omega t\sin kx$$

for some constants ω, k. Substituting into Eq.(1),

$$\begin{aligned}
-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\left(A\sin\omega t\sin kx\right)+\frac{\partial^2}{\partial x^2}\left(A\sin\omega t\sin kx\right) &= 0\\
-\frac{1}{c^2}\left(-\omega^2\right)\left(A\sin\omega t\sin kx\right)-k^2\left(A\sin\omega t\sin kx\right) &= 0
\end{aligned}$$

Collecting terms

$$\left(\frac{\omega^2}{c^2}-k^2\right)A\sin\omega t\sin kx = 0$$

we see that we have a solution as long as ω and k are related by

$$k=\pm\frac{\omega}{c}$$

Notice that ω has units of frequency and k, called the wave number, has units of inverse length. The amplitude, A, and the frequency, ω, are arbitrary, so we may write

$$\begin{aligned}
q\left(x,t\right) &= A\sin\omega t\sin kx\\
&= A\sin\omega t\sin\frac{\omega x}{c}
\end{aligned}$$

for any ω, A.

1.1 Boundary conditions and normal modes

In deriving Eq.(1) we let the number of oscillators become infinite, $N \to \infty$. Since the number of normal modes of an N oscillator system is N, we might guess that there will now be infinitely many normal modes. This is exactly right.

To find the normal modes, we must impose boundary conditions. Let the wave be a guitar string of length L, fixed at both ends. Measuring x from one end, we therefore require at $x = 0$ and $x = L$

$$\begin{aligned} q(0,t) &= 0 \\ q(L,t) &= 0 \end{aligned}$$

The first of these is automatically satisfied, $q(0,t) = A\sin\omega t \sin 0 = 0$. For the second

$$\begin{aligned} 0 &= q(L,t) \\ &= A\sin\omega t \sin kL \end{aligned}$$

Since this must hold at all times t, we demand

$$\sin kL = 0$$

so that $\frac{\omega L}{c}$ must be a multiple of π,

$$\begin{aligned} kL &= n\pi \\ k &= \frac{n\pi}{L} \end{aligned}$$

and for the possible frequencies,

$$\omega = \frac{n\pi c}{L}$$

The normal modes of the string are therefore

$$\begin{aligned} q(x,t) &= A\sin\omega t \sin kx \\ &= A\sin\left(\frac{n\pi ct}{L}\right)\sin\left(\frac{n\pi x}{L}\right) \end{aligned}$$

There is a simple way to think of this. The x-dependent terms gives the shape of an envelope of the motion, while the time dependence produces oscillation within this spatial limit. For example, when $n = 1$ the spatial dependence is

$$A\sin\left(\frac{\pi x}{L}\right)$$

As x runs from 0 to L, $A\sin\left(\frac{\pi x}{L}\right)$ runs through $\frac{1}{2}$ cycle, from a zero through a single maximum then back to zero–a half wavelength. The remaining time dependence $\sin\left(\frac{n\pi ct}{L}\right)$ causes movement of the string up and down within the limits of this half wave.

Changing n increases the number of nodes. For $n = 2$ the envelope comprises an entire wavelength, for $n = 3$ there are two nodes and $1\frac{1}{2}$ wavelengths. In general, the envelope includes $\frac{n}{2}$ wavelengths, so the wavelength is $\lambda = \frac{2L}{n}$. Therefore,

$$\begin{aligned} \lambda &= \frac{2L}{n} \\ &= \frac{2\pi}{k} \end{aligned}$$

since $k = \frac{n\pi}{L}$. Thus, the wave number k is related to the wavelength by

$$k = \frac{2\pi}{\lambda}$$

If instead, we look at a fixed position and ask for the time between successive peaks, we find the period, T, satisfies

$$\begin{aligned} \omega T &= 2\pi \\ \omega &= \frac{2\pi}{T} \end{aligned}$$

1.2 Traveling waves

A standing wave may be understood as a superposition of two traveling waves moving in opposite directions.

Suppose we write another solution to the wave equation,

$$q(x,t) = Re\left\{Ae^{i\omega t}\cos kx\right\} \tag{2}$$

with A complex. Letting $A = 2A_0e^{-i\varphi_0}$ so that

$$q(x,t) = Re\left\{2A_0e^{i(\omega t-\varphi_0)}\cos kx\right\}$$

and the real part is just

$$q(x,t) = 2A_0\cos(\omega t-\varphi_0)\cos kx$$

For simplicity, choose zero phase at the initial time, $t_0 = 0$, so that $\varphi_0 = 0$. This is another standing wave with a shifted phase for the spatial part (since $\cos kx = \sin\left(kx+\frac{\pi}{2}\right)$).

However, suppose we start again with Eq.(2) and write the cosine as

$$\cos kx = \frac{1}{2}\left(e^{ikx}+e^{-ikx}\right)$$

Then (still setting $\varphi_0 = 0$ for simplicity) $q(x,t)$ becomes

$$\begin{aligned} q(x,t) &= Re\left\{2A_0e^{i\omega t}\left(\frac{1}{2}\left(e^{ikx}+e^{-ikx}\right)\right)\right\} \\ &= Re\left\{A_0e^{i\omega t}\left(e^{ikx}+e^{-ikx}\right)\right\} \\ &= Re\left\{A_0\left(e^{i(\omega t+kx)}+e^{i(\omega t-kx)}\right)\right\} \\ &= A_0\cos(\omega t+kx)+A_0\cos(\omega t-kx) \end{aligned}$$

First, we check that $A_0\cos(\omega t\pm kx)$ satisfies the wave equation,

$$\begin{aligned} -\frac{1}{c^2}\frac{\partial^2 q}{\partial t^2}+\frac{\partial^2 q}{\partial x^2} &= -\frac{A_0}{c^2}\frac{\partial^2}{\partial t^2}\left(\cos(\omega t\pm kx)\right)+A_0\frac{\partial^2}{\partial x^2}\left(\cos(\omega t\pm kx)\right) \\ &= -\frac{A_0}{c^2}\left(-\omega^2\right)\cos(\omega t\pm kx)+A_0\left(-(\pm k)^2\right)\cos(\omega t\pm kx) \\ &= \left(\frac{\omega^2}{c^2}-k^2\right)A_0\cos(\omega t\pm kx) \end{aligned}$$

so once again we have a solution as long as $k = \frac{\omega}{c}$.

The solution

$$q(x,t) = A_0\cos(\omega t+kx)+A_0\cos(\omega t-kx)$$

is a superposition of a right moving wave and a left moving wave. We can see this as follows.

How do these waves appear to move? Since a maximum of the cosine occurs when its argument is a multiple of 2π, the maximum of the first term occurs at any x and t such that

$$\omega t + kx = 2n\pi$$

Solving for x,

$$x^L_{max} = \frac{2n\pi}{k} - \frac{\omega}{k}t$$

we see that the position of any one maximum of the first cosine moves with velocity

$$\frac{dx^L_{max}}{dt} = -\frac{\omega}{k} = -c$$

For the second cosine, the position of any given maximum moves with velocity

$$\frac{dx^R_{max}}{dt} = +c$$

The solution therefore represents a superposition of a wave with crests moving to the left and a wave with crests moving to the right.

We have shown that a standing wave may also be seen as a superposition of a right moving wave and a left moving wave.

2 General solutions

2.1 Superposition

Because the wave equation is linear, we may take arbitrary linear combinations of solutions. Most generally, we may sum solutions with any frequency ω, and the amplitude A may be different for each different ω. We may add together n solutions with frequencies $\omega_m, m = 1, \ldots, n$,

$$q(x,t) = \sum_{m=1}^{n} A(\omega_m) e^{i\omega_m t} \sin \frac{\omega_m x}{c}$$

While this can give a wide variety of waveforms, we can write an even more general wave by taking a superposition of *all* values of ω and integrating. Let $\mathcal{A}(\omega)$ be the average amplitude per unit frequency in an infinitisimal range $d\omega$. Then

$$q(x,t) = \int_0^\infty \mathcal{A}(\omega) e^{i\omega t} \sin \frac{\omega x}{c} d\omega$$

is still a solution to the wave equation. The amplitude density can give the wave any shape we like.

2.2 Integration

We can sometimes directly integrate partial differential equations. The 2-dimensional wave equation takes a simpler form if we change variables. Let

$$\begin{aligned} u &= x - ct \\ v &= x + ct \end{aligned}$$

We use the chain rule to transform the derivatives of any function $q(x,t)$:

$$\begin{aligned}
\frac{\partial q}{\partial x} &= \frac{\partial u}{\partial x}\frac{\partial q}{\partial u} + \frac{\partial v}{\partial x}\frac{\partial q}{\partial v} \\
&= \frac{\partial q}{\partial u} + \frac{\partial q}{\partial v} \\
\frac{\partial q}{\partial t} &= \frac{\partial u}{\partial t}\frac{\partial q}{\partial u} + \frac{\partial v}{\partial t}\frac{\partial q}{\partial v} \\
&= -c\frac{\partial q}{\partial u} + c\frac{\partial q}{\partial v}
\end{aligned}$$

Since this is true for any function q, we may write it as a relation between the derivative operators,

$$\begin{aligned}
\frac{\partial}{\partial x} &= \frac{\partial u}{\partial x}\frac{\partial}{\partial u} + \frac{\partial v}{\partial x}\frac{\partial}{\partial v} \\
&= \frac{\partial}{\partial u} + \frac{\partial}{\partial v} \\
\frac{\partial}{\partial t} &= \frac{\partial u}{\partial t}\frac{\partial}{\partial u} + \frac{\partial v}{\partial t}\frac{\partial}{\partial v} \\
&= -c\frac{\partial}{\partial u} + c\frac{\partial}{\partial v}
\end{aligned}$$

Therefore, in terms of u and v the wave equation becomes

$$\begin{aligned}
0 &= -\frac{1}{c^2}\frac{\partial^2 q}{\partial t^2} + \frac{\partial^2 q}{\partial x^2} \\
&= -\frac{1}{c^2}\left(-c\frac{\partial}{\partial u} + c\frac{\partial}{\partial v}\right)\left(-c\frac{\partial q}{\partial u} + c\frac{\partial q}{\partial v}\right) + \left(\frac{\partial}{\partial u} + \frac{\partial}{\partial v}\right)\left(\frac{\partial q}{\partial u} + \frac{\partial q}{\partial v}\right) \\
&= -\left(\frac{\partial^2 q}{\partial u^2} - \frac{\partial^2 q}{\partial v \partial u} - \frac{\partial^2 q}{\partial u \partial v} + \frac{\partial^2 q}{\partial v^2}\right) + \left(\frac{\partial^2 q}{\partial u^2} + \frac{\partial^2 q}{\partial v \partial u} + \frac{\partial^2 q}{\partial u \partial v} + \frac{\partial^2 q}{\partial v^2}\right) \\
&= 4\frac{\partial^2 q}{\partial v \partial u}
\end{aligned}$$

We may divide away the 4 and look at the resulting equation as

$$\frac{\partial}{\partial v}\left(\frac{\partial q}{\partial u}\right) = 0$$

This shows that $\frac{\partial q}{\partial u}$ is independent of v so we may write

$$\frac{\partial q}{\partial u} = F(u)$$

for an arbitrary function $F(u)$. We may now integrate over u,

$$q(u,v) = \int F(u)\, du + g$$

but we had only a partial derivative. This means that the "constant" we add may be a function of v:

$$q(u,v) = \int F(u)\, du + g(v)$$

Check that this does satisfy $\frac{\partial q}{\partial u} = F(u)$, since the partial derivative of $g(v)$ with respect to u is zero. Finally, we may rewrite the indefinite integral over u as a new function of u,

$$f(u) := \int F(u)\, du$$

so that the full solution is

$$q(u,v) = f(u) + g(v)$$

with the functions f and g arbitrary. Substituting back,

$$q(x,t) = f(x-ct) + g(x+ct) \tag{3}$$

This form has a clear physical interpretation for a wave on a string. At $t=0$ the disturbance of the string is given by $f(x) + g(x)$; as time progresses, two waveforms move apart. One has the same shape as $f(x)$ but moves to the right at speed c, while the other maintains the shape $g(x)$ but moves to the left with speed c.

Either $f(x-ct)$ alone or $g(x+ct)$ alone still solves the wave equation. This means that if we choose a solution with $f(x-ct)$ alone, it describes a wave with an arbitrary shape $f(x)$ when $t=0$. *This shape is preserved* the wave moves, but the shape moves off to the right with speed c. Similarly, $g(x+ct)$ alone describes an initial shape $g(x)$ that moves to the left without distorting.

2.3 Initial value solution and d'Alembert's formula

Our solution for a wave on a string becomes unique when we impose boundary conditions such as holding the endpoints of the string fixed. Another common form for specifying a unique solution–as you have seen in solving Newton's second law–is the *initial value formulation.* For Newton's second law, specifying initial values amounts to giving the initial position and velocity vectors for each particle in the system. When we change to a continuum equation, this means we need a continuum of initial values.

Looking at our general solution, Eq.(3) we see that at time $t=0$ we still have

$$q(x,0) = f(x) + g(x)$$

where $f(x)$ and $g(x)$ remain arbitrary. This sum corresponds to the initial displacement of every particle of the system (string, slinky, organ pipe, etc.). We may also ask for the initial velocity,

$$\begin{aligned} \dot{q}(x,0) &= \left.\frac{df}{du}\right|_{t=0} \frac{\partial u}{\partial t} + \left.\frac{df}{dv}\right|_{t=0} \frac{\partial v}{\partial t} \\ &= -cf'(x) + cg'(x) \end{aligned}$$

We now show that specifying these two functions gives a unique solution via a general formula due to d'Alembert.

Let

$$\begin{aligned} q(x,0) &= : a(x) \\ \dot{q}(x,0) &= : b(x) \end{aligned}$$

be given functions. Then, writing equations for $f(x)$ and $g(x)$, we have

$$\begin{aligned} f(x) + g(x) &= a(x) \\ c(-f'(x) + g'(x)) &= b(x) \end{aligned}$$

where f' and g' denote ordinary derivatives of f and g with respect to their arguments, then evaluated at $u(x,0) = x$ and $v(x,0) = x$. We may rewrite the second equation using an arbitrary variable y as

$$c\left(-\frac{df}{dy}(y) + \frac{dg}{dy}(y)\right) = b(y)$$

Multiply by dy and integrate from any initial value x_0 to x,

$$\begin{aligned}
c\left(-\frac{df}{dy}(y)+\frac{dg}{dy}(y)\right)dy &= b(y)\,dy \\
-\int_{x_0}^{x} df(y)+\int_{x_0}^{x} dg(y) &= \frac{1}{c}\int_{x_0}^{x} b(y)\,dy \\
-f(x)+f(x_0)+g(x)-g(x_0) &= \frac{1}{c}\int_{x_0}^{x} b(y)\,dy \\
-f(x)+g(x) &= \frac{1}{c}\int_{x_0}^{x} b(y)\,dy-f(x_0)+g(x_0)
\end{aligned}$$

Now we can solve,

$$\begin{aligned}
f(x)+g(x) &= a(x) \\
-f(x)+g(x) &= \frac{1}{c}\int_{x_0}^{x} b(y)\,dy-f(x_0)+g(x_0) \\
g(x) &= \frac{1}{2}\left[a(x)+\frac{1}{c}\int_{x_0}^{x} b(y)\,dy-f(x_0)+g(x_0)\right] \\
f(x) &= \frac{1}{2}\left[a(x)-\frac{1}{c}\int_{x_0}^{x} b(y)\,dy+f(x_0)-g(x_0)\right]
\end{aligned}$$

where

$$g(x_0)+f(x_0) = a(x_0)$$

The full solution for the displacement at *all* times is therefore,

$$\begin{aligned}
q(x,t) &= f(x-ct)+g(x+ct) \\
&= \frac{1}{2}\left[a(x-ct)-\frac{1}{c}\int_{x_0}^{x-ct} b(y)\,dy+f(x_0)-g(x_0)\right]+\frac{1}{2}\left[a(x+ct)+\frac{1}{c}\int_{x_0}^{x+ct} b(y)\,dy-f(x_0)+g(x_0)\right] \\
&= \frac{1}{2}\left(a(x-ct)+a(x+ct)-\frac{1}{c}\int_{x_0}^{x-ct} b(y)\,dy+f(x_0)-g(x_0)+\frac{1}{c}\int_{x_0}^{x+ct} b(y)\,dy-f(x_0)+g(x_0)\right) \\
&= \frac{1}{2}\left(a(x-ct)+a(x+ct)-\frac{1}{c}\int_{x_0}^{x-ct} b(y)\,dy+\frac{1}{c}\int_{x_0}^{x+ct} b(y)\,dy\right)
\end{aligned}$$

The difference of the two integrals is

$$\begin{aligned}
-\frac{1}{c}\int_{x_0}^{x-ct} b(y)\,dy+\frac{1}{c}\int_{x_0}^{x+ct} b(y)\,dy &= \frac{1}{c}\int_{x-ct}^{x_0} b(y)\,dy+\frac{1}{c}\int_{x_0}^{x+ct} b(y)\,dy \\
&= \frac{1}{c}\int_{x-ct}^{x+ct} b(y)\,dy
\end{aligned}$$

so the final expression for $q(x,t)$ is

$$q(x,t) = \frac{1}{2}\left(a(x-ct) + a(x+ct) + \frac{1}{c}\int\limits_{x-ct}^{x+ct} b(y)\,dy\right)$$

This is d'Alembert's formula. Notice that it depends only on the two functions $a(x)$ and $b(x)$ with all reference to x_0 dropping out.

It is now clear that

3 Gaussian wave packets

An important class of solutions is described by a *Gaussian* or *normal* distribution:

$$a(x) = Ae^{-\frac{x^2}{2\sigma^2}}$$

This function has a maximum at $x = 0$ and falls off smoothly to zero as x approaches infinity. The speed of the fall-off is determined by the constant σ (called the standard deviation in statistics). When $x = \sqrt{2}\sigma$ the value has dropped to $\frac{1}{e} = .37...$ of its original value A.

3.1 Normalizing

Often, the constant A is chosen so that the area under the Gaussian is one,

$$1 = A\int\limits_{-\infty}^{\infty} e^{-\frac{x^2}{2\sigma^2}}\,dx$$

There is a clever trick for evaluating this integral. Let

$$I = e^{-\frac{x^2}{2\sigma^2}}$$

and consider I^2, written as two independent integrals,

$$\begin{aligned} I^2 &= \int\limits_{-\infty}^{\infty} e^{-\frac{x^2}{2\sigma^2}}\,dx \int\limits_{-\infty}^{\infty} e^{-\frac{y^2}{2\sigma^2}}\,dy \\ &= \int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty} e^{-\frac{\left(x^2+y^2\right)}{2\sigma^2}}\,dxdy \end{aligned}$$

If we think of this as an integration over the whole xy plane, we may rewrite it in polar coordinates. Since $r^2 = x^2 + y^2$ and an area element in polar coordinates is $rdrd\varphi$, we may change coordinates and write

$$\begin{aligned} I^2 &= \int\limits_{0}^{\infty} e^{-\frac{r^2}{2\sigma^2}}\,rdr \int\limits_{0}^{2\pi} d\varphi \\ &= 2\pi\int\limits_{0}^{\infty} e^{-\frac{r^2}{2\sigma^2}}\,rdr \end{aligned}$$

where the angular integral is trivial. A simple substitution, $w = r^2$ with $dw = 2rdr$, lets us perform the second integral,

$$\begin{aligned}
\int_0^\infty e^{-\frac{r^2}{2\sigma^2}} rdr &= \int_0^\infty e^{-\frac{w}{2\sigma^2}} \left(\frac{1}{2}dw\right) \\
&= \frac{1}{2}\int_0^\infty e^{-\frac{w}{2\sigma^2}} dw \\
&= -\sigma^2 e^{-\frac{w}{2\sigma^2}} \Big|_0^\infty \\
&= -\sigma^2 e^{-\frac{\infty}{2\sigma^2}} + \sigma^2 e^{-0} \\
&= \sigma^2
\end{aligned}$$

Therefore,

$$\begin{aligned}
I^2 &= 2\pi\sigma^2 \\
I &= \sqrt{2\pi\sigma^2}
\end{aligned}$$

Choosing $A = \frac{1}{\sqrt{2\pi\sigma^2}}$ gives us the result

$$\frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} e^{-\frac{x^2}{2\sigma^2}} dx = 1$$

This normalized Gaussian is used in probability and in quantum mechanics. We will use it soon to give a description of the Dirac delta function.

3.2 Gaussian waves

Choose a wave with the initial conditions

$$\begin{aligned}
a(x) &= Ae^{-\frac{x^2}{2\sigma^2}} \\
b(x) &= 0
\end{aligned}$$

Then we have the immediate solution

$$q(x,t) = \frac{A}{2}\left(e^{-\frac{(x-ct)^2}{2\sigma^2}} + e^{-\frac{(x+ct)^2}{2\sigma^2}}\right)$$

This describes a pair of Gaussian shapes of half the amplitude, moving off in opposite directions.