Three dimensional waves

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1 Waves in new directions

We have seen that a 1-dimensional wave may be described by

$$-\frac{1}{c^2}\frac{\partial^2 q}{\partial t^2}+\frac{\partial^2 q}{\partial x^2}=0$$

In three dimensions, we need to replace the spatial derivative $\frac{\partial^2 q}{\partial x^2}$ with a combination of x, y and z derivatives. Since we still expect to have 1-dimensional solutions, but in any direction it is natural to guess the form[1]

$$-\frac{1}{c^2}\frac{\partial^2 q}{\partial t^2}+\frac{\partial^2 q}{\partial x^2}+\frac{\partial^2 q}{\partial y^2}+\frac{\partial^2 q}{\partial z^2}=0 \tag{1}$$

where $q = q(\vec{\mathbf{x}}, t)$. This lets us immediately write 1-dimensional solutions of the form

$$q \quad = \quad f(z-ct)+g(z+ct)$$

for x, y or z. This combination of spatial derivatives, called the *Laplacian*, is written as

$$\nabla^2 \equiv \frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}$$

This simplifies writing the wave equation Eq.(1) to

$$-\frac{1}{c^2}\frac{\partial^2 q}{\partial t^2}+\nabla^2 q=0 \tag{2}$$

This wave equation, in Cartesian, cylindrical, and spherical coordinates, will occupy us for some time.

2 Grad and div

There are three different geometric derivatives we will need:

1. The *gradient*, called *grad*.
2. The *divergence*, often simply called *div*.
3. The *curl*.

Each of these has definite geometric content. The gradient tells us the magnitude and direction of the change in a function; the divergence tells us how strongly vectors spread out from any given point, while the curl tells how much vectors circle around a point. We will discuss the gradient and divergence now, posponing the curl until we need it.

[1] You might wonder about more general forms involving mixed derivatives. It turns out that by a proper choice of coordinates, any constant mix of second derivatives may be transformed to this form.

2.1 The gradient

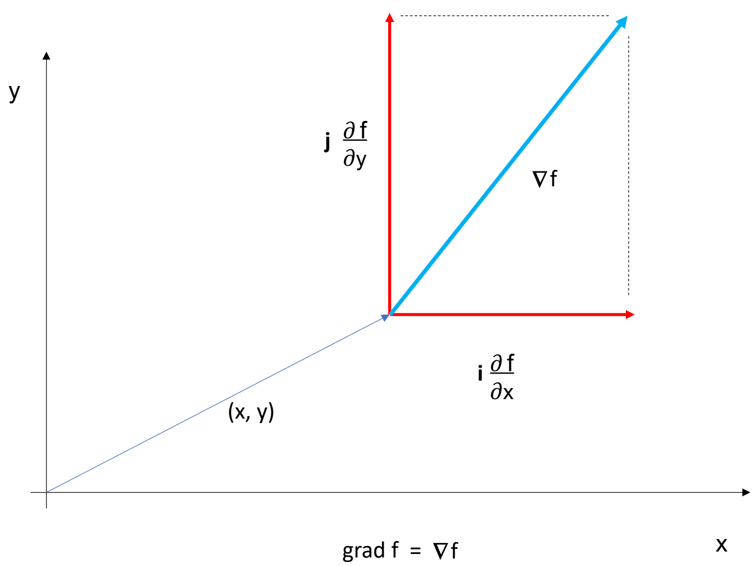
Suppose we associate the direction of increase of a function with its derivative. For a function of one variable we could write

$$\hat{\mathbf{i}}\frac{df}{dx}$$

to indicate that$f$ is increasing in the x-direction when the derivative is positive, and the negative x-direction when $\frac{df}{dx}$ is negative. This makes more sense in a plane, where we might write

$$\hat{\mathbf{i}}\frac{\partial f}{\partial x}+\hat{\mathbf{j}}\frac{\partial f}{\partial y}$$

for a function of two variables, $f(x,y)$. This is now a vector in the plane, with components equal to the rate of change of f in the corresponding directions.



The resulting vector is called the *gradient*, or *grad* f and written with an upside-down delta symbol called *del*. The combination *del* f = *grad* $f = \vec{\boldsymbol{\nabla}} f$ is a vector. In 3-dimensions, for any function $f(x,y,z)$, we have the obvious extension,

$$\vec{\boldsymbol{\nabla}} f=\hat{\mathbf{i}}\frac{\partial f}{\partial x}+\hat{\mathbf{j}}\frac{\partial f}{\partial y}+\hat{\mathbf{k}}\frac{\partial f}{\partial z}$$

Sometimes we refer to the gradient operator alone,

$$\vec{\boldsymbol{\nabla}}=\hat{\mathbf{i}}\frac{\partial}{\partial x}+\hat{\mathbf{j}}\frac{\partial}{\partial y}+\hat{\mathbf{k}}\frac{\partial}{\partial z}$$

We can use the gradient to find the total change of a function in any direction. If $\hat{\mathbf{n}}$ is a unit vector in the desired direction then the usual dot product of two vectors

$$\hat{\mathbf{n}}\cdot\vec{\boldsymbol{\nabla}} f$$

is the directional derivative, that is, the rate of change of the function f in the $\hat{\mathbf{n}}$ direction. As with any vector, $\hat{\mathbf{n}}\cdot\vec{\boldsymbol{\nabla}} f$ gives the component of $\vec{\boldsymbol{\nabla}} f$ in the $\hat{\mathbf{n}}$ direction. The operator

$$\hat{\mathbf{n}}\cdot\vec{\boldsymbol{\nabla}}$$

is called a directional derivative operator.

If we find a vector $\hat{\mathbf{n}}$ orthogonal to grad f, the dot product gives zero. This $\hat{\mathbf{n}}$ gives a direction in which the function f is not changing. The collection of all such directions gives the level surfaces of the function f, surfaces on which f is constant.

2.2 The divergence

Suppose we have a vector field, $\vec{\mathbf{v}}\left(\vec{\mathbf{x}}\right)$, i.e., a vector assigned to each point of space in a smoothly changing way. We may apply the del operator, $\vec{\boldsymbol{\nabla}}$, to a vector so that the derivatives operate on the vector field. Instead of a directional derivative $\vec{\mathbf{v}}\cdot\vec{\boldsymbol{\nabla}}$ we have the *divergence*,

$$div\,\vec{\mathbf{v}} = \vec{\boldsymbol{\nabla}}\cdot\vec{\mathbf{v}}$$

Writing out the derivatives and the dot product, we have

$$\vec{\boldsymbol{\nabla}}\cdot\vec{\mathbf{v}} = \left(\hat{\mathbf{i}}\frac{\partial}{\partial x}+\hat{\mathbf{j}}\frac{\partial}{\partial y}+\hat{\mathbf{k}}\frac{\partial}{\partial z}\right)\cdot\left(\hat{\mathbf{i}}v_x+\hat{\mathbf{j}}v_y+\hat{\mathbf{k}}v_z\right)$$

Since the Cartesian unit vectors, $\hat{\mathbf{i}},\hat{\mathbf{j}},\hat{\mathbf{k}}$ are the same at all points, their derivatives are zero. Carrying out the dot product in the usual way we have

$$\vec{\boldsymbol{\nabla}}\cdot\vec{\mathbf{v}} = \frac{\partial v_x}{\partial x}+\frac{\partial v_y}{\partial y}+\frac{\partial v_z}{\partial z}$$

The divergence tells us how much the vector field $\vec{\mathbf{v}}$ moves away from a given point or region. This is easily seen if we integrate it over some volume.

$$\iiint\limits_V \vec{\boldsymbol{\nabla}}\cdot\vec{\mathbf{v}}\,dx\,dy\,dz$$

To evaluate this over an arbitrary volume, we first start with a simple box with one corner at the origin and sides of lengths a, b and c respectively. Then we have

$$\iiint\limits_{box} \vec{\boldsymbol{\nabla}}\cdot\vec{\mathbf{v}}\,dx\,dy\,dz = \int_0^a dx\int_0^b dy\int_0^c dz\left(\frac{\partial v_x}{\partial x}+\frac{\partial v_y}{\partial y}+\frac{\partial v_z}{\partial z}\right)$$

The trick here is choosing which integral to do first. Rearrange so that we do the x integral of $\frac{\partial v_x}{\partial x}$, the y integral of $\frac{\partial v_y}{\partial y}$ and the z integral of $\frac{\partial v_z}{\partial z}$ first:

$$\begin{aligned}
\iiint\limits_{box} \vec{\boldsymbol{\nabla}}\cdot\vec{\mathbf{v}}\,dx\,dy\,dz &= \int_0^b dy\int_0^c dz\int_0^a dx\frac{\partial v_x}{\partial x}+\int_0^a dx\int_0^c dz\int_0^b dy\frac{\partial v_y}{\partial y}+\int_0^a dx\int_0^b dy\int_0^c dz\frac{\partial v_z}{\partial z} \\
&= \int_0^b dy\int_0^c dz\left[v_x\left(a,y,z\right)-v_x\left(0,y,z\right)\right] \\
&\quad+\int_0^a dx\int_0^c dz\left[v_y\left(x,b,z\right)-v_y\left(x,0,z\right)\right] \\
&\quad+\int_0^a dx\int_0^b dy\left[v_z\left(x,y,c\right)-v_z\left(x,y,0\right)\right]
\end{aligned}$$

Next, notice that the remaining integrals are over the boundary of the box. For example, the final double integral

$$\int_0^a dx \int_0^b dy \left[v_z(x,y,c) - v_z(x,y,0)\right] = \int_0^a dx \int_0^b dy\, v_z(x,y,c) - \int_0^a dx \int_0^b dy\, v_z(x,y,0)$$

is the difference between the z component of the vector at the top of the box, integrated over the top, minus the z-component of the vector at the bottom of the box, integrated over the bottom. We can write the right side of the equation as

$$\int_0^a dx \int_0^b dy \left[v_z(x,y,c) - v_z(x,y,0)\right] = \iint_{top} d^2x\, \hat{\mathbf{n}} \cdot \vec{\mathbf{v}}(x,y,c) + \iint_{bottom} d^2x\, \hat{\mathbf{n}} \cdot \vec{\mathbf{v}}(x,y,0)$$

where we understand the unit vector $\hat{\mathbf{n}}$ to be the *outward normal* to the surface. By specifying outward, we may change the sign of the integral over the bottom of the box. The remaining four terms may be written in the same way, always understanding $\hat{\mathbf{n}}$ to be the outward normal and the vector $\vec{\mathbf{v}}$ to be restricted to the remaining surface of integration:

$$\iiint_{box} \vec{\boldsymbol{\nabla}} \cdot \vec{\mathbf{v}}\, dx\, dy\, dz = \iint_{surface} d^2x\, \hat{\mathbf{n}} \cdot \vec{\mathbf{v}}$$

Now that we have the result for a box, we generalize to an arbitrary volume by dividing it into infinitesimal boxes and adding the results. Wherever to boxes abut one another the integrals cancel because the outward normals are in opposite directions, with the integrals over $\vec{\mathbf{v}}$ being otherwise the same. The result is that on the left, the sum of integrals over all the boxes gives the integral of the divergence over the whole volume, while on the right the integral extends over exactly the unmatched part of the boundary. Thus, for any volume V with surface S,

$$\iiint_V \vec{\boldsymbol{\nabla}} \cdot \vec{\mathbf{v}}\, dx\, dy\, dz = \iint_S d^2x\, \hat{\mathbf{n}} \cdot \vec{\mathbf{v}}$$

where $\hat{\mathbf{n}}$ at any given point of S is the outward normal to S at that point. This is the *divergence theorem.*

The divergence theorem gives the best interpretation of the meaning of the divergence. The divergence of a vector field summed over any volume is equal to the amount of the vector field crossing out of the volume across its boundary.

2.3 The Laplacian

Using the divergence and the gradient together gives us the Laplacian. Starting with any function, $f(\hat{\mathbf{x}})$ we first find the gradient,

$$\vec{\boldsymbol{\nabla}} f = \hat{\mathbf{i}}\frac{\partial f}{\partial x} + \hat{\mathbf{j}}\frac{\partial f}{\partial y} + \hat{\mathbf{k}}\frac{\partial f}{\partial z}$$

Since the gradient is a vector field, we may find its divergence,

$$\begin{aligned}
\vec{\boldsymbol{\nabla}} \cdot \vec{\boldsymbol{\nabla}} f &= \left(\hat{\mathbf{i}}\frac{\partial}{\partial x} + \hat{\mathbf{j}}\frac{\partial}{\partial y} + \hat{\mathbf{k}}\frac{\partial}{\partial z}\right) \cdot \left(\hat{\mathbf{i}}\frac{\partial f}{\partial x} + \hat{\mathbf{j}}\frac{\partial f}{\partial y} + \hat{\mathbf{k}}\frac{\partial f}{\partial z}\right) \\
&= \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} \\
&= \nabla^2 f
\end{aligned}$$

The result is the Laplacian. Thus, the Laplacian tells us how much the change of a function is spreading out from any point. You are now allowed to say "div grad equals del-squared".

3 Solutions to the wave equation

Now return to the wave equation, Eq.(2),

$$-\frac{1}{c^2}\frac{\partial^2 q}{\partial t^2}+\nabla^2 q = 0$$

3.1 Plane waves

Generalizing our solution, $q(x,t)=Ae^{i(kx-\omega t)}$ for the 1-dimensional case, we introduce a vector $\overrightarrow{\mathbf{k}}$ (the wave vector) and write

$$q(\mathbf{x},t)=A\exp i\left(\overrightarrow{\mathbf{k}}\cdot\overrightarrow{\mathbf{x}}-\omega t\right)$$

Substituting into the wave equation, we find

$$\begin{aligned}
0 &= -\frac{1}{c^2}\frac{\partial^2 q}{\partial t^2}+\nabla^2 q\\
0 &= -\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\left(A\exp i\left(\overrightarrow{\mathbf{k}}\cdot\overrightarrow{\mathbf{x}}-\omega t\right)\right)+\nabla^2\left(A\exp i\left(\overrightarrow{\mathbf{k}}\cdot\overrightarrow{\mathbf{x}}-\omega t\right)\right)\\
&= -\frac{1}{c^2}(-i\omega)^2 A\exp i\left(\overrightarrow{\mathbf{k}}\cdot\overrightarrow{\mathbf{x}}-\omega t\right)+\left(\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}+\frac{\partial^2 f}{\partial z^2}\right)\left(A\exp i\left(\overrightarrow{\mathbf{k}}\cdot\overrightarrow{\mathbf{x}}-\omega t\right)\right)\\
&= -\frac{1}{c^2}(-i\omega)^2 A\exp i\left(\overrightarrow{\mathbf{k}}\cdot\overrightarrow{\mathbf{x}}-\omega t\right)+\left((ik_x)^2+(ik_y)^2+(ik_z)^2\right)\left(A\exp i\left(\overrightarrow{\mathbf{k}}\cdot\overrightarrow{\mathbf{x}}-\omega t\right)\right)\\
&= \left(\frac{\omega^2}{c^2}-\overrightarrow{\mathbf{k}}^2\right)\left(A\exp i\left(\overrightarrow{\mathbf{k}}\cdot\overrightarrow{\mathbf{x}}-\omega t\right)\right)
\end{aligned}$$

Therefore, we have a solution as long as the magnitude of $\overrightarrow{\mathbf{k}}$ equals $\frac{\omega}{c}$. If we write $\overrightarrow{\mathbf{k}}=k\hat{\mathbf{n}}$ then the unit vector $\hat{\mathbf{n}}$ gives the direction of the wave vector and the magnitude satisfies

$$\omega = kc$$

This solution is called a plane wave, because in any of the planes defined by

$$\overrightarrow{\mathbf{k}}\cdot\overrightarrow{\mathbf{x}}=k\hat{\mathbf{n}}\cdot\overrightarrow{\mathbf{x}}=0$$

all points simply oscillate in unison. For example, if we choose $\hat{\mathbf{n}}=\hat{\mathbf{j}}$ then the wave moves in the y-direction. Points on the $y=0$ plane all have the solution

$$q(x,0,z,t)=Ae^{-i\omega t}$$

independently of x,z. The same is true for any fixed value $y=y_0$.

3.2 General solution

Plane waves are normal mode solutions. We may take superpositions in three dimensions in much the same way as for simpler systems. If the waves extend over all space, this leads to a Fourier transform. It is instructive to perform the transformation of all four variables, x,y,z,t. Writing this out in exquisite detail, we may write any real function of $(\mathbf{x},t)$ as

$$\begin{aligned}
q(\mathbf{x},t) &= \frac{1}{\left(\sqrt{2\pi}\right)^4}\int_{-\infty}^{\infty}dk_x\int_{-\infty}^{\infty}dk_y\int_{-\infty}^{\infty}dk_z\int_{-\infty}^{\infty}d\omega\left[A(\mathbf{k},\omega)e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)}+A^*(\mathbf{k},\omega)e^{-i(\mathbf{k}\cdot\mathbf{x}-\omega t)}\right]\\
&= \frac{1}{4\pi^2}\int_{-\infty}^{\infty}d^3k\,d\omega\left[A(\mathbf{k},\omega)e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)}+A^*(\mathbf{k},\omega)e^{-i(\mathbf{k}\cdot\mathbf{x}-\omega t)}\right]
\end{aligned}$$

To restrict this general expression to solve the wave equation we apply the wave equation to both sides, demanding

$$\begin{aligned}
0 &= -\frac{1}{c^2}\frac{\partial^2 q}{\partial t^2}(\mathbf{x},t) + \nabla^2 q(\mathbf{x},t) \\
&= \left(-\frac{1}{c^2}\frac{\partial^2}{\partial t^2} + \nabla^2\right)\frac{1}{4\pi^2}\int_{-\infty}^{\infty} dk_x \int_{-\infty}^{\infty} dk_y \int_{-\infty}^{\infty} dk_z \int_{-\infty}^{\infty} d\omega \left[A(\mathbf{k},\omega)\, e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)} + A^*(\mathbf{k},\omega)\, e^{-i(\mathbf{k}\cdot\mathbf{x}-\omega t)}\right]
\end{aligned}$$

Interchanging the order of integration and differentiation and taking the derivatives leads to

$$\begin{aligned}
0 &= \frac{1}{4\pi^2}\int_{-\infty}^{\infty} d^3k\, d\omega \left(-\frac{1}{c^2}\frac{\partial^2}{\partial t^2} + \nabla^2\right)\left[A(\mathbf{k},\omega)\, e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)} + A^*(\mathbf{k},\omega)\, e^{-i(\mathbf{k}\cdot\mathbf{x}-\omega t)}\right] \\
&= \frac{1}{4\pi^2}\int_{-\infty}^{\infty} d^3k\, d\omega \left[\left(-\frac{(-i\omega)^2}{c^2} + (i\mathbf{k})^2\right) A(\mathbf{k},\omega)\, e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)} + \left(-\frac{(i\omega)^2}{c^2} + (-i\mathbf{k})^2\right) A^*(\mathbf{k},\omega)\, e^{-i(\mathbf{k}\cdot\mathbf{x}-\omega t)}\right] \\
&= \frac{1}{4\pi^2}\int_{-\infty}^{\infty} d^3k\, d\omega \left(\frac{\omega^2}{c^2} - \mathbf{k}^2\right)\left[A(\mathbf{k},\omega)\, e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)} + A^*(\mathbf{k},\omega)\, e^{-i(\mathbf{k}\cdot\mathbf{x}-\omega t)}\right]
\end{aligned}$$

The equation is satisfied if we require $\frac{\omega^2}{c^2} - \mathbf{k}^2$. To go further, we need to restrict the frequency in the general expression to $\omega = kc$. We can do this with a Dirac delta function. Then

$$q(\mathbf{x},t) = \frac{1}{(2\pi)^{3/2}}\int_{-\infty}^{\infty} d^3k\, d\omega \left[A(\mathbf{k},\omega)\, e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)} + A^*(\mathbf{k},\omega)\, e^{-i(\mathbf{k}\cdot\mathbf{x}-\omega t)}\right]\delta(\omega - kc)$$

is a general solution to the wave equation. Notice that we omit one factor of $\frac{1}{\sqrt{2\pi}}$ because the delta function is already normalized to one.

We can immediately integrate over ω, because the Dirac delta simply replaces all the ωs with kc. If we write $\mathbf{k} = k\mathbf{n}$ for a unit vector $\mathbf{n}$, we have a complete solution.

$$q(\mathbf{x},t) = \frac{1}{(2\pi)^{3/2}}\int_{-\infty}^{\infty} d^3k \left[A(\mathbf{k})\, e^{ik(\mathbf{n}\cdot\mathbf{x}-ct)} + A^*(\mathbf{k})\, e^{-ik(\mathbf{n}\cdot\mathbf{x}-ct)}\right] \tag{3}$$

3.3 Initial conditions

The initial conditions follow from the general solution, Eq.(3) in two steps. First, take the time derivative to find the velocity at each point,

$$\dot{q}(\mathbf{x},t) = \frac{ic}{(2\pi)^{3/2}}\int_{-\infty}^{\infty} d^3k \left[-kA(\mathbf{k})\, e^{ik(\mathbf{n}\cdot\mathbf{x}-ct)} + kA^*(\mathbf{k})\, e^{-ik(\mathbf{n}\cdot\mathbf{x}-ct)}\right]$$

Second, set the time to zero. If we are given

$$\begin{aligned}
q(\mathbf{x},0) &= a(\mathbf{x}) \\
\dot{q}(\mathbf{x},0) &= b(\mathbf{x})
\end{aligned}$$

Then at $t = 0$ we find

$$a(\mathbf{x}) = \frac{1}{(2\pi)^{3/2}}\int_{-\infty}^{\infty} d^3k \left[A(\mathbf{k})\, e^{i\mathbf{k}\cdot\mathbf{x}} + A^*(\mathbf{k})\, e^{-i\mathbf{k}\cdot\mathbf{x}}\right]$$

$$b(\mathbf{x}) = -\frac{ic}{(2\pi)^{3/2}} \int_{-\infty}^{\infty} d^3k \left[kA(\mathbf{k}) e^{ik(\mathbf{k}\cdot\mathbf{x})} - kA^*(\mathbf{k}) e^{-i\mathbf{k}\cdot\mathbf{x}} \right]$$

These are ordinary Fourier transforms of $a(\mathbf{x})$ and $b(\mathbf{x})$. We may invert the transforms to solve for $A(\mathbf{k})$ and its conjugate $A^*(\mathbf{k})$ in terms of the initial conditions.

4 Separation of variables

While we were able to guess solutions to the wave equation in Cartesian coordinates, we still need to develop techniques that will let us solve the equation in other coordinate systems. One very powerful technique is separation of variables.

Consider the wave equation, Eq.(1), again,

$$-\frac{1}{c^2}\frac{\partial^2 q}{\partial t^2} + \frac{\partial^2 q}{\partial x^2} + \frac{\partial^2 q}{\partial y^2} + \frac{\partial^2 q}{\partial z^2} = 0$$

We wish to find normal mode solutions, so we may assume a single frequency, $q(\mathbf{x},t) = q(\mathbf{x}) e^{i\omega t}$. Eq.(1) becomes

$$\frac{\omega^2}{c^2} q(\mathbf{x}) + \frac{\partial^2 q(\mathbf{x})}{\partial x^2} + \frac{\partial^2 q(\mathbf{x})}{\partial y^2} + \frac{\partial^2 q(\mathbf{x})}{\partial z^2} = 0$$

Now notice that the x, y and z derivatives are in distinct terms. This makes it possible to assume a solution of the product form

$$q(\mathbf{x}) = X(x) Y(y) Z(z)$$

Substituting, the partial derivatives become ordinary derivatives,

$$\frac{\omega^2}{c^2} XYZ + YZ\frac{d^2X}{dx^2} + XZ\frac{d^2Y}{dy^2} + XY\frac{d^2Z}{dz^2} = 0$$

Now divide by XYZ,

$$\frac{\omega^2}{c^2} + \frac{1}{X}\frac{d^2X}{dx^2} + \frac{1}{Y}\frac{d^2Y}{dy^2} + \frac{1}{Z}\frac{d^2Z}{dz^2} = 0 \tag{4}$$

Although the full equation still depends on x, y and z, each term now depends on a single independent variable. Consider the partial derivative of the whole equation with respect to x:

$$\begin{aligned} \frac{\partial}{\partial x}\left[\frac{\omega^2}{c^2} + \frac{1}{X}\frac{d^2X}{dx^2} + \frac{1}{Y}\frac{d^2Y}{dy^2} + \frac{1}{Z}\frac{d^2Z}{dz^2}\right] &= 0 \\ \frac{d}{dx}\left[\frac{1}{X}\frac{d^2X}{dx^2}\right] &= 0 \end{aligned}$$

Since $\frac{1}{X}\frac{d^2X}{dx^2}$ depends only on x, the vanishing derivative means that the term in brackets must be constant,

$$\frac{1}{X}\frac{d^2X}{dx^2} = \mp\alpha^2$$

We may also take derivatives with respect to y, z to show that

$$\begin{aligned} \frac{1}{Y}\frac{d^2Y}{dy^2} &= \mp\beta^2 \\ \frac{1}{Z}\frac{d^2Z}{dz^2} &= \mp\gamma^2 \end{aligned}$$

Substituting these constants into the original equation, Eq.(4), we see that

$$\frac{\omega^2}{c^2} \mp \alpha^2 \mp \beta^2 \mp \gamma^2 = 0$$

Each of these sign choices is independent. Which sign we choose for each constant depends on the boundary conditions. If we want a purely wavelike solution, we choose the upper sign for each, giving

$$\begin{aligned} \frac{d^2X}{dx^2} + \alpha^2 X &= 0 \\ \frac{d^2Y}{dy^2} + \beta^2 Y &= 0 \\ \frac{d^2Z}{dz^2} + \gamma^2 Z &= 0 \end{aligned}$$

where

$$\frac{\omega^2}{c^2} = \alpha^2 + \beta^2 + \gamma^2$$

The solutions are immediate,

$$\begin{aligned} X &= X_0 e^{i\alpha x} \\ Y &= Y_0 e^{i\beta y} \\ Z &= Z_0 e^{i\gamma z} \end{aligned}$$

and we recognize α, β, γ as the components of the wave vector. Setting

$$\mathbf{k} = (\alpha, \beta, \gamma) = (k_1, k_2, k_3)$$

the reassembled answer XYZ is just the normal mode solution we guessed,

$$q(\mathbf{x}, t) = Ae^{i(\mathbf{k}\cdot\mathbf{x} - \omega t)}$$

where the three constants combine into a single complex amplitude, $A = X_0 Y_0 Z_0$. We have a solution of this form for each choice of wave vector $\mathbf{k}$, leading to the same Fourier integral found previously.

When we consider the wave equation in cylindrical and spherical coordinates, this separation technique will be essential.