Separation of variables: Spherical coordinates

1 The wave equation in spherical coordinates.

Perhaps the most important class of waves are represented in spherical coordinates. Any disturbance can produce waves which travel outward in 3-dimensions, becoming more and more spherical as they propagate outward. This is also true of scattering experiments in particle accelerators, where the outgoing state after collision is treated as a spherical wave.

1.1 Spherical coordinates

The spherical coordinates are (r, θ, φ). r gives the distance from the origin, θ is the angle measured downward from the z-axis, and φ is the same azimuthal angle as in cylindrical coordinates. The relation to Cartesian coordinates given by

$$r = \sqrt{x^2 + y^2 + z^2} \quad (1)$$

$$\varphi = \tan^{-1}\left(\frac{y}{x}\right) \quad (2)$$

$$\theta = \tan^{-1}\left(\frac{\sqrt{x^2+y^2}}{z}\right) \quad (3)$$

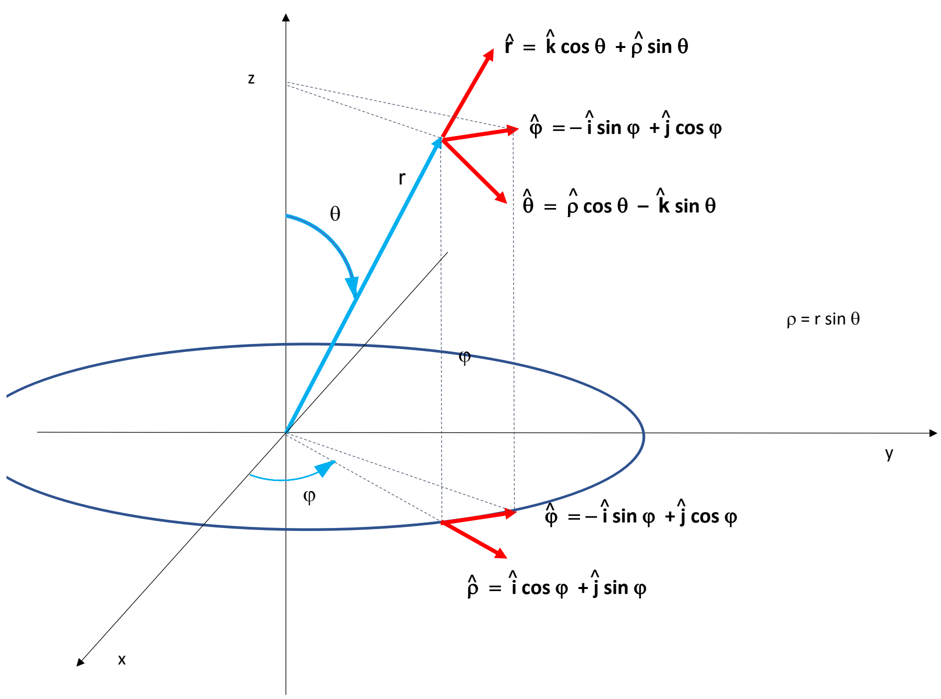
while z remains the usual Cartesian coordinate. For the inverse transformation,

$$x = r\sin\theta\cos\varphi \quad (4)$$

$$y = r\sin\theta\sin\varphi \quad (5)$$

$$z = r\cos\theta \quad (6)$$

These relations are not hard to see from a diagram:



The relations of the various unit vectors are seen in the diagram above. Study the diagram until you can see that

$$
\begin{aligned}
\hat{\mathbf{r}} &= \hat{\mathbf{i}}\sin\theta\cos\varphi + \hat{\mathbf{j}}\sin\theta\sin\varphi + \hat{\mathbf{k}}\cos\theta \\
\hat{\boldsymbol{\theta}} &= \hat{\mathbf{i}}\cos\theta\cos\varphi + \hat{\mathbf{j}}\cos\theta\sin\varphi - \hat{\mathbf{k}}\sin\theta \\
\hat{\boldsymbol{\varphi}} &= -\hat{\mathbf{i}}\sin\varphi + \hat{\mathbf{j}}\cos\varphi
\end{aligned}
\tag{7}
$$

Exercise: Verify that the three vectors in Eq.(7) form an orthonormal set.

Exercise: Show from Eqs.(7) that the Cartesian unit vectors $\hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}}$ are given in terms of $\hat{\mathbf{r}}, \hat{\boldsymbol{\theta}}, \hat{\boldsymbol{\varphi}}$, by

$$
\begin{aligned}
\hat{\mathbf{i}} &= \hat{\mathbf{r}}\sin\theta\cos\varphi + \hat{\boldsymbol{\theta}}\cos\theta\cos\varphi - \hat{\boldsymbol{\varphi}}\sin\varphi \\
\hat{\mathbf{j}} &= \hat{\mathbf{r}}\sin\theta\sin\varphi + \hat{\boldsymbol{\theta}}\cos\theta\sin\varphi + \hat{\boldsymbol{\varphi}}\cos\varphi \\
\hat{\mathbf{k}} &= \hat{\mathbf{r}}\cos\theta - \hat{\boldsymbol{\theta}}\sin\theta
\end{aligned}
\tag{8}
$$

and check that they form an orthonormal set.

1.2 The gradient

Starting from the gradient operator in Cartesian coordinates,

$$
\boldsymbol{\nabla} = \hat{\mathbf{i}}\frac{\partial}{\partial x} + \hat{\mathbf{j}}\frac{\partial}{\partial y} + \hat{\mathbf{k}}\frac{\partial}{\partial z}
$$

the chain rule and relations between the basis vectors, Eqs.(8), allow us to write the gradient in spherical coordinates

$$
\begin{aligned}
\boldsymbol{\nabla} &= \hat{\mathbf{i}}\left(\hat{\mathbf{r}}, \hat{\boldsymbol{\theta}}, \hat{\boldsymbol{\varphi}}\right)\left(\frac{\partial\rho}{\partial x}\frac{\partial}{\partial\rho} + \frac{\partial\theta}{\partial x}\frac{\partial}{\partial\theta} + \frac{\partial\varphi}{\partial x}\frac{\partial}{\partial\varphi}\right) \\
&\quad + \hat{\mathbf{j}}\left(\hat{\mathbf{r}}, \hat{\boldsymbol{\theta}}, \hat{\boldsymbol{\varphi}}\right)\left(\frac{\partial\rho}{\partial y}\frac{\partial}{\partial\rho} + \frac{\partial\theta}{\partial y}\frac{\partial}{\partial\theta} + \frac{\partial\varphi}{\partial y}\frac{\partial}{\partial\varphi}\right) \\
&\quad + \hat{\mathbf{k}}\left(\hat{\mathbf{r}}, \hat{\boldsymbol{\theta}}, \hat{\boldsymbol{\varphi}}\right)\left(\frac{\partial\rho}{\partial z}\frac{\partial}{\partial\rho} + \frac{\partial\theta}{\partial z}\frac{\partial}{\partial\theta} + \frac{\partial\varphi}{\partial z}\frac{\partial}{\partial\varphi}\right)
\end{aligned}
\tag{9}
$$

where the partial derivatives are given by

$$
\begin{aligned}
\frac{\partial r}{\partial x} &= \sin\theta\cos\varphi \\
\frac{\partial r}{\partial y} &= \sin\theta\sin\varphi \\
\frac{\partial r}{\partial z} &= \cos\theta
\end{aligned}
\tag{10}
$$

$$
\begin{aligned}
\frac{\partial\theta}{\partial x} &= \frac{1}{r}\cos\theta\cos\varphi \\
\frac{\partial\theta}{\partial y} &= \frac{1}{r}\cos\theta\sin\varphi \\
\frac{\partial\theta}{\partial z} &= -\frac{1}{r}\sin\theta
\end{aligned}
\tag{11}
$$

and

$$
\frac{\partial\varphi}{\partial x} = -\frac{1}{\rho}\sin\varphi = -\frac{\sin\varphi}{r\sin\theta}
$$

$$
\begin{aligned}
\frac{\partial \varphi}{\partial y} &= \frac{\cos\varphi}{r\sin\theta} \\
\frac{\partial \varphi}{\partial z} &= 0
\end{aligned} \tag{12}
$$

Exercise: By substituting the partial derivatives and basis vectors, show that the gradient in spherical coordinates is given by

$$
\boldsymbol{\nabla} = \hat{\mathbf{r}}\frac{\partial}{\partial r} + \hat{\boldsymbol{\theta}}\frac{1}{r}\frac{\partial}{\partial \theta} + \hat{\boldsymbol{\varphi}}\frac{1}{r\sin\theta}\frac{\partial}{\partial \varphi} \tag{13}
$$

1.3 The Laplacian

We find the Laplacian of a function by taking the divergence of the gradient of the function,

$$
\begin{aligned}
\nabla^2 f &= \boldsymbol{\nabla}\cdot\boldsymbol{\nabla} f \\
&= \left(\hat{\mathbf{r}}\frac{\partial}{\partial r} + \hat{\boldsymbol{\theta}}\frac{1}{r}\frac{\partial}{\partial \theta} + \hat{\boldsymbol{\varphi}}\frac{1}{r\sin\theta}\frac{\partial}{\partial \varphi}\right)\cdot\left(\hat{\mathbf{r}}\frac{\partial}{\partial r} + \hat{\boldsymbol{\theta}}\frac{1}{r}\frac{\partial}{\partial \theta} + \hat{\boldsymbol{\varphi}}\frac{1}{r\sin\theta}\frac{\partial}{\partial \varphi}\right)
\end{aligned} \tag{14}
$$

To carry out the divergence we will need the derivatives of the basis vectors.

Exercise: Using Eqs.(7), show that

$$
\begin{aligned}
\frac{\partial}{\partial \theta}\hat{\mathbf{r}} &= \hat{\boldsymbol{\theta}} \\
\frac{\partial}{\partial \theta}\hat{\boldsymbol{\theta}} &= -\hat{\mathbf{r}} \\
\frac{\partial}{\partial \varphi}\hat{\mathbf{r}} &= \hat{\boldsymbol{\varphi}}\sin\theta \\
\frac{\partial}{\partial \varphi}\hat{\boldsymbol{\theta}} &= \hat{\boldsymbol{\varphi}}\cos\theta \\
\frac{\partial}{\partial \varphi}\hat{\boldsymbol{\varphi}} &= -\hat{\mathbf{r}}\sin\theta - \hat{\boldsymbol{\theta}}\cos\theta
\end{aligned}
$$

Exercise: Carry out the divergence of the gradient of f, Eq.(14), to prove that the Laplacian is given by

$$
\nabla^2 f = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial f}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 f}{\partial \varphi^2} \tag{15}
$$

Whew! Now let's solve the Laplace and wave equations.

2 The Laplace equation in spherical coordinates

Before moving to the wave equation, we consider the Laplace equation,

$$
\nabla^2 f = 0
$$

In spherical coordinates we may solve this with separation of variables, setting $f = R(r)\,\Theta(\theta)\,\Phi(\varphi)$. Substituting in the usual way gives

$$
\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right)\Theta\Phi + \frac{1}{r^2\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right)R\Phi + \frac{1}{r^2\sin^2\theta}\frac{d^2\Phi}{d\varphi^2}R\Theta = 0
$$

Now divide by $q = R\Phi\Theta$,

$$
\frac{1}{R}\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) + \frac{1}{\Theta}\frac{1}{r^2\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{1}{\Phi}\frac{d^2\Phi}{d\varphi^2} = 0 \tag{16}
$$

2.1 The equation for R

Multiplying Eq.(16) by r^2 we see that the first term contains the only r dependence and must therefore be constant,

$$\frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) = \lambda\left(\lambda+1\right)$$

where the constant is written as $\lambda\left(\lambda+1\right)$ with some foreknowledge of the right answer.

Expanding the derivative and multiplying by R,

$$r^2\frac{d^2R}{dr^2} + 2r\frac{dR}{dr} - \lambda\left(\lambda+1\right)R = 0$$

Each term in this equation has the same dimensions in r. Where there is a first derivative (scaling as $\frac{1}{r}$), there is a single r; where there is a second derivative there is a factor of r^2. This means that the equation can be solved by r^σ, that is, by a power of r.

Exercise: Let $R_\lambda\left(r\right) = r^\sigma$. Find two independent values for σ for any given λ.

Notice that at this point we have not assumed that λ is an integer.

2.2 Spherical harmonics

After separating $R\left(r\right)$, the Laplace equation takes the form

$$\lambda\left(\lambda+1\right) + \frac{1}{\Theta}\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) + \frac{1}{\sin^2\theta}\frac{1}{\Phi}\frac{d^2\Phi}{d\varphi^2} = 0$$

Multiplication by $\sin^2\theta$ completes the separation, showing immediately that the azimuthal equation is

$$\frac{1}{\Phi}\frac{d^2\Phi}{d\varphi^2} = \beta$$

With the full range of φ, we must choose $\beta = -m^2$ as we did in cylindrical coordinates. This gives harmonic solutions with solutions

$$\Phi\left(\varphi\right) = e^{im\varphi}, m = \pm1, \pm2, \pm3, \ldots$$

There is no periodic solution for $m = 0$ other than $\Phi = \textit{constant}$.

2.2.1 The Legendre equation

The remaining equation for $\Theta\left(\theta\right)$ is

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta_{\lambda,m}}{d\theta}\right) + \left(\lambda\left(\lambda+1\right) - \frac{m^2}{\sin^2\theta}\right)\Theta_{\lambda,m} = 0$$

A change of variable helps with the solution. If we let $x = \cos\theta$, then

$$\begin{aligned}\frac{d}{d\theta} &= \frac{dx}{d\theta}\frac{d}{dx}\\ &= -\sin\theta\frac{d}{dx}\end{aligned}$$

Therefore, using $\sin^2\theta = 1-\cos^2\theta = 1-x^2$, the derivative term becomes

$$\begin{aligned}
\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta_{\lambda,m}}{d\theta}\right) &= \frac{1}{\sin\theta}\frac{d}{d\theta}\left(-\sin^2\theta\frac{d\Theta}{dx}\right) \\
&= \frac{1}{\sin\theta}\left(-\sin\theta\frac{d}{dx}\right)\left(-\left(1-x^2\right)\frac{d\Theta}{dx}\right) \\
&= \frac{d}{dx}\left(\left(1-x^2\right)\frac{d\Theta}{dx}\right) \\
&= \left(1-x^2\right)\frac{d^2\Theta}{dx^2} - 2x\frac{d\Theta}{dx}
\end{aligned}$$

and we have the *associated Legendre equation*,

$$\left(1-x^2\right)\frac{d^2\Theta}{dx^2} - 2x\frac{d\Theta}{dx} + \left(\lambda\left(\lambda+1\right) - \frac{m^2}{1-x^2}\right)\Theta_\lambda = 0$$

Setting $m = 0$ now gives the *Legendre equation*,

$$\left(1-x^2\right)\frac{d^2\Theta}{dx^2} - 2x\frac{d\Theta}{dx} + \lambda\left(\lambda+1\right)\Theta_\lambda = 0 \tag{17}$$

2.2.2 Solving the Legendre equation

To solve Eq.(17), we try a polynomial or power series,

$$\Theta_\lambda = \sum_{k=0}^{\infty} a_k x^k$$

Substituting and combining with the $1-x^2$ factor,

$$\begin{aligned}
\left(1-x^2\right)\sum_{k=2}^{\infty} a_k k\left(k-1\right)x^{k-2} - 2x\sum_{k=1}^{\infty} k a_k x^{k-1} + \lambda\left(\lambda+1\right)\sum_{k=0}^{\infty} a_k x^k &= 0 \\
\sum_{k=2}^{\infty} a_k k\left(k-1\right)x^{k-2} - \sum_{k=2}^{\infty} a_k k\left(k-1\right)x^k - \sum_{k=1}^{\infty} 2k a_k x^k + \lambda\left(\lambda+1\right)\sum_{k=0}^{\infty} a_k x^k &= 0
\end{aligned}$$

we see that every term but the first has the same powers of x. We change the index on the first sum so make it the same, but this changes the index on the coefficient. Let $m = k-2$ so we can rewrite

$$\sum_{k=2}^{\infty} a_k k\left(k-1\right)x^{k-2} = \sum_{m=0}^{\infty} a_{m+2}\left(m+2\right)\left(m+1\right)x^m$$

It doesn't matter what we call the summation index, so we can rename m as k so that all indices are the same:

$$\sum_{k=0}^{\infty} a_{k+2}\left(k+2\right)\left(k+1\right)x^k - \sum_{k=2}^{\infty} a_k k\left(k-1\right)x^k - \sum_{k=1}^{\infty} 2k a_k x^k + \lambda\left(\lambda+1\right)\sum_{k=0}^{\infty} a_k x^k = 0$$

Finally notice that we can start all of the summations at $k = 0$ because the first two terms (i.e., $k = 0$ and $k = 1$) of

$$\sum_{k=2}^{\infty} a_k k\left(k-1\right)x^k = \sum_{k=0}^{\infty} a_k k\left(k-1\right)x^k$$

vanish, and the first term of $\sum_{k=1}^{\infty} 2ka_k x^k = \sum_{k=0}^{\infty} 2ka_k x^k$ vanishes. Therefore, we may combine everything as a single sum,

$$\sum_{k=0}^{\infty} \left(a_{k+2} (k+2)(k+1) - a_k k (k-1) - 2ka_k x^k + \lambda (\lambda + 1) a_k \right) x^k = 0$$

Since different powers of x are independent, each coefficient must vanish separately. Setting each coefficient to zero

$$a_{k+2} (k+2)(k+1) - a_k k (k-1) - 2ka_k x^k + \lambda (\lambda + 1) a_k = 0$$

and solving for a_{k+2} gives a recursion relation:

$$a_{k+2} = \frac{(k (k+1) - \lambda (\lambda + 1))}{(k+2)(k+1)} a_k \tag{18}$$

This allows us to solve for all even a_k once a_0 is given, and for all odd a_k once a_1 is given. These two (a_0 and a_1) remain as an overall scale.

Different choices of λ give distinct solutions, and we can see that the series terminates in a polynomial if we choose λ to be an integer. The resulting polynomials are the Legendre polynomials. Let $P_n(x)$ be the solution with $\lambda = n$.

Look at the first two values for k. When $k = 0$ or $k = 1$, we have $\lambda = 0$ or $\lambda = 1$ respectively. Since the recursion gives no higher terms we must have

$$\begin{aligned} 0 &= \frac{-\lambda (\lambda + 1)}{2} a_0 \\ 0 &= \frac{2 - \lambda (\lambda + 1)}{2} a_1 \end{aligned}$$

and it is clear that for $\lambda = 0$, $a_1 = 0$ while a_0 is undetermined, while for $\lambda = 1$ it is a_0 that must vanish while a_1 is arbitrary. This holds in general–exactly one of a_0, a_1 can be nonzero, leading to polynomials with only even or odd powers of x, respectively.

For $\lambda = 0$, the series terminates immediately and we have

$$P_0(x) = a_0$$

It is conventional and convenient to choose the normalization so that $P_n(1) = 1$, so we set

$$P_0(x) = 1$$

For $\lambda = 1$, there is also only a single term, a_1, so the polynomial is $P_1(x) = a_1 x$. Normalizing gives

$$P_1(x) = x$$

For $\lambda = 2$, we finally get a second term. Starting with a_0, we see that

$$\begin{aligned} a_2 &= \frac{0(0+1) - 2(2+1)}{(0+2)(0+1)} a_0 \\ &= -3a_0 \end{aligned}$$

while $a_4 = 0$. Therefore,

$$P_2(x) = -3a_0 x^2 + a_0$$

and our convention requires $a_0 = -\frac{1}{2}$:

$$P_2(x) = \frac{1}{2}\left(3x^2 - 1\right)$$

We recognize the same series we constructed when we started to build an orthogonal set of polynomials from the powers of x. With the recursion relation, Eq.(18), the process is much easier.

Exercise: Construct $P_4(x), P_5(x)$, and $P_6(x)$ from the recursion relations, choosing a_0 or a_1 so that $P_n(1) = 1$.

2.2.3 Associated Legendre polynomials

When $m \neq 0$, we have the *associated Legendre polynomials*, satisfying

$$\left(1-x^2\right)\frac{d^2P_l^m(x)}{dx^2} - 2x\frac{dP_l^m(x)}{dx} + \left(l(l+1) - \frac{m^2}{1-x^2}\right)P_l^m(x) = 0$$

This is solved by

$$P_l^m(x) = \left(1-x^2\right)^{\frac{m}{2}}\frac{d^m}{dx^m}P_l(x) \tag{19}$$

where m may take on any of $2l+1$ values, $m = -l, -l+1, \ldots -1, 0, 1, \ldots, l-1, l$. Notice that with $x = \cos\theta$, the derivatives will depend on both $\cos\theta$ and $\sin\theta$.

2.2.4 Spherical harmonics

If we fix the r-dependence for a moment, we see that the solution of the Laplace equation at fixed r

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dY}{d\theta}\right) + \frac{1}{\sin^2\theta}\frac{d^2Y}{d\varphi^2} + l(l+1)Y = 0$$

is given by the *spherical harmonics*,

$$Y_{lm}(\theta,\varphi) = A_{lm}e^{im\varphi}P_l^m(\cos\theta)$$

If we choose the constants

$$A_{lm} = \sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}}$$

then the spherical harmonics are orthonormal,

$$\int_0^\pi \sin\theta d\theta \int_0^{2\pi} d\varphi Y_{lm}^*(\theta,\varphi)Y_{l'm'}(\theta,\varphi) = \delta_{ll'}\delta_{mm'}$$

where $Y_{lm}^*(\theta,\varphi) = A_{lm}e^{-im\varphi}P_l^m(\cos\theta)$ is the complex conjugate. Any piecewise continuous, bounded function $f(\theta,\varphi)$ defined on a sphere may be expanded in spherical harmonics. They are very important for describing radiation patterns or the quantum states of atoms.

2.3 Solutions to the Laplace equation

In terms of the spherical harmonics, we now have a complete solution to the Laplace equation in spherical coordinates:

$$f(r,\theta,\varphi) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l}\left(A_{lm}r^l + \frac{B_{lm}}{r^{l+1}}\right)Y_{lm}(\theta,\varphi) \tag{20}$$

Here, the spherical harmonics satisfy

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dY_{lm}}{d\theta}\right) + \frac{1}{\sin^2\theta}\frac{d^2Y_{lm}}{d\varphi^2} + l(l+1)Y_{lm} = 0 \tag{21}$$

2.4 Example of the use of spherical harmonics

Consider an orange, with its many sections running from the north pole to the south. Let's define a function that oscillates between $+1$ and -1 with alternate quadrants of the orange:

$$f(\theta,\varphi) = \begin{cases} +1 & \frac{2m\pi}{2} < \varphi < \frac{(2m+1)\pi}{2} \\ -1 & \frac{(2m+1)\pi}{2} < \varphi < \frac{(2m+2)\pi}{2} \end{cases}$$

for $m = 0, 1$, (or $\frac{N}{2} - 1$ if we set $\frac{2m\pi}{N/2} < \varphi < \frac{(2m+1)\pi}{N/2}$) for a total of 4 (or N) sections. Since this is a function on a sphere, we can expand it in spherical harmonics. Let

$$f(\theta,\varphi) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} A_{lm} Y_{lm}(\theta,\varphi) \tag{22}$$

If we try to figure out in advance which values of l and m we will need, we see an immediate problem–it seems like we need $e^{im\varphi}$ for m increasing in increments of 4, while there is no θ dependence. But we cannot have nonzero values of m without nonzero values of l!

The way around the problem is to rotate the coordinates so that the function is

$$f(\theta,\varphi) = \begin{cases} +1 & 0 < \varphi < \pi; 0 < \theta < \frac{\pi}{2} \\ -1 & \pi < \varphi < 2\pi; 0 < \theta < \frac{\pi}{2} \\ +1 & \pi < \varphi < 2\pi; \frac{\pi}{2} < \theta < \pi \\ -1 & 0 < \varphi < \pi; \frac{\pi}{2} < \theta < \pi \end{cases}$$

It's the same physical problem, but now it depends on both θ and φ.

Now, multiplying Eq.(22) by $Y^*_{kq}(\theta,\varphi)$ and integrating over all angles,

$$\sum_{l=0}^{\infty} \sum_{m=-l}^{l} A_{lm} \int_0^{2\pi} d\varphi \int_0^{\pi} \sin\theta d\theta\, Y_{lm}(\theta,\varphi) Y^*_{kq}(\theta,\varphi) = \int_0^{2\pi} d\varphi \int_0^{\pi} \sin\theta d\theta\, f(\theta,\varphi)\, Y^*_{kq}(\theta,\varphi)$$

We can immediately evaluate the integrals on the left, since the spherical harmonics are orthonormal. We must divide the right side into the four regions,

$$\begin{aligned} \sum_{l=0}^{\infty} \sum_{m=-l}^{l} A_{lm} \delta_{lk} \delta_{mq} &= \int_0^{\pi} d\varphi \int_0^{\frac{\pi}{2}} \sin\theta\, d\theta\, Y^*_{kq}(\theta,\varphi) - \int_{\pi}^{2\pi} d\varphi \int_0^{\frac{\pi}{2}} \sin\theta\, d\theta\, Y^*_{kq}(\theta,\varphi) \\ &\quad + \int_{\pi}^{2\pi} d\varphi \int_{\frac{\pi}{2}}^{\pi} \sin\theta\, d\theta\, Y^*_{kq}(\theta,\varphi) - \int_0^{\pi} d\varphi \int_{\frac{\pi}{2}}^{\pi} \sin\theta\, d\theta\, Y^*_{kq}(\theta,\varphi) \end{aligned}$$

The Kronecker deltas reduce the sums on the left to a single term. For the right side, we break the spherical harmonics into φ and θ parts, writing the Legendre polynomials as functions of $x = \cos\theta$.

$$\begin{aligned} A_{kq} &= \sqrt{\frac{2k+1}{4\pi} \frac{(k-q)!}{(k+q)!}} \left[\int_0^{\pi} d\varphi e^{-iq\varphi} \int_0^1 dx\, P^q_k(x) - \int_{\pi}^{2\pi} d\varphi e^{-iq\varphi} \int_0^1 P^q_k(x) \right] \\ &\quad + \sqrt{\frac{2k+1}{4\pi} \frac{(k-q)!}{(k+q)!}} \left[\int_{\pi}^{2\pi} d\varphi e^{-iq\varphi} \int_{-1}^0 P^q_k(x) - \int_0^{\pi} d\varphi e^{-iq\varphi} \int_{-1}^0 P^q_k(x)\, dx \right] \end{aligned}$$

$$
\begin{aligned}
&= \sqrt{\frac{2k+1}{4\pi}\frac{(k-q)!}{(k+q)!}}\left[\left(\frac{(-1)^q-1}{-iq}\right)\int_0^1 dx\, P_k^q(x) - \left(\frac{1-(-1)^q}{-iq}\right)\int_0^1 P_k^q(x)\right] \\
&\quad + \sqrt{\frac{2k+1}{4\pi}\frac{(k-q)!}{(k+q)!}}\left[\left(\frac{1-(-1)^q}{-iq}\right)\int_{-1}^0 P_k^q(x) - \left(\frac{(-1)^q-1}{-iq}\right)\int_{-1}^0 P_k^q(x)\,dx\right]
\end{aligned}
$$

Recursion relations or power series for the associated Legendre polynomials could be used to evaluate the final two integrals. The symmetry of the associated Legendre polynomials is given by

$$
P_l^m(-x) = (-1)^{l+m} P_l^m(x)
$$

so letting $y = -x$, we may write

$$
\begin{aligned}
\int_{-1}^0 P_k^q(x)\,dx &= \int_1^0 P_k^q(-y)\,d(-y) \\
&= (-1)^{k+q}\int_0^1 P_k^q(y)\,dy
\end{aligned}
$$

and since the factor $1-(-1)^q$ is nonzero only for odd q we may write

$$
\int_{-1}^0 P_k^q(x)\,dx = -(-1)^k \int_0^1 P_k^q(x)\,dx
$$

and combine the last two integrals as one. The final integral may be evaluated using further identities, for example, by integrating the power series term by term.

Therefore, the coefficients become

$$
A_{kq} = -\frac{2i}{q}\left(1-(-1)^q\right)\left(1+(-1)^k\right)\sqrt{\frac{2k+1}{4\pi}\frac{(k-q)!}{(k+q)!}}\int_0^1 P_k^q(x)\,dx
$$

The final series therefore involves even only k and odd q. The full solution is

$$
f(\theta,\varphi) = -2i\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{1}{q}\left(1-(-1)^q\right)\left(1+(-1)^k\right)\sqrt{\frac{2k+1}{4\pi}\frac{(k-q)!}{(k+q)!}}\left(\int_0^1 P_k^q(x)\,dx\right)Y_{lm}(\theta,\varphi)
$$

3 The wave equation in spherical coordinates

3.1 The wave equation

We may immediately write the wave equation $-\frac{1}{c^2}\frac{\partial^2 q}{\partial t^2} + \nabla^2 q = 0$ in spherical coordinates

$$
-\frac{1}{c^2}\frac{\partial^2 q}{\partial t^2} + \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial f}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 f}{\partial\varphi^2} = 0 \tag{23}
$$

Exercise: Use separation of variables to reduce the 3-dimensional wave equation in spherical coordinates to four ordinary differential equations.

3.2 Separation of variables

Rather than going directly to separation of variables, we observe that the angular portion of the wave equation is still satisfied by spherical harmonics. Therefore, we write

$$q(\mathbf{x},t) = T(t)\,R(r)\,Y_{lm}(\theta,\varphi)$$

Substituting, the partial derivatives become ordinary derivatives,

$$-\frac{1}{c^2}\frac{d^2T}{dt^2}RY_{lm} + \frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right)Y_{lm}T + \frac{RT}{r^2}\left(\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dY_{lm}}{d\theta}\right) + \frac{1}{\sin^2\theta}\frac{d^2Y_{lm}}{d\varphi^2}\right) = 0$$

Using Eq.(21) for the spherical harmonics, we get the simpler expression

$$\left(-\frac{1}{c^2}\frac{d^2T}{dt^2}R + \frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right)T - l(l+1)\frac{RT}{r^2}\right)Y_{lm} = 0$$

Dividing by $q = RTY_{lm}$ yields

$$-\frac{1}{c^2}\frac{1}{T}\frac{d^2T}{dt^2} + \frac{1}{R}\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) + \frac{1}{r^2}l(l+1) = 0 \tag{24}$$

and we find that r and t have separated.

3.2.1 Solve for T first

The first term clearly depends only on t, with no time dependence elsewhere in the wave equation. Therefore, we set

$$-\frac{1}{c^2}\frac{1}{T}\frac{d^2T}{dt^2} = k^2$$

so that T satisfies the harmonic equation

$$\frac{d^2T}{dt^2} + k^2c^2T = 0$$

We immediately have any of the alternative solutions

$$\begin{aligned}
T &= A\cos(kc(t-t_0)) \\
T &= A\cos(kct) + B\sin(kct) \\
T &= Ae^{ikct} + Be^{-ikct} \\
T &= \mathcal{A}e^{ikct}
\end{aligned}$$

where A, B are real and $\mathcal{A}$ complex.

Substituting this equation into the wave equation, gives the *Helmholz equation*,

$$\left(\nabla^2 + k^2\right)q = 0$$

and when substituted into Eq.(24) leaves the radial equation,

$$\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) + \left(l(l+1) + k^2r^2\right)R = 0$$

3.2.2 Solving for R

Expanding the derivatives,

$$r^2 \frac{d^2 R}{dr^2} + 2r \frac{dR}{dr} + \left(k^2 r^2 - l\,(l+1)\right) R = 0 \tag{25}$$

As we did for the Bessel equation, we let $x = kr$, so the equation become

$$x^2 \frac{d^2 R}{dx^2} + 2x \frac{dR}{dx} + \left(x^2 - l\,(l+1)\right) R = 0$$

This is very similar to the Bessel equation, and the solutions are called *spherical Bessel functions*. These are related to the Bessel functions by

$$j_n\,(x) = \sqrt{\frac{\pi}{2}} \frac{1}{\sqrt{x}} J_{n+\frac{1}{2}}\,(x) \tag{26}$$

where $J_{n+\frac{1}{2}}\,(x)$ is the Bessel function satisfying

$$x^2 \frac{d^2}{dx^2} J_{n+\frac{1}{2}} + x \frac{d}{dx} J_{n+\frac{1}{2}} + \left(x^2 - \left(n + \frac{1}{2}\right)^2\right) J_{n+\frac{1}{2}} = 0 \tag{27}$$

Note the factor of 2 difference for the middle term.

3.2.3 Proof that the spherical Bessel functions solve the radial equation

We show that the spherical Bessel functions of Eq.(26) solve the radial equation.

The factor of $\sqrt{\frac{\pi}{2}}$ is merely a constant normalization of $j_n\,(x)$, and is common to every term. We may therefore drop it from the proof and show that

$$j_n\,(x) = \frac{1}{\sqrt{x}} J_{n+\frac{1}{2}}\,(x)$$

solves the radial equation.

The first two derivatives of $j_n\,(x)$ are

$$\begin{aligned} \frac{dj_n}{dx} &= \frac{d}{dx}\left(x^{-\frac{1}{2}} J_{n+\frac{1}{2}}\right) \\ &= -\frac{1}{2} x^{-\frac{3}{2}} J_{n+\frac{1}{2}} + x^{-\frac{1}{2}} \frac{d}{dx} J_{n+\frac{1}{2}} \end{aligned}$$

and

$$\begin{aligned} \frac{d^2 j_n}{dx^2} &= \frac{d}{dx}\left(-\frac{1}{2} x^{-\frac{3}{2}} J_{n+\frac{1}{2}} + x^{-\frac{1}{2}} \frac{d}{dx} J_{n+\frac{1}{2}}\right) \\ &= \frac{3}{4} x^{-\frac{5}{2}} J_{n+\frac{1}{2}} - \frac{1}{2} x^{-\frac{3}{2}} \frac{d}{dx} J_{n+\frac{1}{2}} - \frac{1}{2} x^{-\frac{3}{2}} \frac{d}{dx} J_{n+\frac{1}{2}} + x^{-\frac{1}{2}} \frac{d^2}{dx^2} J_{n+\frac{1}{2}} \\ &= \frac{3}{4} x^{-\frac{5}{2}} J_{n+\frac{1}{2}} - x^{-\frac{3}{2}} \frac{d}{dx} J_{n+\frac{1}{2}} + x^{-\frac{1}{2}} \frac{d^2}{dx^2} J_{n+\frac{1}{2}} \end{aligned}$$

Now substuting $j_n\,(x)$ and its derivatives into Eq.(25) we find

$$\begin{aligned} x^2 \frac{d^2 j_n}{dx^2} + 2x \frac{dj_n}{dx} + \left(x^2 - l\,(l+1)\right) j_n &= x^2 \left(\frac{3}{4} x^{-\frac{5}{2}} J_{n+\frac{1}{2}} - x^{-\frac{3}{2}} \frac{d}{dx} J_{n+\frac{1}{2}} + x^{-\frac{1}{2}} \frac{d^2}{dx^2} J_{n+\frac{1}{2}}\right) \\ &\quad + 2x \left(-\frac{1}{2} x^{-\frac{3}{2}} J_{n+\frac{1}{2}} + x^{-\frac{1}{2}} \frac{d}{dx} J_{n+\frac{1}{2}}\right) + \left(x^2 - l\,(l+1)\right)\left(x^{-\frac{1}{2}} J_{n+\frac{1}{2}}\right) \end{aligned}$$

$$\begin{aligned}
&= \frac{3}{4}x^{-\frac{1}{2}}J_{n+\frac{1}{2}} - xx^{-\frac{1}{2}}\frac{d}{dx}J_{n+\frac{1}{2}} + x^{-\frac{1}{2}}x^2\frac{d^2}{dx^2}J_{n+\frac{1}{2}} \\
&\quad + \left(-x^{-\frac{1}{2}}J_{n+\frac{1}{2}} + x^{-\frac{1}{2}}2x\frac{d}{dx}J_{n+\frac{1}{2}}\right) + \left(x^2 - l\left(l+1\right)\right)\left(x^{-\frac{1}{2}}J_{n+\frac{1}{2}}\right) \\
&= x^{-\frac{1}{2}}\left(x^2\frac{d^2}{dx^2}J_{n+\frac{1}{2}} + x\frac{d}{dx}J_{n+\frac{1}{2}}\right) \\
&\quad + \frac{3}{4}j_n - j_n + \left(x^2 - l\left(l+1\right)\right)j_n
\end{aligned}$$

Now, using the Bessel equation for $n+\frac{1}{2}$, (Eq.(27)) we replace the derivative terms,

$$\begin{aligned}
x^2\frac{d^2j_n}{dx^2} + 2x\frac{dj_n}{dx} + \left(x^2 - n\left(n+1\right)\right)j_n &= \sqrt{\frac{\pi}{2}}x^{-\frac{1}{2}}\left(-\left(x^2 - \left(n+\frac{1}{2}\right)^2\right)J_{n+\frac{1}{2}}\right) \\
&\quad + \frac{3}{4}j_n - j_n + \left(x^2 - n\left(n+1\right)\right)j_n \\
&= \left(\frac{3}{4} - 1 + x^2 - n\left(n+1\right) - x^2 + n^2 + n + \frac{1}{4}\right)j_n \\
&= 0
\end{aligned}$$

We may now restore the normalization without affecting the result. Similar results hold for the second solution and other forms of the Bessel functions.

The spherical Bessel functions have the general form

$$j_n\left(x\right) = P\left(\frac{1}{x}\right)\sin x + Q\left(\frac{1}{x}\right)\cos x$$

where P and Q are polynomials in $\frac{1}{x}$.

3.3 Solutions to the wave equation

Suppose we have a source oscillating at a single frequency $\omega = kc$. Then we have found the solution

$$\begin{aligned}
q\left(r,\theta,\varphi,t\right) &= T\left(t\right)R\left(r\right)Y_{lm}\left(\theta,\varphi\right) \\
&= \sum_{l=0}^{\infty}\sum_{m=-l}^{l}\left(A_{lm}\left(k\right)\cos\left(kct\right) + B_{lm}\left(k\right)\sin\left(kct\right)\right)j_l\left(kr\right)Y_{lm}\left(\theta,\varphi\right)
\end{aligned}$$

to the wave equation in spherical coordinates. More complicated sources can be handled by letting the coefficients depend on k, $A_{lm}\left(k\right), B_{lm}\left(k\right)$ and including a sum or integral over k.

Let initial conditions be given at time $t=0$, on the sphere $r=R$, by

$$\begin{aligned}
q\left(R,\theta,\varphi,0\right) &= 0 \\
\dot{q}\left(R,\theta,\varphi,0\right) &= f\left(\theta,\varphi\right)
\end{aligned}$$

The coefficients now satisfy

$$\begin{aligned}
0 = q\left(R,\theta,\varphi,0\right) &= \sum_{l=0}^{\infty}\sum_{m=-l}^{l}A_{lm}j_l\left(kR\right)Y_{lm}\left(\theta,\varphi\right) \\
&= \sum_{l=0}^{\infty}\sum_{m=-l}^{l}A_{lm}j_l\left(kR\right)Y_{lm}\left(\theta,\varphi\right) \\
f\left(\theta,\varphi\right) = q\left(R,\theta,\varphi,0\right) &= kc\sum_{l=0}^{\infty}\sum_{m=-l}^{l}B_{lm}j_l\left(kR\right)Y_{lm}\left(\theta,\varphi\right)
\end{aligned}$$

Since the spherical harmonics $Y_{lm}(\theta,\varphi)$ are all independent and complete, we must have $A_{lm}=0$ for all l,m . To find the B_{lm} we need multiply by $Y^*_{kq}(\theta,\varphi)$ and integrate:

$$\begin{aligned}\int_0^{\pi}\sin\theta d\theta\int_0^{2\pi}d\varphi f(\theta,\varphi)Y^*_{kq}(\theta,\varphi) &= kc\sum_{l=0}^{\infty}\sum_{m=-l}^{l}B_{lm}j_l(kR)\int_0^{\pi}\sin\theta d\theta\int_0^{2\pi}d\varphi Y^*_{kq}(\theta,\varphi)Y_{lm}(\theta,\varphi)\\ &= kc\sum_{l=0}^{\infty}\sum_{m=-l}^{l}B_{lm}j_l(kR)\delta_{kl}\delta_{mq}\\ &= kcj_l(kR)B_{kq}\end{aligned}$$

so (changing the name of the eigenvalues back to l,m) the B_{lm} are given by

$$B_{lm}=\frac{1}{kc}\frac{1}{j_l(kR)}\int_0^{\pi}\sin\theta d\theta\int_0^{2\pi}d\varphi f(\theta,\varphi)Y^*_{lm}(\theta,\varphi)$$

and the full wave solution is

$$q(r,\theta,\varphi,t) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l}\left[\int_0^{\pi}\sin\theta' d\theta'\int_0^{2\pi}d\varphi' f(\theta',\varphi')Y^*_{lm}(\theta',\varphi')\right]Y_{lm}(\theta,\varphi)\frac{j_l(kr)}{j_l(kR)}\omega\sin(\omega t)$$

We give an example below.

4 Example

4.1 Spherical shell at constant potential

Use the general solution in spherical coordinates to find the electric potential inside and outside a charged spherical shell of radius R, held at a constant potential, $\phi=V$.

Except on the sphere itself, the electric potential satisfies the Laplace equation,

$$\nabla^2\phi=0$$

for which we have the general solution given in Eq.(20),

$$\phi(r,\theta,\varphi)=\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\left(A_{lm}r^l+\frac{B_{lm}}{r^{l+1}}\right)Y_{lm}(\theta,\varphi)$$

Since there is no angular dependence, we have only solutions with $l=0$ and $m=0$, hence proportional to $Y_{00}=\frac{1}{\sqrt{4\pi}}$,

$$\phi(r,\theta,\varphi)=\frac{1}{\sqrt{4\pi}}\left(A_{00}+\frac{B_{00}}{r}\right)$$

For the region outside the sphere, we need the potential at infinity to go to zero, so we require $A_{00}=0$. Then, at $r=R$, the potential must approach V so we have

$$V=\frac{1}{\sqrt{4\pi}}\frac{B_{00}}{R}$$

Solving for B_{00},

$$B_{00}=\sqrt{4\pi}RV$$

Therefore

$$\phi_{outside}(r,\theta,\varphi) = V\frac{R}{r}$$

For the interior region, we have the same expansion but new constants. Now the boundary condition is that ϕ must be finite at the origin, $r = 0$. With

$$\phi(r,\theta,\varphi) = \frac{1}{\sqrt{4\pi}}\left(A_{00} + \frac{B_{00}}{r}\right)$$

we therefore need $B_{00} = 0$ and the potential is constant. Using the boundary condition at $r = R$ we have $A_{00} = \sqrt{4\pi}V$.

The full solution is

$$\phi(r) = \begin{cases} V & r < R \\ \frac{RV}{r} & r > R \end{cases}$$

4.2 Spherical dipole

Now suppose we have a sphere of radius R with a potential $+V$ on the upper hemisphere and $-V$ on the lower. Find the potential outside the sphere.

This is still independent of φ, but now depends on θ. There is no m dependence and our general solution to the Laplace equation becomes

$$\phi(r,\theta) = \sum_{l=0}^{\infty}\left(A_l r^l + \frac{B_l}{r^{l+1}}\right)Y_{l0}(\theta)$$

Since the solution must extend to infinity, we cannot have the terms that grow with $r$so ϕ reduces further to

$$\phi(r,\theta) = \sum_{l=0}^{\infty}\frac{B_l}{r^{l+1}}Y_{l0}(\theta,)$$

Finally, the spherical harmonics are just proportional to Legendre polynomials,

$$Y_{l0}(\theta) = \sqrt{\frac{2l+1}{4\pi}}P_l(\cos\theta)$$

Because the potential is odd, we expect only odd l. To check, and to find the remaining coefficients, we set $r = R$ and $\phi = \phi(R,\theta)$, then multiply both sides by $Y^*_{k0}(\theta) = Y_{k0}(\theta)$ and integrate:

$$\begin{aligned}
\int_0^{\pi}\phi(R,\theta)Y^*_{k0}(\theta)\sin\theta d\theta &= \sum_{l=0}^{\infty}\frac{B_l}{R^{l+1}}\int_0^{\pi}Y_{l0}(\theta)Y^*_{k0}(\theta)\sin\theta d\theta \\
V\int_0^{\frac{\pi}{2}}Y^*_{k0}(\theta)\sin\theta d\theta - V\int_{\frac{\pi}{2}}^{\pi}Y^*_{k0}(\theta)\sin\theta d\theta &= \sum_{l=0}^{\infty}\frac{B_l}{R^{l+1}}\delta_{lk}
\end{aligned}$$

The Kronecker delta on the right makes the sum collapse to the single term with $l = k$. For the integrals on the right we need a recursion formula. It can be shown that

$$P_n(x) = \frac{1}{2n+1}\frac{d}{dx}\left(P_{n+1}(x) - P_{n-1}(x)\right)$$

Rewriting the integrals in terms of x using $dx = d(\cos\theta) = -\sin\theta d\theta$, the sign is absorbed by exchanging the limits:

$$\begin{aligned}\int_0^{\frac{\pi}{2}} P_k(\cos\theta)\sin\theta d\theta &= \int_1^0 P_k(x)(-dx) \\ &= \int_0^1 P_k(x)\,dx\end{aligned}$$

Substituting the recursion relation, the coefficient is therefore given by

$$\begin{aligned}\frac{B_k}{R^{k+1}} &= V\int_0^{\frac{\pi}{2}} Y_{k0}^*(\theta)\sin\theta d\theta - V\int_{\frac{\pi}{2}}^{\pi} Y_{k0}^*(\theta)\sin\theta d\theta \\ &= \sqrt{\frac{2k+1}{4\pi}}V\int_0^1 P_k(x)\,dx - V\int_{-1}^0 P_k(x)\,dx \\ &= \frac{1}{\sqrt{4\pi}}\frac{V}{\sqrt{2k+1}}\int_0^1\left(\frac{d}{dx}\left(P_{k+1}(x) - P_{k-1}(x)\right)\right)dx \\ &\quad -\frac{1}{\sqrt{4\pi}}\frac{V}{\sqrt{2k+1}}\int_{-1}^0\left(\frac{d}{dx}\left(P_{k+1}(x) - P_{k-1}(x)\right)\right)dx \\ &= \frac{1}{\sqrt{4\pi}}\frac{V}{\sqrt{2k+1}}\left[P_{k+1}(1) - P_{k-1}(1) - P_{k+1}(0) + P_{k-1}(0)\right] \\ &\quad -\frac{1}{\sqrt{4\pi}}\frac{V}{\sqrt{2k+1}}\left[P_{k+1}(0) - P_{k-1}(0) - P_{k+1}(-1) + P_{k-1}(-1)\right]\end{aligned}$$

Since $P_n(1) = 1$ and $P_n(-1) = (-1)^n$ the first and last terms cancel, leaving

$$B_k = \frac{1}{\sqrt{4\pi}}\frac{2VR^{k+1}}{\sqrt{2k+1}}\left[P_{k-1}(0) - P_{k+1}(0)\right]$$

There are various power series expansions that reduce to simple combinatoric factors when $x = 0$, but it is simplest to just leave it this way. The potential is therefore

$$\begin{aligned}\phi(r,\theta) &= -2V\sum_{l=0}^{\infty}\frac{1}{2k+1}\left[P_{k+1}(0) - P_{k-1}(0)\right]\left(\frac{R}{r}\right)^{l+1} P_l(\cos\theta) \\ \phi(r,\theta) &= \sum_{l=0}^{\infty}\frac{B_l}{r^{l+1}}Y_{l0}(\theta,) \\ \phi(r,\theta) &= \sum_{l=0}^{\infty}\frac{1}{\sqrt{4\pi}}\frac{2V}{\sqrt{2l+1}}\left[P_{k-1}(0) - P_{k+1}(0)\right]\left(\frac{R}{r}\right)^{l+1}\sqrt{\frac{2l+1}{4\pi}}P_l(\cos\theta) \\ &= \frac{V}{2\pi}\sum_{l=0}^{\infty}\left[P_{k-1}(0) - P_{k+1}(0)\right]\left(\frac{R}{r}\right)^{l+1} P_l(\cos\theta)\end{aligned}$$

4.3 Dipole waves

Suppose this same potential at $r = R$ oscillates harmonically at frequency $\omega = kc$,

$$\begin{aligned} \phi(\theta) &= \begin{cases} V & \theta < \frac{\pi}{2} \\ -V & \theta > \frac{\pi}{2} \end{cases} \\ \phi(\theta, t) &= \phi(\theta)\sin\omega t \end{aligned}$$

Find the resulting waves for $r > R$.

We need to solve the wave equation in spherical coordinates, Eq.(23) with this time-dependent boundary condition. We found the solution to be of the form

$$q(r,\theta,\varphi,t) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l}\left(A_{lm}\cos(kct) + B_{lm}\sin(kct)\right) j_l(kr)\, Y_{lm}(\theta,\varphi)$$

and since we are given the time dependence as $\sin\omega t$, we only need the B_{lm} terms,

$$q(r,\theta,\varphi,t) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} B_{lm}\sin(kct)\, j_l(kr)\, Y_{lm}(\theta,\varphi)$$

Now fit the boundary condition at time $t = \frac{\pi}{2\omega}$ (so that the $\sin(\omega t)$ term gives 1). then we must have

$$\begin{aligned} \phi(\theta) &= \sum_{l=0}^{\infty}\sum_{m=-l}^{l} B_{lm}\sin\left(\frac{\pi}{2}\right) j_l(kR)\, Y_{lm}(\theta,\varphi) \\ &= \sum_{l=0}^{\infty}\sum_{m=-l}^{l} B_{lm} j_l(kR)\, Y_{lm}(\theta,\varphi) \end{aligned}$$

and we have effectively solved this above with R^{2l+1} now replaced by $j_l(kR)$. We have $m = 0$ because there is no φ dependence. Then

$$B_{l0} j_l(kR) = \frac{1}{\sqrt{4\pi}}\frac{2V}{\sqrt{2l+1}}\left[P_{l-1}(0) - P_{l+1}(0)\right]$$

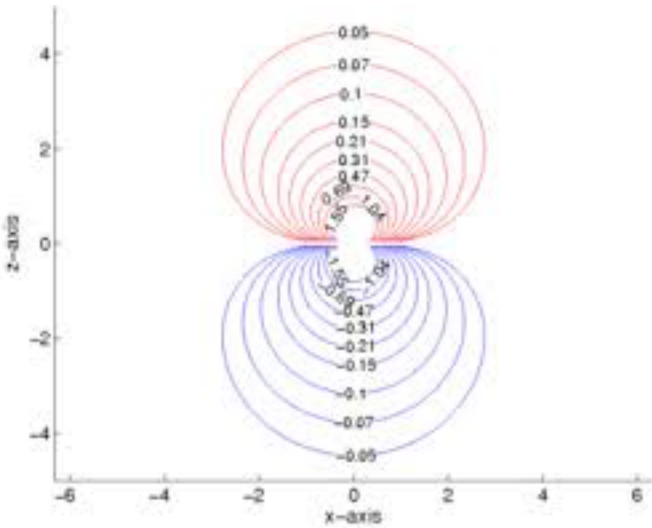
and we write

$$q(r,\theta,\varphi,t) = \frac{2V}{\sqrt{4\pi}}\sum_{l=0}^{\infty}\frac{1}{\sqrt{2l+1}}\left[P_{l-1}(0) - P_{l+1}(0)\right]\sin(kct)\,\frac{j_l(kr)}{j_l(kR)} Y_{lm}(\theta,\varphi)$$

Since the series contains only odd terms, and is dominated by the lowest value of l, we may approximate the series far from the source by the $l = 1$ term:

$$\begin{aligned} \lim_{r>>R} q(r,\theta,\varphi,t) &= \frac{2V}{\sqrt{4\pi}}\sum_{l=0}^{\infty}\frac{1}{3}\left[P_0(0) - P_2(0)\right]\sin(kct)\,\frac{j_1(kr)}{j_1(kR)} Y_{10}(\theta,\varphi) \\ &= \frac{2V}{\sqrt{4\pi}}\sum_{l=0}^{\infty}\frac{1}{3}\left[P_0(0) - P_2(0)\right]\sin(kct)\,\frac{j_1(kr)}{j_1(kR)}\frac{1}{2}\sqrt{\frac{3}{\pi}}\cos\theta \\ &= \frac{V}{2\sqrt{3}\pi}\sum_{l=0}^{\infty}\left(\frac{P_0(0) - P_2(0)}{j_1(kR)}\right) j_1\left(\frac{\omega r}{c}\right)\cos\theta\sin(\omega t) \end{aligned}$$

This is the potential for a dipole field. (Image from Wikipedia)


0.05
0.07
0.1
0.15
0.21
0.31
0.47
−0.47
−0.31
−0.21
−0.15
−0.1
−0.07
−0.05
z−axis
x−axis
4
2
0
−2
−4
−6
6