Separation of variables: Cylindrical coordinates

1 The wave equation in cylindrical coordinates.

Not all waves are plane waves; in fact, most waves we encounter have a localized source and emanate cylindrically from a linear antenna or spherically from a point source. In order to discuss such waves it is desirable to write the wave equation in coordinates with the corresponding symmetry.

1.1 Change of coordinates

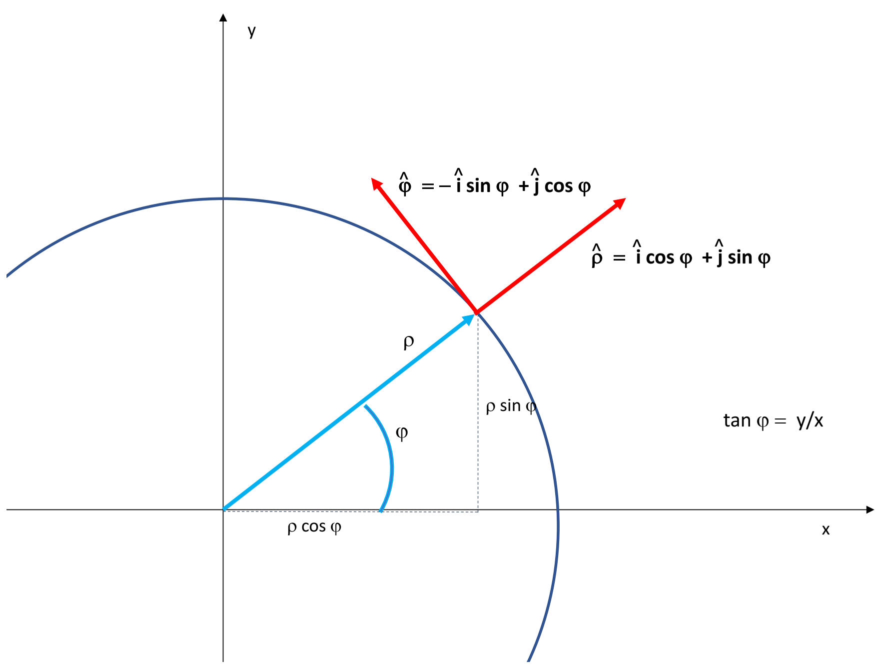
We will call our cylindrical coordinates (ρ, φ, z). The xy-plane is described in polar coordinates, with the relation to Cartesian coordinates given by

$$\begin{aligned} \rho &= \sqrt{x^2 + y^2} \\ \varphi &= \tan^{-1}\left(\frac{y}{x}\right) \\ z &= z \end{aligned} \qquad (1)$$

while z remains the usual Cartesian coordinate. For the inverse transformation,

$$\begin{aligned} x &= \rho \cos\varphi \\ y &= \rho \sin\varphi \\ z &= z \end{aligned}$$

These relations are not hard to see from a diagram:



1.1.1 The gradient in cylindrical coordinates

The gradient operator in Cartesian coordinates is given by

$$\boldsymbol{\nabla} = \hat{\mathbf{i}}\frac{\partial}{\partial x} + \hat{\mathbf{j}}\frac{\partial}{\partial y} + \hat{\mathbf{k}}\frac{\partial}{\partial z}$$

To write this in cylindrical coordinates we must change both the unit vectors and the partial derivatives. Since the z direction is unchanged, we first work in the xy plane, which is the same as the $\rho\varphi$ plane.

It is not hard to see from the picture that a unit vector in the ρ direction is given by

$$\hat{\boldsymbol{\rho}} = \hat{\mathbf{i}}\cos\varphi + \hat{\mathbf{j}}\sin\varphi$$

Again from the diagram, or alternatively by writing a unit vector orthogonal to $\hat{\boldsymbol{\rho}}$, we find

$$\hat{\boldsymbol{\varphi}} = -\hat{\mathbf{i}}\sin\varphi + \hat{\mathbf{j}}\cos\varphi$$

Inverting these to write $\hat{\mathbf{i}}, \hat{\mathbf{j}}$ in terms of $\hat{\boldsymbol{\rho}}, \hat{\boldsymbol{\varphi}}$ we get similar linear combinations:

$$\begin{aligned}
\hat{\mathbf{i}} &= \hat{\boldsymbol{\rho}}\cos\varphi - \hat{\boldsymbol{\varphi}}\sin\varphi \\
\hat{\mathbf{j}} &= \hat{\boldsymbol{\rho}}\sin\varphi + \hat{\boldsymbol{\varphi}}\cos\varphi
\end{aligned} \tag{2}$$

We find the partial derivatives using the chain rule

$$\begin{aligned}
\frac{\partial}{\partial x} &= \frac{\partial\rho}{\partial x}\frac{\partial}{\partial\rho} + \frac{\partial\varphi}{\partial x}\frac{\partial}{\partial\varphi} \\
\frac{\partial}{\partial y} &= \frac{\partial\rho}{\partial y}\frac{\partial}{\partial\rho} + \frac{\partial\varphi}{\partial y}\frac{\partial}{\partial\varphi}
\end{aligned} \tag{3}$$

but for this we need the partial derivatives of Eqs.(1). These are immediate for the derivatives of ρ,

$$\begin{aligned}
\frac{\partial\rho}{\partial x} &= \frac{1}{2}\cdot\frac{1}{\rho}\cdot 2x = \cos\varphi \\
\frac{\partial\rho}{\partial y} &= \frac{1}{2}\cdot\frac{1}{\rho}\cdot 2y = \sin\varphi
\end{aligned} \tag{4}$$

but for φ we use implicit differentiation. Write the relationship between φ, x and y as

$$\tan\varphi = \frac{y}{x}$$

and notice that

$$d\varphi = \frac{\partial\varphi}{\partial x}dx + \frac{\partial\varphi}{\partial y}dy \tag{5}$$

Therefore, if we can solve for $d\varphi$ we can read the partials directly. Therefore, start with the differential of $\tan\varphi$,

$$\begin{aligned}
d(\tan\varphi) &= d\left(\frac{\sin\varphi}{\cos\varphi}\right) \\
&= \left(\frac{\cos\varphi}{\cos\varphi} - \frac{\sin\varphi(-\sin\varphi)}{\cos^2\varphi}\right)d\varphi \\
&= \left(\frac{\cos^2\varphi + \sin^2\varphi}{\cos^2\varphi}\right)d\varphi \\
&= \frac{1}{\cos^2\varphi}d\varphi
\end{aligned}$$

On the other side of the equation we have

$$\begin{aligned} d\left(\frac{y}{x}\right) &= \frac{1}{x}dy - \frac{y}{x^2}dx \\ &= \frac{1}{\rho\cos\varphi}dy - \frac{\sin\varphi}{\rho\cos^2\varphi}dx \end{aligned}$$

Equating and simplifying,

$$\begin{aligned} d(\tan\varphi) &= d\left(\frac{y}{x}\right) \\ \frac{1}{\cos^2\varphi}d\varphi &= \frac{1}{\rho\cos\varphi}dy - \frac{\sin\varphi}{\rho\cos^2\varphi}dx \end{aligned}$$

we multiply through by $\cos^2\varphi$ to give

$$d\varphi = \frac{1}{\rho}\cos\varphi dy - \frac{1}{\rho}\sin\varphi dx \tag{6}$$

Comparing Eq.(5) and Eq.(6) we see that

$$\begin{aligned} \frac{\partial\varphi}{\partial x} &= -\frac{1}{\rho}\sin\varphi \\ \frac{\partial\varphi}{\partial y} &= \frac{1}{\rho}\cos\varphi \end{aligned} \tag{7}$$

Now we substitute the partials of ρ from Eq.(4) and the partials of φ from Eq.(7) into the Cartesian derivatives, Eqs.(3),

$$\begin{aligned} \frac{\partial}{\partial x} &= \cos\varphi\frac{\partial}{\partial\rho} - \frac{1}{\rho}\sin\varphi\frac{\partial}{\partial\varphi} \\ \frac{\partial}{\partial y} &= \sin\varphi\frac{\partial}{\partial\rho} + \frac{1}{\rho}\cos\varphi\frac{\partial}{\partial\varphi} \end{aligned}$$

Substitution into the Cartesian form of the gradient simplifies quickly,

$$\begin{aligned} \boldsymbol{\nabla} &= \hat{\mathbf{i}}\frac{\partial}{\partial x} + \hat{\mathbf{j}}\frac{\partial}{\partial y} + \hat{\mathbf{k}}\frac{\partial}{\partial z} \\ &= (\hat{\boldsymbol{\rho}}\cos\varphi - \hat{\boldsymbol{\varphi}}\sin\varphi)\left(\cos\varphi\frac{\partial}{\partial\rho} - \frac{1}{\rho}\sin\varphi\frac{\partial}{\partial\varphi}\right) + (\hat{\boldsymbol{\rho}}\sin\varphi + \hat{\boldsymbol{\varphi}}\cos\varphi)\left(\sin\varphi\frac{\partial}{\partial\rho} + \frac{1}{\rho}\cos\varphi\frac{\partial}{\partial\varphi}\right) + \hat{\mathbf{k}}\frac{\partial}{\partial z} \\ &= \hat{\boldsymbol{\rho}}\left(\cos\varphi\left(\cos\varphi\frac{\partial}{\partial\rho} - \frac{1}{\rho}\sin\varphi\frac{\partial}{\partial\varphi}\right) + \sin\varphi\left(\sin\varphi\frac{\partial}{\partial\rho} + \frac{1}{\rho}\cos\varphi\frac{\partial}{\partial\varphi}\right)\right) \\ &\quad -\hat{\boldsymbol{\varphi}}\left(\sin\varphi\left(\cos\varphi\frac{\partial}{\partial\rho} - \frac{1}{\rho}\sin\varphi\frac{\partial}{\partial\varphi}\right) - \cos\varphi\left(\sin\varphi\frac{\partial}{\partial\rho} + \frac{1}{\rho}\cos\varphi\frac{\partial}{\partial\varphi}\right)\right) + \hat{\mathbf{k}}\frac{\partial}{\partial z} \\ &= \hat{\boldsymbol{\rho}}\left(\cos^2\varphi\frac{\partial}{\partial\rho} - \frac{1}{\rho}\cos\varphi\sin\varphi\frac{\partial}{\partial\varphi} + \sin^2\varphi\frac{\partial}{\partial\rho} + \frac{1}{\rho}\sin\varphi\cos\varphi\frac{\partial}{\partial\varphi}\right) \\ &\quad -\hat{\boldsymbol{\varphi}}\left(\sin\varphi\cos\varphi\frac{\partial}{\partial\rho} - \frac{1}{\rho}\sin^2\varphi\frac{\partial}{\partial\varphi} - \cos\varphi\sin\varphi\frac{\partial}{\partial\rho} - \frac{1}{\rho}\cos^2\varphi\frac{\partial}{\partial\varphi}\right) + \hat{\mathbf{k}}\frac{\partial}{\partial z} \end{aligned}$$

Cancelling the cross terms and adding $\cos^2\varphi + \sin^2\varphi = 1$, gives the del operator,

$$\boldsymbol{\nabla} = \hat{\boldsymbol{\rho}}\frac{\partial}{\partial\rho} + \hat{\boldsymbol{\varphi}}\frac{1}{\rho}\frac{\partial}{\partial\varphi} + \hat{\mathbf{k}}\frac{\partial}{\partial z} \tag{8}$$

To write the wave equation, we now need the Laplacian

1.2 The Laplacian

Directly writing the Laplacian in a different coordinate system can get daunting. It is by far easiest to use our relation

$$\nabla^2 = \boldsymbol{\nabla} \cdot \boldsymbol{\nabla}$$

It is easiest to see which terms to keep if we let ∇^2 act on a function, f. Substituting $\boldsymbol{\nabla} f = \hat{\boldsymbol{\rho}}\frac{\partial f}{\partial \rho} + \hat{\boldsymbol{\varphi}}\frac{1}{\rho}\frac{\partial f}{\partial \varphi} + \hat{\mathbf{k}}\frac{\partial f}{\partial z}$ into the divergence,

$$\nabla^2 f = \left(\hat{\boldsymbol{\rho}}\frac{\partial}{\partial \rho} + \hat{\boldsymbol{\varphi}}\frac{1}{\rho}\frac{\partial}{\partial \varphi} + \hat{\mathbf{k}}\frac{\partial}{\partial z}\right) \cdot \left(\hat{\boldsymbol{\rho}}\frac{\partial f}{\partial \rho} + \hat{\boldsymbol{\varphi}}\frac{1}{\rho}\frac{\partial f}{\partial \varphi} + \hat{\mathbf{k}}\frac{\partial f}{\partial z}\right)$$

Since we need to differentiate not only the first derivatives of f, but also the position-dependent unit vectors $\hat{\boldsymbol{\rho}}$ and $\hat{\boldsymbol{\varphi}}$, it is easiest to differentiate these when written back in terms of the constant Cartesian unit vectors:

$$\begin{aligned} \hat{\boldsymbol{\rho}} &= \hat{\mathbf{i}}\cos\varphi + \hat{\mathbf{j}}\sin\varphi \\ \hat{\boldsymbol{\varphi}} &= -\hat{\mathbf{i}}\sin\varphi + \hat{\mathbf{j}}\cos\varphi \end{aligned}$$

Then we only have φ derivatives, and notice that

$$\begin{aligned} \frac{\partial \hat{\boldsymbol{\rho}}}{\partial \varphi} &= -\hat{\mathbf{i}}\sin\varphi + \hat{\mathbf{j}}\cos\varphi = \hat{\boldsymbol{\varphi}} \\ \frac{\partial \hat{\boldsymbol{\varphi}}}{\partial \varphi} &= -\hat{\mathbf{i}}\cos\varphi - \hat{\mathbf{j}}\sin\varphi = -\hat{\boldsymbol{\rho}} \end{aligned} \tag{9}$$

Now, carrying out the divergence,

$$\begin{aligned} \nabla^2 f &= \hat{\boldsymbol{\rho}}\frac{\partial}{\partial \rho} \cdot \left(\hat{\boldsymbol{\rho}}\frac{\partial f}{\partial \rho} + \hat{\boldsymbol{\varphi}}\frac{1}{\rho}\frac{\partial f}{\partial \varphi} + \hat{\mathbf{k}}\frac{\partial f}{\partial z}\right) + \hat{\boldsymbol{\varphi}}\frac{1}{\rho}\frac{\partial}{\partial \varphi} \cdot \left(\hat{\boldsymbol{\rho}}\frac{\partial f}{\partial \rho} + \hat{\boldsymbol{\varphi}}\frac{1}{\rho}\frac{\partial f}{\partial \varphi} + \hat{\mathbf{k}}\frac{\partial f}{\partial z}\right) \\ &\quad + \hat{\mathbf{k}}\frac{\partial}{\partial z} \cdot \left(\hat{\boldsymbol{\rho}}\frac{\partial f}{\partial \rho} + \hat{\boldsymbol{\varphi}}\frac{1}{\rho}\frac{\partial f}{\partial \varphi} + \hat{\mathbf{k}}\frac{\partial f}{\partial z}\right) \\ &= \left(\hat{\boldsymbol{\rho}} \cdot \hat{\boldsymbol{\rho}}\frac{\partial^2 f}{\partial \rho^2} + \hat{\boldsymbol{\rho}} \cdot \hat{\boldsymbol{\varphi}}\frac{\partial}{\partial \rho}\left(\frac{1}{\rho}\frac{\partial f}{\partial \varphi}\right) + \hat{\boldsymbol{\rho}} \cdot \hat{\mathbf{k}}\frac{\partial}{\partial \rho}\left(\frac{\partial f}{\partial z}\right)\right) \\ &\quad + \left(\hat{\boldsymbol{\varphi}}\frac{1}{\rho}\frac{\partial}{\partial \varphi} \cdot \left(\hat{\boldsymbol{\rho}}\frac{\partial f}{\partial \rho}\right) + \hat{\boldsymbol{\varphi}}\frac{1}{\rho}\frac{\partial}{\partial \varphi} \cdot \left(\hat{\boldsymbol{\varphi}}\frac{1}{\rho}\frac{\partial f}{\partial \varphi}\right) + \hat{\boldsymbol{\varphi}} \cdot \hat{\mathbf{k}}\frac{1}{\rho}\frac{\partial}{\partial \varphi}\left(\frac{\partial f}{\partial z}\right)\right) \\ &\quad + \left(\hat{\mathbf{k}} \cdot \hat{\boldsymbol{\rho}}\frac{\partial}{\partial z}\left(\frac{\partial f}{\partial \rho}\right) + \hat{\mathbf{k}} \cdot \hat{\boldsymbol{\varphi}}\frac{\partial}{\partial z}\left(\frac{1}{\rho}\frac{\partial f}{\partial \varphi}\right) + \hat{\mathbf{k}} \cdot \hat{\mathbf{k}}\frac{\partial^2 f}{\partial z^2}\right) \end{aligned}$$

Most of the dot products vanish, leaving

$$\nabla^2 f = \frac{\partial^2 f}{\partial \rho^2} + \hat{\boldsymbol{\varphi}} \cdot \frac{1}{\rho}\left(\frac{\partial \hat{\boldsymbol{\rho}}}{\partial \varphi}\frac{\partial f}{\partial \rho} + \hat{\boldsymbol{\rho}}\frac{\partial^2 f}{\partial \varphi \partial \rho}\right) + \hat{\boldsymbol{\varphi}}\frac{1}{\rho} \cdot \left(\frac{\partial \hat{\boldsymbol{\varphi}}}{\partial \varphi}\frac{1}{\rho}\frac{\partial f}{\partial \varphi} + \hat{\boldsymbol{\varphi}}\frac{1}{\rho}\frac{\partial^2 f}{\partial \varphi^2}\right) + \frac{\partial^2 f}{\partial z^2}$$

Using Eqs(9), we arrive at the Laplacian in cylindrical coordinates,

$$\nabla^2 f = \frac{\partial^2 f}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial f}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2 f}{\partial \varphi^2} + \frac{\partial^2 f}{\partial z^2} \tag{10}$$

Notice that the ρ derivatives may be combined as a single term

$$\frac{\partial^2 q}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial q}{\partial \rho} = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho\frac{\partial q}{\partial \rho}\right)$$

1.3 Separation of variables in cylindrical coordinates

1.3.1 The wave equation

We may immediately write the wave equation $-\frac{1}{c^2}\frac{\partial^2 q}{\partial t^2} + \nabla^2 q = 0$ in cylindrical coordinates

$$-\frac{1}{c^2}\frac{\partial^2 q}{\partial t^2} + \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho\frac{\partial q}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 q}{\partial \varphi^2} + \frac{\partial^2 q}{\partial z^2} = 0 \quad (11)$$

We wish to find normal mode solutions, so we may assume a single frequency, $q(\mathbf{x}, t) = Q(\mathbf{x})\, e^{i\omega t}$. Eq.(11) becomes

$$\frac{\omega^2}{c^2}Q(\mathbf{x}) + \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho\frac{\partial q}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 q}{\partial \varphi^2} + \frac{\partial^2 q}{\partial z^2} = 0$$

This is just one form of separation of variables, replacing the time derivatives with a constant multiple of Q.

1.3.2 Separation of variables

Now notice that the ρ, φ and z derivatives are in distinct terms. Even though the coefficient of the φ-derivatives depends on ρ, it is still possible to assume a solution of the product form

$$q(\mathbf{x}) = R(\rho)\,\Phi(\varphi)\,Z(z)$$

Substituting, the partial derivatives become ordinary derivatives,

$$\frac{\omega^2}{c^2}R\Phi Z + \Phi Z\frac{1}{\rho}\frac{d}{d\rho}\left(\rho\frac{dR}{d\rho}\right) + RZ\frac{1}{\rho^2}\frac{d^2\Phi}{d\varphi^2} + R\Phi\frac{d^2 Z}{dz^2} = 0$$

Now divide by $Q = R\Phi Z$,

$$\frac{\omega^2}{c^2} + \frac{1}{R}\frac{1}{\rho}\frac{d}{d\rho}\left(\rho\frac{dR}{d\rho}\right) + \frac{1}{\Phi}\frac{1}{\rho^2}\frac{d^2\Phi}{d\varphi^2} + \frac{1}{Z}\frac{d^2 Z}{dz^2} = 0 \quad (12)$$

1.3.3 Solving for Z

Eq.(12) is only partially separated. Taking a partial derivative with respect to z immediately shows the last term to be constant, so we may set

$$\frac{d^2 Z}{dz^2} \mp \alpha^2 Z = 0$$

where the sign of the constant is determined by the boundary conditions.

If $Z(z)$ is not required to be periodic, it may be most convenient to use the upper sign, giving hyperbolic or exponential solutions,

$$Z(z) = A\cosh\alpha z + B\sinh\alpha z$$

or

$$Z(z) = Ae^{\alpha z} + Be^{-\alpha z}$$

When we choose the lower sign, the solutions are periodic,

$$Z(z) = A\cos\alpha z + B\sin\alpha z$$

or $Z(z) = \mathcal{B}e^{i\alpha z}$. If we wish a solution with wavelength $z = L$, then we may set $\alpha = \frac{2\pi}{L}$.

1.3.4 Separating Φ and R

Using the constancy, $\frac{1}{Z}\frac{d^2Z}{dz^2} = \pm\alpha^2$, we rewrite the equation as

$$\frac{1}{R}\frac{1}{\rho}\frac{d}{d\rho}\left(\rho\frac{dR}{d\rho}\right) + \frac{1}{\Phi}\frac{1}{\rho^2}\frac{d^2\Phi}{d\varphi^2} + \left(\frac{\omega^2}{c^2} \pm \alpha^2\right) = 0$$

To complete the separation of $\Phi(\varphi)$, we multiply the entire equation by ρ^2,

$$\left[\frac{\rho}{R}\frac{d}{d\rho}\left(\rho\frac{dR}{d\rho}\right) + \left(\frac{\omega^2}{c^2} \pm \alpha^2\right)\rho^2\right] + \frac{1}{\Phi}\frac{d^2\Phi}{d\varphi^2} = 0$$

This shows that the last term is a constant, β, so we may write

$$\frac{d^2\Phi}{d\varphi^2} - \beta\Phi = 0$$

When the full range of φ is allowed, Φ must be periodic. Therefore, we choose β negative, $\beta = -m^2$, with m an integer. We have immediate solutions,

$$\Phi(\varphi) = A\cos m\varphi + B\sin m\varphi$$

or $\Phi = \mathcal{A}e^{im\varphi}$ with a complex constant $\mathcal{A}$.

Now we substitute $-m^2 = \frac{1}{\Phi}\frac{d^2\Phi}{d\varphi^2}$, define

$$\begin{aligned}\kappa^2 &:= \frac{\omega^2}{c^2} \pm \alpha^2 \\ &= k^2 \pm \alpha^2\end{aligned}$$

and multiply by R to find the Bessel equation,

$$\rho^2\frac{d^2R}{d\rho^2} + \rho\frac{dR}{d\rho} + \left(\kappa^2\rho^2 - m^2\right)R = 0$$

Expanding the condensed derivative, $\rho\frac{d}{d\rho}\left(\rho\frac{dR}{d\rho}\right) = \rho^2\frac{d^2R}{d\rho^2} + \rho\frac{dR}{d\rho}$ and defining a dimensionless variable $x = \kappa\rho$, we find the standard form of the Bessel equation

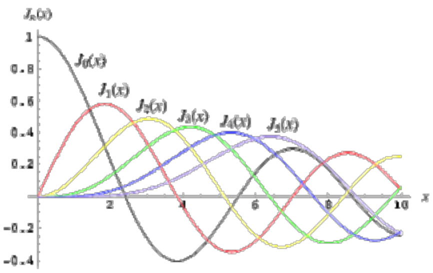
$$x^2\frac{d^2R}{dx^2} + x\frac{dR}{dx} + \left(x^2 - m^2\right)R = 0 \tag{13}$$

We note that the Bessel equation is an ordinary, second order, linear differential equation. As a result, for each value of m, we have two solutions

$J_m(x)$ — *Bessel functions*

$N_m(x)$ — *Neumann functions (or Weber functions, or Bessel functions of the 2^{nd} kind)*

These differ in their behavior near $x = k\rho = 0$, the axis of the cylindrical system. $J_0(0) = 1$, while the remaining Bessel functions vanish on the axis, $J_m(0) = 0, m > 0$. The Neumann functions, however, are divergent on the axis and are therefore used for exterior solutions, i.e., solutions only valid for some $\rho \geq \rho_0 > 0$. Like the sine and cosine, Bessel functions oscillate, but with decreasing amplitude (*image created, developed, and nurtured by Eric Weissteinat Wolfram Research*):



The oscillations of $J_m(x)$ lead to an infinite sequence of zeros $x_{mn}, n = 1, 2, 3, \ldots$ of each Bessel function, $J_m(x_{mn}) = 0$. Knowing these zeros (and we do, they're tabulated) we can easily fit boundary conditions. For example, if we know that our solution of the wave equation vanishes on a cylinder at $x = x_0$, then we may write radial functions in the form

$$J_m\left(\frac{x_{mn}x}{x_0}\right) = J_m\left(\frac{x_{mn}\rho}{\rho_0}\right)$$

Because the Bessel equation is linear, any superposition of these will satisfy the m^{th} Bessel equation,

$$R(x) = \sum_{n=1}^{\infty} a_n J_m\left(\frac{x_{mn}x}{x_0}\right)$$

and the boundary condition $R(x_0) = 0$. Like the sine and cosine, Bessel and Neumann functions form a complete, orthogonal set and we may write any (reasonable) function as an infinite series of them.

There are many variants of Bessel and Neumann functions. Most important are the Hankel functions. These are complex combinations

$$\begin{aligned} H_m^{(1)} &= J_m + iY_m \\ H_m^{(2)} &= J_m - iY_m \end{aligned}$$

The Hankel functions play a role relative to the Bessel functions much like the Euler relation $e^{\pm ix} = \cos x \pm i \sin x$ does for trigonometric functions.

1.3.5 Relations between the constants

Returning to the full wave equation, setting $\omega = kc$, and substituting in the constants, we have

$$\begin{aligned} k^2 + \frac{1}{R}\frac{1}{\rho}\frac{d}{d\rho}\left(\rho\frac{dR}{d\rho}\right) + \frac{1}{\Phi}\frac{1}{\rho^2}\frac{d^2\Phi}{d\varphi^2} + \frac{1}{Z}\frac{d^2Z}{dz^2} &= 0 \\ k^2 + \frac{1}{R}\frac{1}{\rho}\frac{d}{d\rho}\left(\rho\frac{dR}{d\rho}\right) - \frac{m^2}{\rho^2} \pm \alpha^2 &= 0 \\ \rho\frac{d}{d\rho}\left(\rho\frac{dR}{d\rho}\right) + \left(\left(k^2 \pm \alpha^2\right)\rho^2 - m^2\right)R &= 0 \end{aligned}$$

1.4 Solutions to the cylindrical wave equation

In general, solutions in cylindrical coordinates are sums over products of the frightening form

$$q(\rho, \varphi, z, t) = \sum_{k,\alpha,m} \left(a_\omega e^{ikct} + b_\omega e^{-ikct}\right)\left(g_\alpha e^{\alpha z} + h_\alpha e^{-\alpha z}\right)\left(c_m e^{im\varphi} + d_m e^{-im\varphi}\right)\left(e_m J_m(\kappa\rho) + f_m N_m(\kappa\rho)\right)$$

but we can simplify this considerably once we know the boundary conditions.

Suppose we need a solution which is valid on the axis, $\rho = 0$. Then we must expand in Bessel, rather than Neumann, functions. If, in addition we have a vanishing boundary condition at ρ_0,

$$q(\rho_0, \varphi, z, t) = 0$$

then we can expand the part of the solution in the xy plane as

$$\sum_{n=1}^{\infty} a_{mn} J_m \left(\frac{x_{mn}\rho}{\rho_0} \right) e^{im\varphi}$$

If we have a two dimensional problem, we can suppress z altogether. This could describe waves on a drumhead or ripples on the surface of water. The simplest solution occurs when there is perfect cylindrical symmetry. Then there is no φ dependence and we must take only the zero mode, $m = 0$. Including the time dependence, the wave is given by

$$q(\rho, t) = \sum_{n=1}^{\infty} a_n J_0 \left(\frac{x_{0n}\rho}{\rho_0} \right) \cos \left(\frac{x_{0n} ct}{\rho_0} + \theta_{0n} \right)$$

Notice that the roots of the Bessel function determine $\kappa = k^2 - \alpha^2$, which in this case reduces to $\kappa_n = k_n = \frac{x_{0n}}{\rho_0}$. This describes a cylindrical oscillation with maximum amplitude at the center, decreasing to zero at ρ_0.

Now suppose we do have φ dependence. We can determine the full solution if we know the φ-dependence at some particular $0 < \rho_1 < \rho_0$. Let the φ-dependence at ρ_1 be given by

$$q(\rho_1, \varphi) = \sum_{m=0}^{\infty} (a_m \sin m\varphi + b_m \cos m\varphi)$$

Then we get the full solution for the disk $\rho \leq \rho_0$ by adjoining Bessel functions that are normalized to give 1 when $\rho = \rho_1$:

$$q(\rho, \varphi) = \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} (a_{mn} \sin m\varphi + b_{mn} \cos m\varphi) \frac{J_m \left(\frac{x_{mn}\rho}{\rho_0} \right)}{J_m \left(\frac{x_{mn}\rho_1}{\rho_0} \right)}$$

where the a_{mn} and b_{mn} may be distinct constants for each m and n, subject to $a_m = \sum_{n=1}^{\infty} a_{mn}$ and $b_m = \sum_{n=1}^{\infty} b_{mn}$.

Finally, suppose the solution varies with z. Since the z solution does not depend on the form of the solution in the plane, we may simply put in one more factor. For solutions periodic in z,

$$q(\rho, \varphi, z) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} (a_{mn} \sin m\varphi + b_{mn} \cos m\varphi) \frac{J_m \left(\frac{x_{mn}\rho}{\rho_0} \right)}{J_m \left(\frac{x_{mn}\rho_1}{\rho_0} \right)} \cos(\alpha z + \theta_0)$$

If the solution vanishes at $z = 0$ and $z = z_0$, we can set $\alpha = \frac{2\pi l}{z_0}, l = 1, 2, \ldots$. Then

$$Z = A_l \sin \frac{2\pi l z}{z_0}$$

and the full time-dependent solution becomes

$$q(\rho, \varphi, z, t) = \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} \sum_{l=1}^{\infty} (a_{mnl} \sin m\varphi + b_{mnl} \cos m\varphi) \frac{J_m \left(\frac{x_{mn}\rho}{\rho_0} \right)}{J_m \left(\frac{x_{mn}\rho_1}{\rho_0} \right)} \sin \frac{2\pi l z}{z_0} \cos kc(t - t_0)$$

The constants are related by

$$
\begin{aligned}
\kappa &= \frac{x_{mn}}{\rho_0} \\
\kappa^2 &= k^2 + \left(\frac{2\pi l}{z_0}\right)^2 \\
k^2 &= \left(\frac{x_{mn}}{\rho_0}\right)^2 - \left(\frac{2\pi l}{z_0}\right)^2
\end{aligned}
$$