Motion of a mass on a spring

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We wish to study the one dimensional motion of a mass, m, attached to a spring of spring constant k. The force on the mass is proportional to the stretch of the spring, as given by Hooke's law,

$$F = -kx$$

Then, substituting this force into Newton's second law, writing the acceleration as the second time derivative of the position, and collecting terms we have:

$$\begin{aligned} F &= ma \\ -kx &= m\frac{d^2x}{dt^2} \\ \frac{d^2x}{dt^2} + \frac{k}{m}x &= 0 \end{aligned}$$

Define $\omega^2 \equiv \frac{k}{m}$ so this takes the form

$$\frac{d^2x}{dt^2} + \omega^2 x = 0$$

Notice that ω has units of frequency, sec^{-1}.

How do we solve this? There is a standard trick to use whenever the force depends only on position. Multiply by $v = \frac{dx}{dt}$, and rewrite the acceleration as $\frac{d^2x}{dt^2} = \frac{dv}{dt}$. Then we have

$$\begin{aligned} \frac{d^2x}{dt^2} + \omega^2 x &= 0 \\ v\frac{dv}{dt} + \frac{dx}{dt}\omega^2 x &= 0 \end{aligned}$$

Multiply by dt (but NEVER ∂t!) and bring the dx term to the right:

$$vdv = -\omega^2 xdx$$

Notice how we have now got dv times a function of v only, and similarly dx multiplying a function of x alone. This means that we can now integrate. It's best to make our integrals definite, with the initial values $x_0 = x(t_0)$ and $v_0 = v(t_0)$ explicit:

$$\int_{v_0}^{v(t)} vdv = -\int_{x_0}^{x(t)} \omega^2 xdx$$

The integration is now easy, giving

$$\frac{1}{2}v^2 - \frac{1}{2}v_0^2 = -\frac{1}{2}\omega^2 x^2 + \frac{1}{2}\omega^2 x_0^2$$

Rearrangement, putting expressions at time t on one side and expressions at time t_0 on the other shows that we have a constant of the motion,

$$\begin{aligned} \frac{1}{2}v^2 + \frac{1}{2}\omega^2 x^2 &= \frac{1}{2}v_0^2 + \frac{1}{2}\omega^2 x_0^2 \\ &\equiv \frac{E}{m} = constant! \end{aligned}$$

The particular form $\frac{1}{2}v^2 + \frac{1}{2}\omega^2 x^2$ gives the same value whether we evaluate it at time t_0, or any generic time t, so we recognize it as a constant of the motion.

Now solve for the velocity

$$v = \sqrt{2E - \omega^2 x^2}$$

and write $v = \frac{dx}{dt}$,

$$\frac{dx}{dt} = \sqrt{2E - \omega^2 x^2}$$

Once again, we multiply by dt. This time the dt stays on once side with dx on the other. We need to bring the x dependent terms to one side before we can integrate.

$$\begin{aligned} \frac{dx}{\sqrt{2E - \omega^2 x^2}} &= dt \\ \int_{x_0}^{x(t)} \frac{dx}{\sqrt{1 - \frac{\omega^2}{2E}x^2}} &= \sqrt{2E}\int_{t_0}^{t} dt \end{aligned}$$

We use the same initial limits as for the first integration–x and t evaluated at t_0.

To integrate we use a trigonometric substitution: let $\sqrt{\frac{\omega^2}{2E}}x = \sin\varphi$ so that the square root on the left becomes simply a cosine,

$$\sqrt{1 - \frac{\omega^2}{2E}x^2} = \sqrt{1 - \sin^2\varphi} = \cos\varphi$$

We also have to replace dx. Taking the differential of our substitution and solving for dx gives

$$\begin{aligned} \sqrt{\frac{\omega^2}{2E}}dx &= d(\sin\varphi) \\ dx &= \frac{\sqrt{2E}}{\omega}\cos\varphi\, d\varphi \end{aligned}$$

Putting it all together the integral becomes

$$\begin{aligned} \frac{\sqrt{2E}}{\omega}\int_{x_0}^{x(t)} \frac{\cos\varphi d\varphi}{\sqrt{1 - \sin^2\varphi}} &= \sqrt{2E}\int_{t_0}^{t} dt \\ \frac{1}{\omega}\int_{x_0}^{x(t)} d\varphi &= (t - t_0) \\ \varphi\big|_{x_0}^{x(t)} &= \omega(t - t_0) \end{aligned}$$

There are two ways we could have handled the limits:

1. Leave the limits in terms of x. Then, after completing the φ integration, transform back to x before evaluating at the limits.

2. Rewrite the limits in terms of φ, where

$$\sqrt{\frac{\omega^2}{2E}}x_0 = \sin\varphi_0$$

This introduces a new constant, so I usually choose the first option. However, either is fine.

Continuing, we replace $\varphi = \sin^{-1}\sqrt{\frac{\omega^2}{2E}}x$ then evaluate at the limits,

$$\begin{aligned}
\sin^{-1}\sqrt{\frac{\omega^2}{2E}}x\Big|_{x_0}^{x(t)} &= \omega\left(t-t_0\right) \\
\sin^{-1}\sqrt{\frac{\omega^2}{2E}}x - \sin^{-1}\sqrt{\frac{\omega^2}{2E}}x_0 &= \omega\left(t-t_0\right) \\
\sin^{-1}\sqrt{\frac{\omega^2}{2E}}x &= \omega t - \omega t_0 + \sin^{-1}\sqrt{\frac{\omega^2}{2E}}x_0
\end{aligned}$$

Define two new constants:

$$\begin{aligned}
\varphi_0 &\equiv \sin^{-1}\sqrt{\frac{\omega^2}{2E}}x_0 - \omega t_0 \\
A &\equiv \frac{\sqrt{2E}}{\omega}
\end{aligned}$$

This cleans up the result considerably,

$$\begin{aligned}
\sqrt{\frac{\omega^2}{2E}}x &= \sin\left[\omega t + \varphi_0\right] \\
x\left(t\right) &= \frac{\sqrt{2E}}{\omega}\sin\left(\omega t + \varphi_0\right)
\end{aligned}$$

and finally

$$x\left(t\right) = A\sin\left(\omega t + \varphi_0\right)$$

The identities

$$\begin{aligned}
\sin\left(a+b\right) &= \sin a\cos b + \sin b\cos a \\
\cos\left(a+b\right) &= \cos a\cos b - \sin a\sin b
\end{aligned}$$

let us write $x\left(t\right)$ in a variety of ways, for example

$$x\left(t\right) = B\cos\omega t + C\sin\omega t$$

where the constants B and C are related to A and φ_0 in a definite way. Since this function is a solution to Newton's second law in one dimension, we know that there will always be two constants, determined by the initial position and the initial velocity.