Fourier Series

October 9, 2019

1 Looking back: N coupled masses

We have seen how the seemingly complicated motion of a coupled system of N masses connected by springs may be described as linear combinations of N simple harmonic oscillations. If these normal modes have frequencies $\omega_i, i = 1, 2, \ldots, N$ and the eigenvectors describing the normal modes are $\vec{\mathbf{u}}_i$ then the most general motion of the system is

$$\vec{\mathbf{x}} = \sum_{i=1}^{N} A_i \vec{\mathbf{u}}_i \sin(\omega_i t + \varphi_i)$$

Notice that the normal modes form a vector basis for the space of solutions.

The initial conditions for the position and velocity are

$$\begin{aligned} \vec{\mathbf{x}}(0) &= \sum_{i=1}^{N} (A_i \sin\varphi_i) \vec{\mathbf{u}}_i \\ \dot{\vec{\mathbf{x}}}(0) &= \sum_{i=1}^{N} \omega_i (A_i \cos\varphi_i) \vec{\mathbf{u}}_i \end{aligned}$$

Each of these conditions is an N-dimensional vector in the solution space, with components

$$\begin{aligned} \vec{\mathbf{x}}(0) &\iff (A_1 \sin\varphi_1, A_2 \sin\varphi_2, \ldots A_N \sin\varphi_N) \\ \dot{\vec{\mathbf{x}}}(0) &\iff (A_1\omega_1 \cos\varphi_1, A_2\omega_2 \cos\varphi_2, \ldots A_N\omega_N \cos\varphi_N) \end{aligned}$$

in the $\vec{\mathbf{u}}_i$ basis.

We now show that the eigenvectors $\vec{\mathbf{u}}_i$ form an orthonormal basis.

Orthonormality of the eigenvector basis Recall how we derived the eigenvalue equation. Starting from our original equation of motion

$$\ddot{\vec{\mathbf{x}}} + M\vec{\mathbf{x}} = 0$$

we performed a linear transformtion to a new set of vectors

$$\vec{\mathbf{q}} = A\vec{\mathbf{x}}$$

which satisfy

$$\ddot{\vec{\mathbf{q}}} + AMA^{-1}\vec{\mathbf{q}} = 0$$

Since M is symmetric, we know that it can be diagonalized by an orthogonal transformation,

$$AMA^{-1} = \begin{pmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & \lambda_N \end{pmatrix}$$

Replacing $\frac{d^2}{dt^2}\left(\overrightarrow{\mathbf{q}}_i e^{i\sqrt{\lambda_i}t}\right) = -\lambda_i\left(\overrightarrow{\mathbf{q}}_i e^{i\sqrt{\lambda_i}t}\right)$ the time dependence drops out and we are left with

$$AMA^{-1}\overrightarrow{\mathbf{q}}_i = \lambda_i \overrightarrow{\mathbf{q}}_i$$

Because AMA^{-1} is diagonal, we immediately solve with the n eigenvectors

$$\overrightarrow{\mathbf{q}}_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \overrightarrow{\mathbf{q}}_2 = \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix}, \ldots, \overrightarrow{\mathbf{q}}_N = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}$$

Then we have N solutions $\overrightarrow{\mathbf{q}}_i e^{i\sqrt{\lambda_i}t}$ for the new position vectors $\overrightarrow{\mathbf{q}}$.

These eigenvectors are clearly orthonormal. But since A^{-1} is an orthogonal transformation, the inner product is preserved and the eigenvectors remain orthonormal in the original basis, that is,

$$\begin{aligned} \overrightarrow{\mathbf{u}}_1 &= A^{-1}\overrightarrow{\mathbf{q}}_1 \\ \overrightarrow{\mathbf{u}}_2 &= A^{-1}\overrightarrow{\mathbf{q}}_2 \\ \vdots \quad &\vdots \quad \vdots \\ \overrightarrow{\mathbf{u}}_N &= A^{-1}\overrightarrow{\mathbf{q}}_N \end{aligned}$$

satisfy

$$\langle \overrightarrow{\mathbf{u}}_i, \overrightarrow{\mathbf{u}}_j \rangle = \delta_{ij}$$

The eigenvectors are therefore orthonormal.

Once we specify the two constant vectors, $\overrightarrow{\mathbf{x}}(0) = (A_i \sin\varphi_i)$ and $\dot{\overrightarrow{\mathbf{x}}}(0) = (\omega_i A_i \cos\varphi_i)$, we have uniquely specified the motion of the system.

2 Vectors in the continuum limit

While we know that the general solution to the wave equation in the continuum limit is given by d'Alembert's formula, it is frequently more useful to divide the motion into normal modes. Despite the continuum limit, we will find that this pattern of a vector description continues.

Considert the motion of a string with endpoints fixed at $x = 0$ and $x = L$. We have seen that we may write solutions of the form

$$\sin\frac{n\pi x}{L}\sin\left(\frac{n\pi ct}{L} + \varphi_n\right)$$

and since this is simple harmonic motion at a single frequency, it is a normal mode.

More general solutions may be specified by taking linear combinations of normal modes, with a general superposition giving a wide range of possible motions,

$$\overrightarrow{\mathbf{q}} = \sum_{n=1}^{\infty} A_n \sin\frac{n\pi x}{L}\sin(\omega_n t + \varphi_n) \tag{1}$$

This is similar to our solution for N masses, but now the upper limit is infinite.

A striking difference between the general solution in the d'Alembert form

$$\overrightarrow{\mathbf{q}}(x,t) = \frac{1}{2}\left[a(x-ct) + a(x+ct) + \frac{1}{c}\int_{x-ct}^{x+ct} g(x')\,dx'\right] \tag{2}$$

and Eq.(1) becomes evident when we look at the initial conditions. For d'Alembert's formula we specify two functions,

$$\begin{aligned}\vec{\mathbf{q}}\,(x,0) &= a\,(x)\\ \dot{\vec{\mathbf{q}}}\,(x,0) &= b\,(x)\end{aligned}$$

while for the mode expansion, it is sufficient to specify two infinite lists,

$$\begin{aligned}A_n \sin\varphi_n &\quad\Longrightarrow\quad \vec{\mathbf{q}}\,(x,0) = \sum_{i=1}^{n} A_i \sin\frac{n\pi x}{L}\sin\varphi_i\\ \omega_n A_n\cos\varphi_n &\quad\Longrightarrow\quad \dot{\vec{\mathbf{q}}}\,(x,0) = \sum_{i=1}^{n}\omega_i A_i \sin\frac{n\pi x}{L}\cos\varphi_i\end{aligned}$$

Our central problem task is to show that these are equivalent. Since we know that functions can be expressed as power series, we know that such a relationship between infinite vectors and functions exists. We need to show that the functions $\sin\frac{n\pi x}{L}$ provide a basis.

2.1 Orthonormality of the sine and cosine

The normal modes on the interval $[0,L]$ are given by $\sin\frac{n\pi x}{L}$. Taking the L^2 inner product, $\int_0^L \sin\frac{n\pi x}{L}\sin\frac{m\pi x}{L}dx$ we add and subtract $\cos\frac{n\pi x}{L}\cos\frac{m\pi x}{L}$ and use the expressions for the cosine of sums and differences.

$$\begin{aligned}\int_0^L \sin\frac{n\pi x}{L}\sin\frac{m\pi x}{L}dx &= \frac{1}{2}\int_0^L\left[\left(\sin\frac{n\pi x}{L}\sin\frac{m\pi x}{L}+\cos\frac{n\pi x}{L}\cos\frac{m\pi x}{L}\right)+\right]dx\\ &\quad+\frac{1}{2}\int_0^L\left(\sin\frac{n\pi x}{L}\sin\frac{m\pi x}{L}-\cos\frac{n\pi x}{L}\cos\frac{m\pi x}{L}\right)dx\\ &= \frac{1}{2}\int_0^L\left[\cos\frac{(n-m)\pi x}{L}+\cos\frac{(n+m)\pi x}{L}\right]dx\end{aligned}$$

For $n\neq m$, the integrals are

$$\begin{aligned}\int_0^L \sin\frac{n\pi x}{L}\sin\frac{m\pi x}{L}dx &= \frac{1}{2}\int_0^L\left[\cos\frac{(n-m)\pi x}{L}+\cos\frac{(n+m)\pi x}{L}\right]dx\\ &= \frac{1}{2}\frac{L}{(n-m)\pi}\sin\frac{(n-m)\pi x}{L}\bigg|_0^L+\frac{1}{2}\frac{L}{(n+m)\pi}\sin\frac{(n+m)\pi x}{L}\bigg|_0^L\\ &= \frac{1}{2}\frac{L}{(n-m)\pi}\sin\frac{(n-m)\pi L}{L}+\frac{1}{2}\frac{L}{(n+m)\pi}\sin\frac{(n+m)\pi L}{L}\\ &= 0\end{aligned}$$

Therefore, $\left\{\sin\frac{n\pi x}{L}\,\middle|\, n=1,2,\ldots\right\}$ form an orthogonal set. For the case when $m=n$, we have

$$\begin{aligned}\int_0^L \sin\frac{n\pi x}{L}\sin\frac{n\pi x}{L}dx &= \int_0^L \sin^2\frac{n\pi x}{L}dx\\ &= \frac{1}{2}\int_0^L\left[\cos 0+\cos\frac{2n\pi x}{L}\right]dx\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2}\left(x - \frac{L}{2n\pi}\sin\frac{2n\pi x}{L}\right)\Big|_0^L \\
&= \frac{L}{2}
\end{aligned}
$$

Therefore, the functions

$$\left\{\sqrt{\frac{2}{L}}\sin\frac{n\pi x}{L}\,\middle|\, n = 1, 2, \ldots\right\}$$

form an orthonormal set.

It makes no difference here that we have chosen the interval $[0, L]$. A linear change of the independent variable

$$
\begin{aligned}
y &= \frac{b-a}{L}x + a \\
x &= \frac{L(y-a)}{b-a}
\end{aligned}
$$

changes the interval to $[a, b]$ and the normal mode series becomes

$$\vec{\mathbf{q}} = \sum_{n=1}^{\infty} A_n \sin\frac{n\pi}{b-a}(y-a)\sin(\omega_n t + \varphi_n)$$

The new amplitudes may be written as

$$
\begin{aligned}
\sin\frac{n\pi}{b-a}(y-a) &= \cos\frac{n\pi a}{b-a}\sin\frac{n\pi y}{b-a} + \sin\frac{n\pi a}{b-a}\cos\frac{n\pi y}{b-a} \\
&= a_n \sin\frac{n\pi y}{b-a} + b_n\cos\frac{n\pi y}{b-a}
\end{aligned}
$$

and we need only show that orthonormality extends to cosines.

Exercise: Complete the proof that $\left\{\sqrt{\frac{2}{L}}\sin\frac{n\pi y}{L}, \sqrt{\frac{2}{L}}\cos\frac{n\pi y}{L}\,\middle|\, n = 0, 1, 2, \ldots\right\}$ is an orthonormal set by showing that

$$
\begin{aligned}
\frac{2}{L}\int_0^L \sin\frac{n\pi x}{L}\cos\frac{m\pi x}{L}dx &= \delta_{nm} \\
\frac{2}{L}\int_0^L \cos\frac{n\pi x}{L}\cos\frac{m\pi x}{L}dx &= \delta_{nm}
\end{aligned}
$$

Exercise: Show that the complex Fourier modes, $\left\{\frac{1}{\sqrt{2\pi}}e^{ikx}\,\middle|\, k = 0, \pm 1, \pm 2, \ldots\right\}$, are orthonormal on the interval $[-\pi, \pi]$, using the complex inner product $\langle f, g\rangle = \int_{-\pi}^{\pi} f^* g\, dx$.

While this shows that we have an orthonormal basis, we do not yet know whether the class of functions we can write as *Fourier series*,

$$f(x) = \sqrt{\frac{2}{L}}\sum_{k=0}^{\infty} a_k \sin\frac{k\pi x}{L}$$

is as general as specifying the arbitrary functions $a(x), b(x)$ for initial conditions. To know this, we must prove *completeness* of the basis. It turns out to be simpler to allow general boundary conditions and write the series as (the real or imaginary part of)

$$f(x) = \frac{1}{\sqrt{2L}}\sum_{k=-\infty}^{\infty} \alpha_k e^{\frac{ik\pi x}{L}}$$

for complex constants α_k.

3 Completeness

By completeness of the Fourier basis we mean that we can write any L^2 function as a Fourier series. Suppose $f(x)$ is square integrable, twice differentiable, and periodic on the interval $[-\pi, \pi]$. We would like to show that we can find constants α_k such that

$$f(x) = \frac{1}{\sqrt{2\pi}} \sum_{k=-\infty}^{\infty} \alpha_k e^{ikx} \tag{3}$$

More precisely we show that

$$\lim_{N\to\infty} f_N(x) = f(x)$$

where $f_N(x)$ is given by the partial sums

$$f_N(x) = \frac{1}{\sqrt{2\pi}} \sum_{k=-N}^{N} \alpha_k e^{ikx}$$

3.1 Determining the constants

The first step is to determine the choice for the constants α_k. Using orthonormality, we multiply both sides of Eq.(3) by $\frac{1}{\sqrt{2L}} e^{-\frac{in\pi x}{L}}$ and integrate over the range $[-L, L]$. This just gives the inner product of f with the basis vector,

$$\begin{aligned}
\left\langle \frac{1}{\sqrt{2L}} e^{inx}, f(x) \right\rangle &= \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} e^{-inx} f(x)\, dx \\
&= \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-inx} \sum_{k=-\infty}^{\infty} \alpha_k e^{ikx} dx \\
&= \sum_{k=-\infty}^{\infty} \alpha_k \delta_{kn} \\
&= \alpha_n
\end{aligned}$$

so the components are given by the integrals

$$\alpha_n = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} e^{-inx} f(x)\, dx$$

This is just how we usually find the components of a finite vector–taking the dot product with the i^{th} basis vector,

$$\langle \vec{\mathbf{v}}, \hat{\mathbf{e}}_i \rangle = v_i$$

Now we study the partial sums $f_N(x)$. Including the specification of the constants $\alpha_k = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} e^{-iny} f(y)\, dy$, these become

$$\begin{aligned}
f_N(x) &= \frac{1}{\sqrt{2\pi}} \sum_{k=-N}^{N} \left(\frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} e^{-iky} f(y)\, dy \right) e^{ikx} \\
&= \frac{1}{2\pi} \int_{-\pi}^{\pi} dy f(y) \sum_{k=-N}^{N} e^{ik(x-y)}
\end{aligned}$$

3.2 Evaluating the sum

The sum is just depends on sums of powers, $\sum_{k=-N}^{N} e^{ik(x-y)} = \sum_{k=-N}^{N} \left[e^{i(x-y)}\right]^k$, so consider such sums in general,

$$\sum_{k=1}^{N} x^k$$

Notice that

$$\begin{aligned}
(1-x)\left(1+x+x^2+\cdots+x^N\right) &= \left(1+x+x^2+\cdots+x^N\right) - x\left(1+x+x^2+\cdots+x^N\right) \\
&= 1+x+x^2+\cdots+x^N - \left(x+x^2+\cdots+x^{N+1}\right) \\
&= 1-x^{N+1}
\end{aligned}$$

Therefore,

$$\sum_{k=0}^{N} x^k = \frac{1-x^{N+1}}{1-x}$$

For the sum from $-N$ we can write

$$\begin{aligned}
\sum_{k=-N}^{N} x^k &= x^{-N}\sum_{k=0}^{2N} x^k \\
&= x^{-N}\frac{1-x^{2N+1}}{1-x} \\
&= \frac{x^{-N}-x^{N+1}}{1-x} \\
&= \frac{x^{1/2}}{x^{1/2}}\frac{x^{-\left(N-\frac{1}{2}\right)}-x^{\left(N+\frac{1}{2}\right)}}{x^{-1/2}-x^{1/2}} \\
&= \frac{x^{\left(N+\frac{1}{2}\right)}-x^{-\left(N-\frac{1}{2}\right)}}{x^{1/2}-x^{-1/2}}
\end{aligned}$$

Therefore, replacing x with $e^{i(x-y)}$,

$$\begin{aligned}
\sum_{k=-N}^{N} e^{ik(x-y)} &= \frac{e^{i\left(N+\frac{1}{2}\right)(x-y)}-e^{-i\left(N-\frac{1}{2}\right)(x-y)}}{x^{i(x-y)/2}-x^{-i(x-y)/2}} \\
&= \frac{\sin\left(\frac{2N+1}{2}(x-y)\right)}{\sin\left(\frac{x-y}{2}\right)}
\end{aligned}$$

3.3 Evaluating the limit

Adding and subtracting $f(x)$ inside the integral, the N^{th} approximation now becomes

$$\begin{aligned}
f_N(x) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(y)\frac{\sin\left(N+\frac{1}{2}\right)(x-y)}{\sin\frac{x-y}{2}}dy \\
&= \frac{1}{2\pi}f(x)\int_{-\pi}^{\pi}\frac{\sin\left(N+\frac{1}{2}\right)(x-y)}{\sin\frac{x-y}{2}}dy + \frac{1}{2\pi}\int_{-\pi}^{\pi}\left(f(y)-f(x)\right)\frac{\sin\left(N+\frac{1}{2}\right)(x-y)}{\sin\frac{x-y}{2}}dy
\end{aligned}$$

Suppose our original function is a constant, α_0. Then every partial sum equals just the zeroth term, so $f_N(x) = \alpha_0$ and $f(y) = \alpha_0$. The second term above drops out leaving

$$\alpha_0 = \frac{1}{2\pi}\alpha_0 \int_{-\pi}^{\pi} \frac{\sin\left(N+\frac{1}{2}\right)(x-y)}{\sin\frac{x-y}{2}} dy$$

showing that the first integral is 1. Therefore, returning to the general case,

$$\begin{aligned} f_N(x) &= f(x) + \frac{1}{2\pi}\int_{-\pi}^{\pi} (f(y) - f(x)) \frac{\sin\left[\left(N+\frac{1}{2}\right)(x-y)\right]}{\sin\frac{x-y}{2}} dy \\ &= f(x) + \frac{1}{2\pi}\int_{-\pi}^{\pi} \left[\frac{(f(y) - f(x))}{\sin\frac{x-y}{2}}\right] \sin\left[\left(N+\frac{1}{2}\right)(x-y)\right] dy \end{aligned}$$

Define

$$h(y) := \frac{(f(y) - f(x))}{\sin\frac{x-y}{2}}$$

and notice that

$$\begin{aligned} h(\pi) &= \frac{f(\pi) - f(x)}{\sin\frac{x-\pi}{2}} \\ &= \frac{f(\pi) - f(x)}{\sin\frac{x}{2}\cos\frac{\pi}{2} - \sin\frac{\pi}{2}\cos\frac{x}{2}} \\ &= -\frac{f(\pi) - f(x)}{\cos\frac{x}{2}} \end{aligned}$$

while

$$\begin{aligned} h(-\pi) &= \frac{f(-\pi) - f(x)}{\sin\frac{x+\pi}{2}} \\ &= \frac{f(-\pi) - f(x)}{\sin\frac{x}{2}\cos\frac{\pi}{2} + \sin\frac{\pi}{2}\cos\frac{x}{2}} \\ &= \frac{f(\pi) - f(x)}{\cos\frac{x}{2}} \end{aligned}$$

where we use the periodicity of the boundary conditions, $f(\pi) = f(-\pi)$. We conclude that $h(\pi) = -h(-\pi)$.

Now integrate the second term by parts,

$$\begin{aligned} f_N(x) &= f(x) + \frac{1}{2\pi}\int_{-\pi}^{\pi} h(y) \sin\left[\left(N+\frac{1}{2}\right)(z-y)\right] \\ &= f(x) + \frac{1}{2\pi}\int_{-\pi}^{\pi} h(y) \left(\frac{-1}{N+\frac{1}{2}}\frac{d}{dy}\cos\left[\left(N+\frac{1}{2}\right)(z-y)\right]\right) \\ &= f(x) - \frac{1}{N+\frac{1}{2}}\frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{d}{dy}\left(h(y)\cos\left[\left(N+\frac{1}{2}\right)(z-y)\right]\right) \\ &\quad + \frac{1}{N+\frac{1}{2}}\frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{dh(y)}{dy}\cos\left[\left(N+\frac{1}{2}\right)(z-y)\right] \end{aligned}$$

The integral of a total derivative gives a boundary term,

$$-\frac{1}{N+\frac{1}{2}}\frac{1}{2\pi}\left(h(y)\cos\left[\left(N+\frac{1}{2}\right)(z-y)\right]\right)\Big|_{-\pi}^{\pi} = -\frac{1}{N+\frac{1}{2}}\frac{1}{2\pi}\left(h(\pi)\cos\left(\frac{2N+1}{2}z-\frac{2N+1}{2}\pi\right)-h(-\pi)\cos\left(\frac{2N+1}{2}z\right.\right.$$

Using the $\cos(a\pm b)=\cos a\cos b\mp\sin a\sin b$ we see that the cosine also changes sign

$$\begin{aligned}\cos\left(\frac{2N+1}{2}z-\frac{2N+1}{2}\pi\right) &= (-1)^N\sin\left(\frac{2N+1}{2}z\right)\\ \cos\left(\frac{2N+1}{2}z+\frac{2N+1}{2}\pi\right) &= -(-1)^N\sin\left(\frac{2N+1}{2}z\right)\end{aligned}$$

The product of the odd cosine terms with the odd $h(\pi)=-h(-\pi)$, gives the same value at $\pm\pi$ and the surface term is zero,

$$\left(h(y)\cos\left[\left(N+\frac{1}{2}\right)(z-y)\right]\right)\Big|_{-\pi}^{\pi}=0$$

We are left with

$$f_N(x) = f(x)+\frac{1}{N+\frac{1}{2}}\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{dh(y)}{dy}\cos\left[\left(N+\frac{1}{2}\right)(z-y)\right]$$

The only remaining concern is the possible divergence of $h(y)$ at $y=x$, since

$$h(y)=\frac{(f(y)-f(x))}{\sin\frac{x-y}{2}}$$

However, for x near y, we let $x=y+\xi$ and expand

$$\begin{aligned}h(\xi) &= \frac{f(y)-f'(y)\xi-f(x)}{\frac{1}{2}\xi}\\ &= -f'(y)\end{aligned}$$

and therefore h' just gives the second derivative. The integrand is bounded, the limit of $\frac{1}{N+\frac{1}{2}}$ is zero, and we have

$$\lim_{N\to\infty}f_N(x)=f(x)$$

4 Fourier series as vectors

Since only second derivatives were assumed in our completeness proof, the result is already more general than the existence of Taylor series, which requires all derivatives. The result can be strengthened further, to show that any bounded, piecewise continuous function on a finite interval may be expanded in a Fourier series. By repeating the interval, the result applies to periodic functions as well.

To summarize, we have shown that like n coupled masses, general solutions of the wave equation may be broken into a sum of simple harmonic motions of normal modes. Solutions may be represented as a elements of a vector space with the normal modes forming an orthonormal basis. The principal difference from the n coupled masses is that the vector space is now infinite dimensional.

We conclude with some examples.

4.1 Example: a square wave

For any $0 < a < b < 1$ let

$$f(x) = \begin{cases} 1 & a < x < b \\ 0 & elsewhere \end{cases}$$

Write $f(x)$ as a Fourier series on the interval $[0, 1]$.

Since $L = 1$ we may write the series in the general form

$$f(x) = \frac{1}{2} b_0 + \sqrt{2} \sum_{k=1}^{\infty} (a_k \sin k\pi x + b_k \cos k\pi x)$$

where the zero mode is just proportional to $\cos(0) = 1$ (the $\frac{1}{2}$ gives the correct weighting) and find the coefficients. Multiplying by $\sqrt{2} \sin n\pi x$ and using orthonormality, we have

$$\begin{aligned} a_n &= \sqrt{2} \int_0^1 f(x) \sin n\pi x \, dx \\ &= \sqrt{2} \int_a^b \sin n\pi x \, dx \\ &= -\frac{\sqrt{2}}{n\pi} (\cos n\pi b - \cos n\pi a) \end{aligned}$$

and multiplying by $\sqrt{2} \cos n\pi x$ and integrating,

$$\begin{aligned} b_n &= \sqrt{2} \int_0^1 f(x) \cos n\pi x \, dx \\ &= \sqrt{2} \int_a^b \cos n\pi x \, dx \\ &= \frac{\sqrt{2}}{n\pi} (\sin n\pi b - \sin n\pi a) \end{aligned}$$

For $n = 0$ we need to check b_0 separately. so $a = 1$.

$$\begin{aligned} b_0 &= \int_0^1 f(x) \cdot 1 \, dx \\ &= \int_a^b dx \\ &= b - a \end{aligned}$$

Therefore,

$$\begin{aligned} f(x) &= \frac{1}{2}(b-a) + \sqrt{2} \sum_{k=1}^{\infty} \left(-\frac{\sqrt{2}}{k\pi} (\cos k\pi b - \cos k\pi a) \sin k\pi x + \frac{\sqrt{2}}{k\pi} (\sin k\pi b - \sin k\pi a) \cos k\pi x \right) \\ &= \frac{1}{2}(b-a) + \frac{2}{\pi} \sum_{k=1}^{\infty} \frac{1}{k} (-\cos k\pi b \sin k\pi x + \cos k\pi a \sin k\pi x + \sin k\pi b \cos k\pi x - \sin k\pi a \cos k\pi x) \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2}(b-a) + \frac{2}{\pi}\sum_{k=1}^{\infty}\frac{1}{k}\left(\frac{1}{2}(\cos k\pi a \sin k\pi x - \sin k\pi a \cos k\pi x) + \frac{1}{2}(\sin k\pi b \cos k\pi x - \cos k\pi b \sin k\pi x)\right) \\
&= \frac{1}{2}(b-a) + \frac{1}{\pi}\sum_{k=1}^{\infty}\frac{1}{k}\left((\sin k\pi (x-a)) - \sin k\pi (x-b)\right)
\end{aligned}
$$

Exercise: For the square wave solution above, let $a = \frac{1}{3}$ and $b = \frac{2}{3}$. Plot the partial sums,

$$
f_N(x) = \frac{1}{2}(b-a) + \frac{1}{\pi}\sum_{k=1}^{N}\frac{1}{k}\left((\sin k\pi (x-a)) - \sin k\pi (x-b)\right)
$$

for $N = 2, 5, 10, 100,$ *and* 1000 to see how the series approaches a unit step.

Exercise: Prove that the only Fourier series that gives the zero function, $f(x) = 0$, has all zero coefficients.

Exercise: Using the previous exercise, and without integrating to calculate any of the coefficients, prove that any symmetric function, $f(-x) = f(x)$, on a symmetric interval (you may take $[-\pi, \pi]$) may be written as a cosine series and that any odd function, $f(-x) = -f(x)$ on the same interval may be written as a sine series.

4.2 Example: A triangle wave

Expand the function

$$
f(x) = \begin{cases} x & 0 < x < \frac{L}{2} \\ L - x & \frac{L}{2} < x < L \\ 0 & elsewhere \end{cases}
$$

on the interval $[0, L]$ in a Fourier series.

Here we can take advantage of the preceeding symmetry exercises to write

$$
f(x) = \sqrt{\frac{2}{L}}\sum_{k=0}^{\infty} a_k \sin\frac{k\pi x}{L}
$$

Notice that $f(x) = f(L-x)$. If we look at the sine on the second half,

$$
\begin{aligned}
\sin\frac{k\pi (L-x)}{L} &= \sin\left(k\pi - \frac{x}{L}\right) \\
&= \sin k\pi \cos\frac{x}{L} - \sin\frac{x}{L}\cos k\pi \\
&= -(-1)^k \sin\frac{x}{L}
\end{aligned}
$$

Therefore, for even k, $\sin\frac{k\pi(L-x)}{L}$ is odd on $[0, L]$, while for k odd, $\sin\frac{k\pi(L-x)}{L}$ is even. Since $f(x)$ is even on this interval, we expect only odd k,

$$
f(x) = \sqrt{\frac{2}{L}}\sum_{m=0}^{\infty} a_{2m+1}\sin\frac{(2m+1)\pi x}{L}
$$

and the coefficients are

$$
\begin{aligned}
a_{2m+1} &= \sqrt{\frac{2}{L}}\int_0^L f(x)\sin\frac{(2m+1)\pi x}{L}dx \\
&= 2\sqrt{\frac{2}{L}}\int_0^{\frac{L}{2}} x\sin\frac{(2m+1)\pi x}{L}dx
\end{aligned}
$$

Now integrate by parts,

$$
\begin{aligned}
a_{2m+1} &= 2\sqrt{\frac{2}{L}}\int\limits_0^{\frac{L}{2}}\left(x\left(-\frac{L}{(2m+1)\pi}\frac{d}{dx}\cos\frac{(2m+1)\pi x}{L}\right)\right)dx \\
&= 2\sqrt{\frac{2}{L}}\left(-\frac{L}{(2m+1)\pi}x\cos\frac{(2m+1)\pi x}{L}\right)\Bigg|_0^{\frac{L}{2}} + \frac{2L}{(2m+1)\pi}\sqrt{\frac{2}{L}}\int\limits_0^{\frac{L}{2}}\cos\frac{(2m+1)\pi x}{L}dx \\
&= 2\sqrt{\frac{2}{L}}\left(-\frac{L^2}{2(2m+1)\pi}\cos\frac{(2m+1)\pi L}{2L}\right) + \frac{2L}{(2m+1)\pi}\sqrt{\frac{2}{L}}\frac{L}{(2m+1)\pi}\sin\frac{(2m+1)\pi L}{2L} \\
&= -\frac{L^2}{(2m+1)\pi}\sqrt{\frac{2}{L}}\cos\frac{(2m+1)\pi}{2} + \sqrt{\frac{2}{L}}\frac{2L^2}{(2m+1)^2\pi^2}\sin\frac{(2m+1)\pi}{2} \\
&= (-1)^m\sqrt{\frac{2}{L}}\frac{2L^2}{(2m+1)^2\pi^2}
\end{aligned}
$$

Therefore,

$$
f(x) = \frac{4L}{\pi^2}\sum_{m=0}^{\infty}\frac{(-1)^m}{(2m+1)^2}\sin\frac{(2m+1)\pi x}{L}
$$

Exercise: We argued by symmetry that the even terms in the Fourier series of

$$
f(x) = \begin{cases} x & 0 < x < \frac{L}{2} \\ L - x & \frac{L}{2} < x < L \\ 0 & elsewhere \end{cases}
$$

on the interval $[0, L]$ must vanish. Compute the coefficients of the even terms

$$
a_{2m} = \sqrt{\frac{2}{L}}\int\limits_0^L f(x)\sin\frac{2m\pi x}{L}dx
$$

explicitly to show that they do indeed all vanish.

Exercise: Compute the Fourier series of the function

$$
f(x) = \begin{cases} A\left(x - \frac{L}{2}\right)^2 & 0 < x < L \\ 0 & elsewhere \end{cases}
$$

on the interval $[0, L]$.