# Fourier Analysis

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## 1 Fourier series as normal mode solutions to an eigenvalue problem

We have seen how any bounded, piecewise continuous function may be written as a Fourier series. Now, we return to our picture of solutions to the wave equation to place this statement in terms of normal modes and eigenvectors.

On an interval $[0, L]$ we begin with the wave equation

$$-\frac{1}{c^2}\frac{\partial^2 q}{\partial t^2}+\frac{\partial^2 q}{\partial x^2}=0$$

and substitute a *single frequency mode* for the time dependence,

$$q(x,t)=\varphi_n(x)\sin(\omega_n t+\varphi_n)$$

This gives

$$\begin{aligned}-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\left(\varphi_n(x)\sin(\omega_n t+\varphi_n)\right)+\frac{\partial^2}{\partial x^2}\left(\varphi_n(x)\sin(\omega_n t+\varphi_n)\right) &= 0\\ \frac{\omega_n^2}{c^2}\varphi_n(x)\sin(\omega_n t+\varphi_n)+\frac{\partial^2\varphi_n(x)}{\partial x^2}\sin(\omega_n t+\varphi_n) &= 0\end{aligned}$$

so cancelling the time dependence we have an eigenvalue equation,

$$\frac{\partial^2\varphi_n(x)}{\partial x^2}=-\frac{\omega_n^2}{c^2}\varphi_n(x)$$

As we found for coupled masses, the eigenvalues are minus the squares of the frequencies, $-\omega_n^2$. This time, in place of an *eigenvector*, we have an *eigenfunction*, $\varphi_n(x)$. We easily see that orthonormal solutions to this eigenfunction equation are

$$\varphi_n=a_n\sin\frac{n\pi x}{L}+b_n\cos\frac{n\pi x}{L}$$

Choosing $a_n$ and $b_n$ to normalize to 1,

$$\int_0^L\varphi_n^2(x)=1$$

we have

$$\sqrt{\frac{2}{L}}\sin\frac{n\pi x}{L},\sqrt{\frac{2}{L}}\cos\frac{n\pi x}{L}$$

as the eigenfunctions, with normal mode frequencies coming from the eigenvalues as

$$\omega_n=\frac{\pi nc}{L}$$

A general solution to the wave equation is a linear superposition of the normal modes,

$$q(x,t)=\sum_{n=0}^{\infty}\left(a_n\sin\frac{n\pi x}{L}+b_n\cos\frac{n\pi x}{L}\right)\sin\left(\omega_n t+\varphi_n\right) \tag{1}$$

Thus, the solution to the wave equation may be described as linear combinations of a countably infinite sum of simple harmonic oscillations.

### 1.1 Importance for finding time evolution

The normal mode form of solutions to a wide class of wave equations allows us to easily predict the time evolution of a solution from an initial spatial distribution. Consider, for example, the 1-dimensional *Schrödinger equation*,

$$-\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2}+V(x)\,\psi=i\hbar\frac{\partial\psi}{\partial t} \tag{2}$$

This equation has only a first time derivative instead of $\frac{\partial^2\psi}{\partial t^2}$ as for string. This has its roots in the uncertainty principle. Because we cannot know both the initial time and initial position of a quantum particle exactly, it is impossible to specify the two initial conditions required for a second order equation. Even though the time derivative is only linear, this equation has wavelike solutions.

To apply the normal mode approach to this more general equation, we again assume single-frequency solutions, setting

$$\psi(x,t)=\varphi_E(x)\,e^{-\frac{i}{\hbar}Et}$$

for some constant $E$. Substituting into the Schrödinger equation,

$$\begin{aligned}-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\left(\varphi_E(x)\,e^{-\frac{i}{\hbar}Et}\right)+V(x)\,\varphi_E(x)\,e^{-\frac{i}{\hbar}Et} &= i\hbar\frac{\partial}{\partial t}\left(\varphi_E(x)\,e^{-\frac{i}{\hbar}Et}\right)\\ \left(-\frac{\hbar^2}{2m}\frac{\partial^2\varphi_E(x)}{\partial x^2}+V(x)\,\varphi_E(x)\right)e^{-\frac{i}{\hbar}Et} &= E\varphi_E(x)\,e^{-\frac{i}{\hbar}Et}\end{aligned}$$

and cancelling the time dependence, arrive at the *stationary state Schrödinger equation*,

$$-\frac{\hbar^2}{2m}\frac{d^2\varphi_E(x)}{dx^2}+V(x)\,\varphi_E(x)=E\varphi_E(x) \tag{3}$$

You may recognize this as an eigenvalue equation. On the left, we have a linear differential operator acting on $\varphi(x)$, and on the right a constant times $\varphi(x)$. Indeed, the operator on the right is often replaced by a matrix operator, depending on the problem. The solutions may no longer be simple sines and cosines, but we have a simpler equation to solve.

Solutions to Eq.(3) may be either continuous, $\varphi_k(x)\,, kreal$, for unbound problems, or discrete, $\varphi_n(x)\,, n=1,2,\ldots$ for bound states. Suppose we have a bound state so that the eigenvalues may be labeled by an integer, $E_n$. Then labelling the solutions as $\varphi_n(x)$ the initial wave function is given by an arbitrary superposition,

$$\psi(x,0)=\sum_{n=0}^{\infty}a_n\varphi_n(x)$$

where the constants $a_n$ determined by the initial conditions.

The remarkable fact is that we can now *immediately* write the full time-dependent solution

$$\psi(x,t)=\sum_{n=0}^{\infty}a_n\varphi_n(x)\,e^{-\frac{i}{\hbar}E_n t}$$

since each normal mode oscillates with the single frequency $\frac{E_n}{\hbar}$.

The method applies to a wide class of wave equations.

## 1.2 Time evolution of wave solutions

As an example of finding the time evolution of an initial solution, suppose we deform a guitar string of unit length with into a square initial pulse and release it from rest. We have already seen that the initial square wave

$$f(x) = \begin{cases} 1 & a < x < b \\ 0 & elsewhere \end{cases}$$

may be written as the Fourier series

$$f(x) = \frac{1}{2}(b-a) + \frac{2}{\pi}\sum_{k=1}^{\infty}\frac{1}{k}\left(-(\cos k\pi b - \cos k\pi a)\sin k\pi x + (\sin k\pi b - \sin k\pi a)\cos k\pi x\right)$$

(We further reduced this to $f(x) = \frac{1}{2}(b-a) + \frac{1}{\pi}\sum_{k=1}^{\infty}\frac{1}{k}\left((\sin k\pi(x-a)) - \sin k\pi(x-b)\right)$, but here we need the pure $\sin k\pi x$ and $\cos k\pi x$ terms.)

To find the time evolution, we need only multiply the $k^{th}$ mode of the initial series by $\sin(\omega_k t + \varphi_k)$, then choose the phases $\varphi_k$ to match the initial conditions. Therefore, the displacement of the guitar string is given by

$$q(x,t) = \frac{1}{2}(b-a) + \frac{2}{\pi}\sum_{k=1}^{\infty}\left(-\frac{1}{k}(\cos k\pi b - \cos k\pi a)\sin k\pi x + \frac{1}{k}(\sin k\pi b - \sin k\pi a)\cos k\pi x\right)\sin(\omega_k t + \varphi_k)$$

For initial conditions, we require $q(x,0) = f(x)$. Therefore

$$\sin(\omega_k t + \varphi_k)|_{t=0} = \sin\varphi_k = 1$$

and we choose the phases to be $\varphi_k = \frac{\pi}{2}$. Then $\sin\left(\omega_k t + \frac{\pi}{2}\right) = \cos\omega_k t$ satisfies both the initial position and velocity conditions, since the time derivative, $\omega_k \sin\omega_k t$, vanishes at $t = 0$.

Next, we write $q(x,t)$ in terms of right and left moving waves. To do this, recall the addition formulas

$$\begin{aligned} \sin a\cos b &= \frac{1}{2}(\sin(a+b) + \sin(a-b)) \\ \cos a\cos b &= \frac{1}{2}(\cos(a-b) + \cos(a+b)) \end{aligned}$$

Using these we have

$$\begin{aligned} q(x,t) &= \frac{1}{2}(b-a) + \frac{2}{\pi}\sum_{k=1}^{\infty}\frac{1}{k}\left(-(\cos k\pi b - \cos k\pi a)\sin k\pi x\cos\omega_k t + (\sin k\pi b - \sin k\pi a)\cos k\pi x\cos\omega_k t\right) \\ &= \frac{1}{2}(b-a) + \frac{1}{\pi}\sum_{k=1}^{\infty}\frac{1}{k}\left(-(\cos k\pi b - \cos k\pi a)(\sin k\pi(x+ct) + \sin k\pi(x-ct))\right) \\ &\quad + \frac{1}{\pi}\sum_{k=1}^{\infty}\frac{1}{k}\left((\sin k\pi b - \sin k\pi a)(\cos(k\pi(x+ct)) + \cos k\pi(x-ct))\right) \end{aligned}$$

Collecting the right and left moving pieces, we have

$$\begin{aligned} q(x,t) &= \frac{1}{4}(b-a) + \frac{1}{\pi}\sum_{k=1}^{\infty}\frac{1}{k}\left(-(\cos k\pi b - \cos k\pi a)\sin k\pi(x+ct) + (\sin k\pi b - \sin k\pi a)\cos(k\pi(x+ct))\right) \\ &\quad + \frac{1}{4}(b-a) + \frac{1}{\pi}\sum_{k=1}^{\infty}\frac{1}{k}\left(-(\cos k\pi b - \cos k\pi a)\sin k\pi(x-ct) + (\sin k\pi b - \sin k\pi a)\cos k\pi(x-ct)\right) \end{aligned}$$

Each part reproduces the original wave with half the amplitude. A half-height square wave moves off to the left, and an idential half-height square wave moves off to the right.

***Exercise:*** Suppose we pluck a guitar string of length $L$ and fixed ends by raising the center to form a triangle, then releasing it from rest. We have already seen that the initial triangle wave

$$f(x) = \begin{cases} x & 0 < x < \frac{L}{2} \\ L - x & \frac{L}{2} < x < L \\ 0 & elsewhere \end{cases}$$

may be represented by a Fourier series,

$$f(x) = \frac{4L}{\pi^2} \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m+1)^2} \sin \frac{(2m+1)\pi x}{L}$$

1. Find the time evolution of the guitar string if we release the it at rest from its stretched triangular position.
2. Write the solution as a sum of right moving and left moving waves.
3. Describe the resulting waves.

## 1.3 Equivalence to the integrated general solution

We would like to show that the general normal mode superpostion, Eq.(1), may be written as a sum of right- and left-moving functions. Begin with the general Fourier series,

$$\begin{aligned} q(x,t) &= \sum_{n=0}^{\infty} (a_n \sin k_n x + b_n \cos k_n x) \sin(\omega_n t + \varphi_n) \\ &= \sum_{n=0}^{\infty} (a_n \sin k_n x \sin(\omega_n t + \varphi_n) + b_n \cos k_n x \sin(\omega_n t + \varphi_n)) \end{aligned}$$

where $k_n = \frac{n\pi x}{L}$ and $\omega_n = \frac{n\pi c x}{L}$. Using $\sin(a+b) = \cos a \sin b + \cos b \sin a$ to separate the initial phases and regrouping, the first $(a_n)$ sum becomes

$$\sum_{n=0}^{\infty} a_n \sin k_n x (\cos \omega_n t \sin \varphi_n + \cos \varphi_n \sin \omega_n t) = \sum_{n=0}^{\infty} (a_n \sin \varphi_n (\sin k_n x \cos \omega_n t) + a_n \cos \varphi_n (\sin k_n x \sin \omega_n t))$$

with a similar result for the $b_n$ sum. Now, grouping the $x$ and $t$ dependent terms, we use the sum and difference formulas to write each of the new products as a sum and difference, for example

$$\sin k_n x \cos \omega_n t = \frac{1}{2}\left(\sin\left(\frac{n\pi x}{L} + \omega_n t\right) - \sin\left(\frac{n\pi x}{L} - \omega_n t\right)\right)$$

All the terms in the sum may be rewritten in this way, leading to

$$\begin{aligned} q(x,t) &= \sum_{n=0}^{\infty} \left(a_n \sin \frac{n\pi x}{L} \sin(\omega_n t + \varphi_n) + b_n \cos \frac{n\pi x}{L} \sin(\omega_n t + \varphi_n)\right) \\ &= \frac{1}{2} \sum_{n=0}^{\infty} a_n \sin \varphi_n \left(\sin\left(\frac{n\pi x}{L} + \omega_n t\right) - \sin\left(\frac{n\pi x}{L} - \omega_n t\right)\right) \\ &\quad + \frac{1}{2} \sum_{n=0}^{\infty} a_n \cos \varphi_n \left(-\cos\left(\frac{n\pi x}{L} + \omega_n t\right) + \cos\left(\frac{n\pi x}{L} - \omega_n t\right)\right) \\ &\quad + \frac{1}{2} \sum_{n=0}^{\infty} b_n \sin \varphi_n \left(\cos\left(\frac{n\pi x}{L} + \omega_n t\right) + \cos\left(\frac{n\pi x}{L} - \omega_n t\right)\right) \\ &\quad + \frac{1}{2} \sum_{n=0}^{\infty} b_n \cos \varphi_n \left(\sin\left(\frac{n\pi x}{L} + \omega_n t\right) + \sin\left(\frac{n\pi x}{L} - \omega_n t\right)\right) \end{aligned}$$

Collecting the right- and left-moving modes, we have

$$\begin{aligned} q(x,t) &= \frac{1}{2}\sum_{n=0}^{\infty}\left((a_n\sin\varphi_n + b_n\cos\varphi_n)\sin\left(\frac{n\pi x}{L}+\omega_n t\right) + (b_n\sin\varphi_n - a_n\cos\varphi_n)\cos\left(\frac{n\pi x}{L}+\omega_n t\right)\right) \\ &\quad +\frac{1}{2}\sum_{n=0}^{\infty}\left((b_n\cos\varphi_n - a_n\sin\varphi_n)\sin\left(\frac{n\pi x}{L}-\omega_n t\right) + (b_n\sin\varphi_n + a_n\cos\varphi_n)\cos\left(\frac{n\pi x}{L}-\omega_n t\right)\right) \end{aligned}$$

Now define

$$\begin{aligned} A_n &= a_n\sin\varphi_n + b_n\cos\varphi_n \\ B_n &= b_n\sin\varphi_n - a_n\cos\varphi_n \\ C_n &= b_n\cos\varphi_n - a_n\sin\varphi_n \\ D_n &= b_n\sin\varphi_n + a_n\cos\varphi_n \end{aligned}$$

and notice that these four linear combinations of $a_n, b_n$ and $\sin\varphi_n, \cos\varphi_n$ are independent. Therefore,

$$\begin{aligned} q(x,t) &= \frac{1}{2}\sum_{n=0}^{\infty}\left(A_n\sin\left(\frac{n\pi x}{L}+\omega_n t\right) + B_n\cos\left(\frac{n\pi x}{L}+\omega_n t\right)\right) \\ &\quad +\frac{1}{2}\sum_{n=0}^{\infty}\left(C_n\sin\left(\frac{n\pi x}{L}-\omega_n t\right) + D_n\cos\left(\frac{n\pi x}{L}-\omega_n t\right)\right) \end{aligned}$$

Because $A_n$ and $B_n$ are independent constants, $\frac{1}{2}\sum_{n=0}^{\infty}\left(A_n\sin\left(\frac{n\pi x}{L}+\omega_n t\right) + B_n\cos\left(\frac{n\pi x}{L}+\omega_n t\right)\right)$ is an arbitrary function of $\frac{n\pi x}{L}+\omega_n t$. Similarly, the independence of $C_n$ and $D_n$make the second sum an arbitrary function of $\frac{n\pi x}{L}-\omega_n t$. Calling these functions $f$ and $g$ respectively,

$$q(x,t) = f\left(\frac{n\pi x}{L}+\omega_n t\right) + g\left(\frac{n\pi x}{L}-\omega_n t\right)$$

in agreement with our direct integration of the wave equation. Furthermore, we have shown that $f$ and $g$ may be any piecewise continuous functions.

## 2 Fourier transform

We have seen how to represent a wide class of functions on a bounded interval, $[-L, L]$. Now we look at what happens when we let the interval expand to the whole real line.

Start with the Fourier series for a function $f(x)$ on a symmetric interval, written as the real or imaginary part of

$$f(x) = \frac{1}{\sqrt{2\pi}}\sum_{n=-\infty}^{\infty} A_n e^{ik_n x}$$

The wave vector $k_n$ is restricted to the normal modes, $k_n = \frac{n\pi}{2L}$ and the frequencies to $\omega_n = k_n c = \frac{n\pi c}{L}$.

The spacing between adjacent wave numbers $k_n$ is

$$\begin{aligned} k_{n+1} - k_n &= \frac{(n+1)\pi}{L} - \frac{n\pi}{L} \\ &= \frac{\pi}{L} \end{aligned}$$

This spacing vanishes as we take the limit $L \longrightarrow \infty$. This means that every wave number $k$ is allowed. In the same limit, the sum over integers becomes an integral over all wave vectors $k$. Since $k_n$ is proportional to $n$ we may think of the constants $A_n$ as dependent on $k_n$, $A_n = A(k_n)$. As $k$ becomes continuous, $A_n$ becomes a continuous complex function, $A(k)$. Then the Fourier series becomes a *Fourier integral,*

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dk\, A(k)\, e^{ikx}$$

We will show that knowing the function $A(k)$ is equivalent to knowing $f(x)$. Because of this, we often give them related names, such as $f(x)$ and $\tilde{f}(k)$,

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dk\, \tilde{f}(k)\, e^{ikx} \tag{4}$$

The new function $\tilde{f}(k)$ is called the *Fourier transform* of $f(x)$.

To see that a function and its Fourier transform are equivalent, we need to show that we may invert Eq.(4), solving for $\tilde{f}(k)$ in terms of $f(x)$. It is not unreasonable to guess that we can do this the same way we found the coefficients in the Fourier series–multiplying by $\frac{1}{\sqrt{2\pi}} e^{-iqx}$ and integrating over $x$. This leads to a somewhat odd requirement which is nonetheless correct:

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{-iqx} dx = \frac{1}{2\pi} \int_{-\infty}^{\infty} dk\, \tilde{f}(k)\, e^{i(k-q)x}$$

If this is to give the Fourier transform $\tilde{f}(q)$, we must have

$$\tilde{f}(q) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dk\, \tilde{f}(k)\, e^{i(k-q)x}$$

This is the defining property of the *Dirac delta function*, $\delta(k-q)$. We need to digress to define this new object.

# 3 The Dirac delta function

The Dirac delta function is, curiously, not a function at all but a *distribution*. For our purposes, a distribution is the limit of an infinite sequence of functions, and will have meaning only under an integral sign. The same distribution may be written in many different ways as a sequence of functions; what is important is how the limit behaves in integrals. Once we understand how the Dirac delta function works in integrals, we usually do not need to write it as a limit of functions.

It is not wrong to think of the Dirac delta function as a continuous generalization of the Kronecker delta, in the following sense. Just as a sum of a series with a Kronecker delta pulls out a particular element of the sum,

$$\sum_{n=0}^{\infty} \alpha_n \delta_{kn} = \alpha_k$$

because $\delta_{kn}$is zero unless $k = n$, the Dirac delta singles out a particular value from an integral

$$\int_{-\infty}^{\infty} f(x)\, \delta(x - x_0)\, dx = f(x_0) \tag{5}$$

In a vague sense, the delta function vanishes except at $x = x_0$ but is so large there that its integral is 1. Notice that if we set $f(x) = 1$ in Eq.(5), we get that the integral of the Dirac delta is 1,

$$\int_{-\infty}^{\infty} \delta(x - x_0)\, dx = 1$$

However, over any interval not containing the point $x = x_0$, the integral gives zero. For any $a > 0$,

$$\begin{aligned} \int_{-\infty}^{x_0 - a} \delta(x - x_0)\, dx &= 0 \\ \int_{x_0 + a}^{\infty} \delta(x - x_0)\, dx &= 0 \end{aligned}$$

We need to write $\delta(x)$ as the limit of a series to make these ideas well-defined.

We define the Dirac delta function as the limit of any sequence of functions $h_n(x)$ such that for any smooth function $f(x)$ which vanishes outside a bounded region (called a *test function*)

$$\lim_{n \longrightarrow \infty} \int_{-\infty}^{\infty} h_n(x) f(x)\, dx = f(0)$$

The functions $h_n(x)$ may be any ones with this property. Loosely, we write

$$\delta(x) = \lim_{n \longrightarrow \infty} h_n(x)$$

but we must remember that this only has meaning when we integrate it.

### 3.1 Gaussian representation of the Dirac delta function

One series that gives a nice intuitive picture of the Dirac delta is as a limit of normalized Gaussians (see Section 3 in the Lecture notes on Solutions to the wave equation). Let

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{x^2}{2\sigma^2}\right)$$

This satisfies

$$\frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} \exp\left(-\frac{x^2}{2\sigma^2}\right) dx = 1$$

for any $\sigma$. The number $\sigma$ characterizes the width of the Gaussian, and also the height since $f(0) = \frac{1}{\sqrt{2\pi}} \frac{1}{\sigma}$. As we take $\sigma$smaller and smaller, our Gaussian grows taller and narrower, but always encloses unit area beneath its curve. Define a sequence of functions given by setting $\sigma = \frac{1}{n}$.

$$h_n(x) = \frac{n}{\sqrt{2\pi}} \exp\left(-\frac{1}{2} n^2 x^2\right)$$

For large $n$, the maximum of $h_n$ increases without bound, but because the exponential decays with $n^2$, $h_n$ drops to negligible values very quickly. To see that

$$\delta(x) = \lim_{n \longrightarrow \infty} h_n(x)$$

we integrate with a test function and evaluate the limit,

$$\lim_{n\longrightarrow\infty}\int\limits_{-\infty}^{\infty} h_n(x) f(x)\, dx = \lim_{n\longrightarrow\infty}\int\limits_{-\infty}^{\infty}\frac{n}{\sqrt{2\pi}}\exp\left(-\frac{1}{2}n^2x^2\right) f(x)\, dx$$

Choose $n$ large enough such that $f(x)$ changes very little for $x \in \left(-\frac{1}{n},\frac{1}{n}\right)$. Since $f(x)$ is continuous, no matter how fast $f(x)$ is changing, we can always find $n$large enough that the change in $f(x)$ is as small as we like on $\left(-\frac{1}{n},\frac{1}{n}\right)$. Concretely, continuity means that for any $\delta > 0$ we can find sufficiently large $n$ such that for all $x \in \left(-\frac{1}{n},\frac{1}{n}\right)$, we have $f(x) \in (f(0) - \delta_n, f(0) + \delta_n)$. We may therefore write

$$f(x) = f(0 + \delta_n g(x))$$

on the entire interval $x \in \left(-\frac{1}{n},\frac{1}{n}\right)$, where $|g(x)| < 1$. Moreover, as $n \longrightarrow \infty, \delta_n \longrightarrow 0$. The Gaussian integral may be written as

$$\begin{aligned}
\lim_{n\longrightarrow\infty}\int\limits_{-\infty}^{\infty} h_n(x) f(x)\, dx &= \lim_{n\longrightarrow\infty} f(0)\int\limits_{-\infty}^{\infty}\frac{n}{\sqrt{2\pi}}\exp\left(-\frac{1}{2}n^2x^2\right) dx \\
&= \lim_{n\longrightarrow\infty}\left[f(0)\int\limits_{-\infty}^{\infty}\frac{n}{\sqrt{2\pi}}\exp\left(-\frac{1}{2}n^2x^2\right) dx + \delta_n\int\limits_{-\infty}^{\infty}\frac{n}{\sqrt{2\pi}}\exp\left(-\frac{1}{2}n^2x^2\right) g(x)\, dx\right]
\end{aligned}$$

Since the Gaussians are all normalized to 1 and $|g(x)| < 1$,

$$\begin{aligned}
\delta_n\int\limits_{-\infty}^{\infty}\frac{n}{\sqrt{2\pi}}\exp\left(-\frac{1}{2}n^2x^2\right) g(x)\, dx &< \delta_n\int\limits_{-\infty}^{\infty}\frac{n}{\sqrt{2\pi}}\exp\left(-\frac{1}{2}n^2x^2\right)|g(x)|\, dx \\
&\leq \delta_n\int\limits_{-\infty}^{\infty}\frac{n}{\sqrt{2\pi}}\exp\left(-\frac{1}{2}n^2x^2\right) dx \\
&= \delta_n
\end{aligned}$$

Because

$$\lim_{n\longrightarrow\infty}\delta_n = 0$$

the second term vanishes in the limit. The Gaussian integral multiplying $f(0)$ gives 1, and we have

$$\lim_{n\longrightarrow\infty}\int\limits_{-\infty}^{\infty} h_n(x) f(x)\, dx = f(0)$$

We may write

$$\delta(x) = \lim_{n\longrightarrow\infty}\frac{n}{\sqrt{2\pi}}\exp\left(-\frac{1}{2}n^2x^2\right)$$

## 3.2 Properties of the Dirac delta function

For any (reasonable) function $f(x)$,

$$\int\limits_{-\infty}^{\infty}\delta(x) f(x)\, dx = f(0)$$

The singular point of $\delta(x)$ occurs when its argument is zero, but we could make this happen at any point by setting

$$\delta(x-x_0) = \lim_{n\longrightarrow\infty} \frac{n}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}n^2(x-x_0)^2\right)$$

Then

$$\int_{-\infty}^{\infty} \delta(x-x_0) f(x)\, dx = f(x_0)$$

Now consider $\delta(ax)$ for any constant $a$. A change of variable makes the result clear. With $y = ax$,

$$\begin{aligned}\int_{-\infty}^{\infty} \delta(ax) f(x)\, dx &= \int_{-\infty}^{\infty} \delta(y) f(ay)(a dy) \\ &= a \int_{-\infty}^{\infty} \delta(y) f(ay)\, dy \\ &= a f(0)\end{aligned}$$

We can even make the argument of $\delta$ into a function. The function $x^2-4 = (x-2)(x+2)$ has zeros at $\pm 2$. If we write

$$\delta\left(x^2-4\right)$$

there will be two points where the Dirac delta becomes infinite. The effect of integrating $f(x)$ with $\delta\left(x^2-4\right)$ is equivalent to having two delta functions,

$$\delta\left(x^2-4\right) = \alpha\delta(x-2) + \beta\delta(x+2)$$

but we need to know the relative weights. To find them, we look in a small neighborhood of one of the roots, say $x \in (2-\varepsilon, 2+\varepsilon)$. Then

$$\begin{aligned}\int_{2-\varepsilon}^{2+\varepsilon} \delta\left(x^2-4\right) f(x)\, dx &= \alpha \int_{2-\varepsilon}^{2+\varepsilon} \delta(x-2) f(x)\, dx \\ \int_{2-\varepsilon}^{2+\varepsilon} \delta\left(x^2-4\right) f(x)\, dx &= \alpha f(2)\end{aligned}$$

Again we change variable, letting $y = x^2-4$. Then $dy = 2x dx$ so that

$$\begin{aligned}dx &= \frac{dy}{2x} \\ &= +\frac{dy}{2\sqrt{y+4}}\end{aligned}$$

Therefore,

$$\begin{aligned}\int_{2-\varepsilon}^{2+\varepsilon} \delta\left(x^2-4\right) f(x)\, dx &= \int_{2-\varepsilon}^{2+\varepsilon} \delta(y) f\left(+\sqrt{y+4}\right) \frac{dy}{2\sqrt{y+4}} \\ &= \left.\frac{f\left(+\sqrt{y+4}\right)}{2\sqrt{y+4}}\right|_{y=0} \\ &= \frac{f(2)}{4}\end{aligned}$$

Exercise: By studying $\int \delta(g(x)) f(x)\, dx$ in a sufficiently small neighborhood of each zero, prove that for any smooth function $g(x)$ with isolated simple zeros at points $x_i, i = 1, 2, \ldots, n$,

$$\delta(g(x)) = \sum_{i=1}^{n} \frac{1}{|g'(x_i)|} \delta(x - x_i)$$

where $g'(x_i)$ is the first derivative of $g(x)$ evaluated at the $i^{th}$ pole.

## 3.3 Derivatives of the Dirac delta

The derivative of a Dirac delta function is defined by using integration by parts. For any test function $f(x)$,

$$\begin{aligned}
\int_{-\infty}^{\infty} \left(\frac{d}{dx}\delta(x)\right) f(x)\, dx &= \int_{-\infty}^{\infty} \frac{d}{dx}(\delta(x) f(x))\, dx - \int_{-\infty}^{\infty} \delta(x) \frac{d}{dx} f(x)\, dx \\
&= (\delta(x) f(x))\big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} \delta(x) \frac{d}{dx} f(x)\, dx \\
&= -\int_{-\infty}^{\infty} \delta(x) \frac{d}{dx} f(x)\, dx \\
&= -\frac{df}{dx}(0)
\end{aligned}$$

where the surface term vanishes because test functions $f(x)$ vanish at $\pm\infty$. Higher derivatives of the Dirac delta function are defined in the same way.

## 3.4 Fourier representation of the Dirac delta

Consider the series of functions

$$f_N(x) = \frac{1}{2\pi} \int_{-N}^{N} dk\, e^{ikx}$$

Carrying out the integration,

$$\begin{aligned}
f_N(x) &= \frac{1}{2\pi} \int_{-N}^{N} dk\, e^{ikx} \\
&= \frac{1}{2\pi} \frac{1}{ix}\, e^{ikx}\Big|_{-N}^{N} \\
&= \frac{1}{2\pi i} \frac{1}{x} \left(e^{iNx} - e^{-iNx}\right) \\
&= \frac{\sin Nx}{\pi x}
\end{aligned}$$

To show that this series defines a delta function, we need to compute $\lim_{N\to\infty} \int_{-\infty}^{\infty} \frac{\sin Nx}{\pi x} f(x)\, dx$ for an arbitrary test function $f(x)$. The integral requires contour integration in the complex plane.

## 4 Contour integration and the residue theorem

In our discussion of complex numbers we noted that any analytic function of $x$ (that is, a function with a convergent power series) has a complex extension. If $f(x)$ is given in some region by

$$f(x)=\sum_{k=0}^{\infty}a_k x^k$$

then the analytic extension is

$$f(z)=\sum_{k=0}^{\infty}a_k z^k$$

where $z=x+iy$.

Suppose we wish to integrate a function $f(z)$ along a curve in any region where this power series converges. Then we can show that the value of the integral does not change if we shift the curve continuously within the region, holding the endpoints fixed.

A complex function fails to be integrable if it has poles–points where the value becomes undefined. This can happen when the function is given by a *Laurent series*,

$$f(z)=\sum_{k=-n}^{\infty}a_k z^k$$

Since the series starts at some $-n<0$, there are terms which diverge,

$$f(z)=\frac{a_{-n}}{z^n}+\frac{a_{-n+1}}{z^{n-2}}+\dots$$

and we ask what happens in an otherwise well-behaved region if our curve of integration encloses one of these poles. Since the region is analytic all around the pole, we may deform the curve to a small circle around the pole. Then the integral of the Laurent series takes the form

$$\begin{aligned}\oint f(z)\,dz &= \sum_{k=-n}^{\infty}a_k z^k dz\\ &= \sum_{k=-n}^{\infty}a_k\oint z^k dz\end{aligned}$$

Writing $z=\varepsilon e^{i\theta}$ and $dz=\varepsilon i e^{i\theta}d\theta$ where $\varepsilon$ is the constant radius of our circle, this becomes

$$\begin{aligned}\oint f(z)\,dz &= \sum_{k=-n}^{\infty}a_k\varepsilon^k\oint e^{ik\theta}i\varepsilon e^{i\theta}d\theta\\ &= \sum_{k=-n}^{\infty}a_k i\varepsilon^{k+1}\oint e^{i(k+1)\theta}d\theta\\ &= \sum_{k=-n}^{\infty}a_k\varepsilon^{k+1}\frac{1}{k+1}\left.e^{i(k+1)\theta}\right|_0^{2\pi}\\ &= \sum_{k=-n}^{\infty}a_k\varepsilon^{k+1}\frac{1}{k+1}\left(e^{2\pi i(k+1)}-1\right)\\ &= 0\end{aligned}$$

even for negative powers, except when $k=-1$. Surprisingly, most orders of pole contribute nothing to closed integrals. The one exception is the *simple pole*, $\frac{a_{-1}}{z}$. For this, we find

$$\begin{aligned}\oint \frac{a_{-1}}{z} dz &= \frac{a_{-1}}{\varepsilon} \oint e^{-i\theta} i\varepsilon e^{i\theta} d\theta \\ &= i a_{-1} \oint d\theta \\ &= 2\pi i a_{-1}\end{aligned}$$

The number $a_{-1}$ is called the *residue* of the integrand. It is all of the integrand evaluated at the singular point, except for the $\frac{1}{z}$ of the simple pole. For example, if $f(z)$ is analytic at a point $z_0$ and we wish to integrate

$$\oint \frac{f(z)}{z-z_0} dz$$

The residue is $f(z)$ evaluated at the pole, i.e., $f(z_0)$. The value of the integral is

$$\begin{aligned}\oint \frac{f(z)}{z-z_0} dz &= 2\pi i \, Res\left(\frac{f(z)}{z-z_0}\right) \\ &= 2\pi i \, f(z_0)\end{aligned}$$

## 4.1 Example

Suppose we which to integrate

$$I = \int_{-\infty}^{\infty} \frac{e^{ikx}}{x^2+4} dx$$

This integrand has a Laurent series expansion, so we can look at the analytic extension to

$$I = \int_{-\infty}^{\infty} \frac{e^{ikz}}{z^2+4} dz$$

If we choose, we can deform the path of integration away from the real axis as long as we do not cross a pole. To see the poles clearly, we rewrite

$$\begin{aligned}\frac{1}{z^2+4} &= \frac{1}{(z-2i)(z+2i)} \\ &= \frac{1}{4i}\left(\frac{1}{z-2i} - \frac{1}{z+2i}\right)\end{aligned}$$

There are simple poles at $2i$ and $-2i$.

To make use of the residue theorem, we need a closed curve. Consider curves that include the real axis along the interval $[-R, R]$ then form a half circle in the upper half plane from $(R, 0)$ counterclockwise along the arc $Re^{i\theta}$ as $\theta$ runs from 0 to $\pi$. This forms a closed contour. The integral along this contour may be written as

$$I = \frac{1}{4i}\int_{-R}^{R}\left(\frac{e^{ikz}}{z-2i} - \frac{e^{ikz}}{z+2i}\right) dz + \frac{1}{4i}\int_{0}^{\pi}\left(\frac{e^{ikz}}{z-2i} - \frac{e^{ikz}}{z+2i}\right) iRe^{i\theta} d\theta$$

Now look at the integrand in the second integral. Writing $z = x+iy$ in the exponential,

$$\left(\frac{e^{ikz}}{z-2i} - \frac{e^{ikz}}{z+2i}\right) iRe^{i\theta} = \left(\frac{1}{z-2i} - \frac{1}{z+2i}\right) iRe^{i\theta} e^{ikx} e^{-ky}$$

Since the half circle always has positive $y$, it is suppressed by $e^{-ky}$. As the radius of the circle increases to infinity, $y$ does as well, and since the rest of the integral is bounded, tends to zero. Therefore, in the limit as $R \to \infty$, the second integral vanishes, while the first part becomes the original integral $I$. The integral around the closed loop therefore equals $I$:

$$I = \frac{1}{4i} \oint \left( \frac{e^{ikz}}{z - 2i} - \frac{e^{ikz}}{z + 2i} \right) dz$$

The closed curve in the upper half plane encloses one of the poles, at $z = 2i$. The second term has no pole in the upper half plane and therefore gives zero. The residue theorem immediately gives the final result,

$$\begin{aligned} I &= \frac{1}{4i} \oint \left( \frac{e^{ikz}}{z - 2i} - \frac{e^{ikz}}{z + 2i} \right) dz \\ &= \frac{1}{4i} \oint \frac{e^{ikz}}{z - 2i} dz \\ &= 2\pi i Res \left[ \frac{1}{4i} \frac{e^{ikz}}{z - 2i} \right] \\ &= 2\pi i \left[ \frac{1}{4i} e^{ikz} \right]_{2i} \\ &= \frac{\pi}{2} e^{-2k} \end{aligned}$$

Pure magic!

***Exercise:*** Compute the integral

$$\int_{-\infty}^{\infty} \frac{e^{ikx}}{x^2 - ix} dx$$

using contour integration.

## 4.2 Returning to the Fourier integral

Now return to our Fourier integral,

$$I = \lim_{N \to \infty} \int_{-\infty}^{\infty} \frac{\sin Nx}{\pi x} f(x)\, dx$$

and consider the analytic continuation. Writing $\sin Nx = \frac{e^{iNx} - e^{-iNx}}{2i}$ and letting $x \to z$,

$$I = \lim_{N \to \infty} \int_{-\infty}^{\infty} \frac{e^{iNz} - e^{-iNz}}{2\pi i z} f(z)\, dz$$

The first step is to displace the pole off the real axis,

$$I = \lim_{N \to \infty} \lim_{\varepsilon \to 0} \int_{-\infty}^{\infty} \frac{e^{iNz} - e^{-iNz}}{2\pi i (z - i\varepsilon)} f(z)\, dz$$

then to split the integral into two

$$I = \lim_{N \to \infty} \lim_{\varepsilon \to 0} \int_{-\infty}^{\infty} \frac{e^{iNz}}{2\pi i (z - i\varepsilon)} f(z)\, dz - \lim_{N \to \infty} \lim_{\varepsilon \to 0} \int_{-\infty}^{\infty} \frac{e^{-iNz}}{2\pi i (z - i\varepsilon)} f(z)\, dz$$

We can complete the first integral in the upper half-plane as above, since for positive $y$, $e^{iNz} = e^{iNx}e^{-Ny}$ converges. For the second integral, the sign in the exponent requires us to close the contour in the lower half plane. However, since the pole is in the upper half plane, the second integral then vanishes. We are left with

$$\begin{aligned} I &= \lim_{N\to\infty}\lim_{\varepsilon\to 0}\oint \frac{e^{iNz}}{2\pi i\left(z-i\varepsilon\right)} f\left(z\right)dz \\ &= \lim_{N\to\infty}\lim_{\varepsilon\to 0} 2\pi i\, Res\left[\frac{e^{iNz}}{2\pi i\left(z-i\varepsilon\right)} f\left(z\right)\right] \\ &= \lim_{N\to\infty}\lim_{\varepsilon\to 0} 2\pi i\left[\frac{e^{iNz}}{2\pi i\left(z-i\varepsilon\right)} f\left(z\right)\right]_{z=i\varepsilon} \\ &= \lim_{N\to\infty}\lim_{\varepsilon\to 0}\left(e^{-N\varepsilon} f\left(i\varepsilon\right)\right) \\ &= \lim_{N\to\infty} f\left(0\right) \\ &= f\left(0\right) \end{aligned}$$

Therefore, combining

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} dk\, e^{ikx} = \lim_{N\to\infty} f_N\left(x\right)$$

with

$$\lim_{N\to\infty}\int_{-\infty}^{\infty} f_N\left(x\right) f\left(x\right)dx = f\left(0\right)$$

we may write

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} dk\, e^{ikx} = \delta\left(x\right)$$

This is the Fourier representation of the Dirac delta function.

## 4.3 Completeness

At last, we have the tools to invert the Fourier transform. Given

$$f\left(x\right) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dk\, \tilde{f}\left(k\right)e^{ikx}$$

we multiply both sides by $\frac{1}{\sqrt{2\pi}}e^{-iqx}$ and integrate over $x$,

$$\begin{aligned} \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-iqx} f\left(x\right)dx &= \frac{1}{2\pi}\int_{-\infty}^{\infty} dx\, e^{-iqx}\int_{-\infty}^{\infty} dk\, \tilde{f}\left(k\right)e^{ikx} \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} dk\, \tilde{f}\left(k\right)\int_{-\infty}^{\infty} dx\, e^{i(k-q)x} \\ &= \int_{-\infty}^{\infty} dk\, \tilde{f}\left(k\right)\delta\left(k-q\right) \\ &= \tilde{f}\left(q\right) \end{aligned}$$

Therefore, given any $f(x)$ we can solve for the transform $\tilde{f}(k)$ and given $\tilde{f}(k)$ we can find $f(x)$. The two are equivalent.

***Exercise***: Show that $f(x) = 0$ if and only if its Fourier transform $\tilde{f}(k)$ vanishes.

It is peculiar that the real function $f(x)$ can be equivalent to a complex function, $\tilde{f}(k)$. Indeed, there is a constraint on $\tilde{f}(k)$. Since $f(x)$ is its own complex conjugate, $f^*(x) = f(x)$, we must have

$$\begin{aligned} f^*(x) &= f(x) \\ \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{\infty} dk\, \tilde{f}^*(k)\, e^{-ikx} &= \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{\infty} dk\, \tilde{f}(k)\, e^{ikx} \end{aligned}$$

Changing the variable on the left from $k$ to $-k$,

$$\begin{aligned} \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{\infty} dk\, \tilde{f}^*(k)\, e^{-ikx} &= \frac{1}{\sqrt{2\pi}} \int\limits_{+\infty}^{-\infty} d(-k)\;\; \tilde{f}^*(-k)\, e^{ikx} \\ &= \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{\infty} dk\, \tilde{f}^*(-k)\, e^{ikx} \end{aligned}$$

The reality of $f(x)$ then requires

$$\begin{aligned} \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{\infty} dk\, \tilde{f}^*(-k)\, e^{ikx} &= \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{\infty} dk\, \tilde{f}(k)\, e^{ikx} \\ 0 &= \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{\infty} dk\, \tilde{f}(k)\, e^{ikx} - \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{\infty} dk\, \tilde{f}^*(-k)\, e^{ikx} \\ 0 &= \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{\infty} dk\, \left(\tilde{f}(k) - \tilde{f}^*(-k)\right) e^{ikx} \end{aligned}$$

The right hand side is just the Fourier transform of $\tilde{f}(k) - \tilde{f}^*(-k)$ and since transforms are invertible, the vanishing of the transform implies

$$\tilde{f}(k) - \tilde{f}^*(-k) = 0$$

***Exercise:*** Find the Fourier transform of the function

$$f(x) = \begin{cases} e^{ax} & x < 0 \\ e^{-ax} & x \geq 0 \end{cases}$$

where $a$ is a positive real number. Verify that $\tilde{f}(k) = \tilde{f}^*(-k)$.

# 5 Using the Fourier transform

We may use Fourier transforms to solve differential equations.

## 5.1 Harmonic source

As an example, consider the 2-dimensionsal wave equation,

$$-\frac{1}{c^2}\frac{\partial^2 q}{\partial t^2} + \frac{\partial^2 q}{\partial x^2} = 0$$

It is possible to add a driving force, $F(x,t)$ to the right side of the equation,

$$-\frac{1}{c^2}\frac{\partial^2 q}{\partial t^2}+\frac{\partial^2 q}{\partial x^2}=F(x,t)$$

We suppose the driving force is given. This could correspond to pushing in various ways on a guitar string, or sending a sound wave through a modulated medium.

To keep the problem simple, suppose we study a single frequency mode. Let the driving force be harmonic,

$$F(x,t)=f(x)\,e^{i\omega_0 t}$$

Then we suppose $q(x,t)$ responds at this same frequency (a particular solution), to which we may add any solution to the sourceless (homogeneous) equation,

$$q(x,t)=q(x)\,e^{i\omega_0 t}+g(x-ct)+h(x+ct)$$

Then the the $g$ and $h$ parts vanish from the wave equation when we substitute in, leaving

$$\begin{aligned}\left(\frac{\omega_0^2}{c^2}q(x)+\frac{\partial^2 q(x)}{\partial x^2}\right)e^{i\omega_0 t} &= f(x)\,e^{i\omega_0 t}\\ \frac{\partial^2 q(x)}{\partial x^2}+\frac{\omega_0^2}{c^2}q(x) &= f(x)\end{aligned}$$

Now substitute the Fourier transforms,

$$\begin{aligned}q(x) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}dk\,\tilde{q}(k)\,e^{ikx}\\ F(x) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}dk\,\tilde{f}(k)\,e^{ikx}\end{aligned}$$

We find

$$\frac{\partial^2}{\partial x^2}\left(\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}dk\,\tilde{q}(k)\,e^{ikx}\right)+\frac{\omega_0^2}{c^2}\left(\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}dk\,\tilde{q}(k)\,e^{ikx}\right) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}dk\,\tilde{f}(k)\,e^{ikx}$$

Taking the $x$-derivatives and collecting terms,

$$\begin{aligned}\int_{-\infty}^{\infty}dk\,\left(-k^2\right)\tilde{q}(k)\,e^{ikx}+\frac{\omega_0^2}{c^2}\int_{-\infty}^{\infty}dk\,\tilde{q}(k)\,e^{ikx} &= \int_{-\infty}^{\infty}dk\,\tilde{f}(k)\,e^{ikx}\\ \int_{-\infty}^{\infty}dk\,\left(-k^2\tilde{q}(k)+\frac{\omega_0^2}{c^2}\tilde{q}(k)-\tilde{f}(k)\right)e^{ikx} &= 0\end{aligned}$$

This is just the Fourier transform of the terms in parentheses; since the transform is invertible the integrand must vanish,

$$-k^2\tilde{q}(k)+\frac{\omega_0^2}{c^2}\tilde{q}(k)-\tilde{f}(k)=0$$

We assume the driving force is given so that its transform $\tilde{f}(k)$ is known. Then solving for $\tilde{q}(k)$

$$\tilde{q}(k)=\frac{c^2\tilde{f}(k)}{\omega_0^2-c^2k^2}$$

we invert to find $q\left(x\right)$,

$$q\left(x\right)=\frac{c^{2}}{\sqrt{2\pi}}\int\limits_{-\infty}^{\infty}dk\,\frac{\tilde{f}\left(k\right)}{\omega_{0}^{2}-c^{2}k^{2}}e^{ikx}$$

and the full solution for $q\left(x,t\right)$, adding the homogeneous solution to this particular one, is

$$q\left(x,t\right)=e^{i\omega_{0}t}\frac{c^{2}}{\sqrt{2\pi}}\int\limits_{-\infty}^{\infty}dk\,\frac{\tilde{f}\left(k\right)}{\omega_{0}^{2}-c^{2}k^{2}}e^{ikx}+g\left(x-ct\right)+h\left(x+ct\right)$$

Notice that the form of the remaining integral may often be solved using contour integration!

## 5.2 Double Fourier integral

The same procedure works even if the driving force depends on both $x$ and $t$ because we may use the Fourier transform on more than one variable at a time. In the present case for a general forcing function $F\left(x,t\right)$ we may solve solve by writing $q\left(x,t\right)$ and $F\left(x,t\right)$ in terms Fourier transforms of both $x$ and of $t$,

$$\begin{aligned}
q\left(x,t\right) &= \frac{1}{\sqrt{2\pi}}\int\limits_{-\infty}^{\infty}dk\frac{1}{\sqrt{2\pi}}\int\limits_{-\infty}^{\infty}d\omega\,h\left(k,\omega\right)e^{i\left(kx-\omega t\right)} \\
&= \frac{1}{2\pi}\int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty}dkd\omega\,h\left(k,\omega\right)e^{i\left(kx-\omega t\right)} \\
F\left(x,t\right) &= \frac{1}{\sqrt{2\pi}}\int\limits_{-\infty}^{\infty}dk\frac{1}{\sqrt{2\pi}}\int\limits_{-\infty}^{\infty}d\omega\,j\left(k,\omega\right)e^{i\left(kx-\omega t\right)}
\end{aligned}$$

where, since $F\left(x,t\right)$ is given, $j\left(k,\omega\right)$ may also be found.

Substituting into the wave equation, and take the derivatives

$$\begin{aligned}
-\frac{1}{c^{2}}\frac{\partial^{2}}{\partial t^{2}}\left(\frac{1}{2\pi}\int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty}dkd\omega\,h\left(k,\omega\right)e^{i\left(kx-\omega t\right)}\right)+\frac{\partial^{2}}{\partial x^{2}}\left(\frac{1}{2\pi}\int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty}dkd\omega\,h\left(k,\omega\right)e^{i\left(kx-\omega t\right)}\right) &= \frac{1}{2\pi}\int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty}dkd\omega\,j\left(k,\omega\right)e^{i(kx-} \\
\frac{1}{2\pi}\int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty}dkd\omega\,h\left(k,\omega\right)\left(\frac{\omega^{2}}{c^{2}}-k^{2}\right)e^{i\left(kx-\omega t\right)} &= \frac{1}{2\pi}\int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty}dkd\omega\,j\left(k,\omega\right)e^{i(kx-}
\end{aligned}$$

Combining the two sides together,

$$\frac{1}{2\pi}\int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty}dkd\omega\,\left(h\left(k,\omega\right)\left(\frac{\omega^{2}}{c^{2}}-k^{2}\right)-j\left(k,\omega\right)\right)e^{i\left(kx-\omega t\right)}=0$$

This is just a (double) Fourier transform, which we may invert to show that the term in parentheses vanishes,

$$h\left(k,\omega\right)\left(\frac{\omega^{2}}{c^{2}}-k^{2}\right)-j\left(k,\omega\right)\;=\;0$$

Since we know $j\left(k,\omega\right)$ is known, we may solve immediately for $h\left(k,\omega\right)$:

$$h\left(k,\omega\right)=\frac{c^{2}j\left(k,\omega\right)}{\omega^{2}-k^{2}c^{2}}$$

and the solution to our problem is given by substituting this into the Fourier transform for $q(x,t)$,

$$\begin{aligned} q(x,t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dk d\omega\, h(k,\omega)\, e^{i(kx-\omega t)} \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dk d\omega\, \frac{c^2 j(k,\omega)\, e^{i(kx-\omega t)}}{\omega^2 - k^2c^2} \end{aligned}$$

This may or may not be simple, depending on the form of the driving force, but it reduces solving the differential equation to a pair of integrals.