Final Exam Notes

December 9, 2019

I will include helpful formulas; these and/or others as needed:

Forms of the gradient:

$$\boldsymbol{\nabla} = \hat{\mathbf{i}}\frac{\partial}{\partial x} + \hat{\mathbf{j}}\frac{\partial}{\partial x} + \hat{\mathbf{k}}\frac{\partial}{\partial z} \quad (1)$$

$$\boldsymbol{\nabla} = \hat{\boldsymbol{\rho}}\frac{\partial}{\partial \rho} + \hat{\boldsymbol{\varphi}}\frac{1}{\rho}\frac{\partial}{\partial \varphi} + \hat{\mathbf{k}}\frac{\partial}{\partial z} \quad (2)$$

$$\boldsymbol{\nabla} = \hat{\mathbf{r}}\frac{\partial}{\partial r} + \hat{\boldsymbol{\theta}}\frac{1}{r}\frac{\partial}{\partial \theta} + \hat{\boldsymbol{\varphi}}\frac{1}{r\sin\theta}\frac{\partial}{\partial \varphi} \quad (3)$$

Laplacian:

$$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

$$\nabla^2 f = \frac{\partial^2 f}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial f}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2 f}{\partial \varphi^2} + \frac{\partial^2 f}{\partial z^2}$$

$$\nabla^2 f = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial f}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 f}{\partial \varphi^2}$$

Spherical harmonics satisfy

$$\frac{1}{\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial Y_{lm}}{\partial \theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2 Y_{lm}}{\partial \varphi^2} = -l\left(l+1\right)Y_{lm}$$

$$Y_{lm}\left(\theta,\varphi\right) = \sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}}e^{im\varphi}P_l^m\left(\cos\theta\right)$$

Basis vectors in terms of Cartesian:

$$\hat{\boldsymbol{\rho}} = \hat{\mathbf{i}}\cos\varphi + \hat{\mathbf{j}}\sin\varphi$$

$$\hat{\boldsymbol{\varphi}} = -\hat{\mathbf{i}}\sin\varphi + \hat{\mathbf{j}}\cos\varphi$$

$$\hat{\mathbf{r}} = \hat{\mathbf{i}}\sin\theta\cos\varphi + \hat{\mathbf{j}}\sin\theta\sin\varphi + \hat{\mathbf{k}}\cos\theta$$

$$\hat{\boldsymbol{\theta}} = \hat{\mathbf{i}}\cos\theta\cos\varphi + \hat{\mathbf{j}}\cos\theta\sin\varphi - \hat{\mathbf{k}}\sin\theta$$

$$\hat{\boldsymbol{\varphi}} = -\hat{\mathbf{i}}\sin\varphi + \hat{\mathbf{j}}\cos\varphi$$

The spherical Bessel functions satisfy

$$\frac{\partial}{\partial r}\left(r^2\frac{\partial j_n\left(kr\right)}{\partial r}\right) + k^2r^2 j_n\left(kr\right) = 0$$

Expect the following problems, modified so that you need to work through them, on the final exam:

1. Find the eigenvalues and eigenvectors of a 2×2 matrix, such as $\begin{pmatrix} 0 & 2 \\ 2 & -3 \end{pmatrix}$, but with different entries.
   ANSWER: To find the eigenvalues, we solve the eigenvalue equation

$$\begin{aligned}
\det(M-\lambda 1) &= 0 \\
0 &= \det\left[\begin{pmatrix} 0 & 2 \\ 2 & -3 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}\right] \\
&= \det\begin{pmatrix} -\lambda & 2 \\ 2 & -3-\lambda \end{pmatrix} \\
&= \lambda^2 + 3\lambda - 4 \\
&= (\lambda-1)(\lambda+4)
\end{aligned}$$

   so the eigenvalues are $\lambda = 1, -4$. Solve the eigenvector equation for each one. For $\lambda = 1$,

$$\begin{aligned}
\begin{pmatrix} 0 & 2 \\ 2 & -3 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} &= 1\begin{pmatrix} a \\ b \end{pmatrix} \\
2b &= a \\
2a - 3b &= b
\end{aligned}$$

   Both of these give the same value, $b = \frac{a}{2}$, so the first eigenvector is

$$\mathbf{v}_1 = a\begin{pmatrix} 1 \\ \frac{1}{2} \end{pmatrix} = \frac{a}{2}\begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

   We can normalize this by choosing $a = \frac{2}{\sqrt{5}}$,

$$\hat{\mathbf{v}}_1 = \frac{1}{\sqrt{5}}\begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

   For the second eigenvector, set $\lambda = -4$,

$$\begin{aligned}
\begin{pmatrix} 0 & 2 \\ 2 & -3 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} &= -4\begin{pmatrix} a \\ b \end{pmatrix} \\
2b &= -4a \\
2a - 3b &= -4b
\end{aligned}$$

   Both of these give $b = -2a$, so

$$\mathbf{v}_{-4} = a\begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

   and the normalized eigenvector is

$$\hat{\mathbf{v}}_{-4} = \frac{1}{\sqrt{5}}\begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

2. Potential energy and small oscillations. Here's an example. The Exam question will be simpler than this and shorter to work. I'm doing a more complicated version to show you all the features that could come up, but your potential will be more like the one in the first midterm, $V = 2x^2 - 5x^4$ which has only one minimum at the origin.

(a) Suppose we have a potential

$$V = x^4 + \frac{4}{3}x^3 - 4x^2 + \frac{5}{3}$$

Sketch the potential. Find the frequency of small oscillations of a mass m about the absolute minimum. ANSWER: First, find all extrema by setting the first derivative to zero:

$$\begin{aligned} 0 &= V' \\ &= 4x^3 + 4x^2 - 8x \\ &= 4x\left(x^2 + x - 2\right) \\ &= 4x\left(x-1\right)\left(x+2\right) \end{aligned}$$

Therefore, the extrema are at $x = -2, 0, 1$. To find which are maxima and which are minima, look at the sign of the second derivative of the potential at each of the three points. Positive V'' indicates a minimum, negative V'' indicates a maximum:

$$V''(x) = 12x^2 + 8x - 8$$

$$\begin{aligned} V''(-2) &= 8 > 0 \\ V''(0) &= -8 < 0 \\ V''(1) &= 12 > 0 \end{aligned}$$

This means that $x = -2, 1$ are minima and we can expect small oscillations around each. To find which is the absolute minimum, we look at the value of the potential at those points:

$$\begin{aligned} V &= x^4 + \frac{4}{3}x^3 - 4x^2 + \frac{5}{3} \\ V(-2) &= (-2)^4 + \frac{4}{3}(-2)^3 - 4(-2)^2 + \frac{5}{3} \\ &= 16 - \frac{32}{3} - 16 + \frac{5}{3} \\ &= -9 \\ V(1) &= (1)^4 + \frac{4}{3}(1)^3 - 4(1)^2 + \frac{5}{3} \\ &= 0 \end{aligned}$$

Since $V(-2)$ is lower, $x = -2$ is the absolute minimum, i.e., the very smallest value of the potential. Next, expand the potential in a Taylor series around $x = -2$, that is, in powers of $(x+2)$. In general, the Taylor series of a function about a point $x = x_0$ is given by

$$\begin{aligned} f(x) &= f(x_0) + f'(x_0)(x - x_0) + \frac{1}{2!}f''(x_0)(x - x_0)^2 + \dots \\ &= \sum_{n=0}^{\infty} \frac{1}{n!}\frac{d^n f}{dx^n}(x - x_0)^n \end{aligned}$$

We only need up to second order to find small oscillations,

$$\begin{aligned} V(x) &= V(-2) + V'(-2)(x - (-2)) + \frac{1}{2!}V''(-2)(x+2)^2 + \dots \\ &= -9 + 0 + \frac{1}{2!}8(x+2)^2 + \dots \end{aligned}$$

This is enough to find the Hooke's law approximation to the force:

$$\begin{aligned} F &= -V' \\ &= -8(x+2) \end{aligned}$$

so the equation of motion is

$$m\ddot{x}+8(x+2)=0$$

Let $y=x+2$. Then

$$m\ddot{y}+8y=0$$

and the frequency of oscillations is

$$\omega=\sqrt{\frac{8}{m}}$$

The full solution is

$$\begin{aligned} x(t) &= y(t)-2 \\ &= A\cos\omega t+B\sin\omega t-2 \end{aligned}$$

3. Find the divergence of the vector field

$$\mathbf{v}(\rho,\varphi,z)=\frac{\rho}{z^2}\hat{\boldsymbol{\rho}}$$

ANSWER:

$$\begin{aligned} \left(\hat{\boldsymbol{\rho}}\frac{\partial}{\partial\rho}+\hat{\boldsymbol{\varphi}}\frac{1}{\rho}\frac{\partial}{\partial\varphi}+\hat{\mathbf{k}}\frac{\partial}{\partial z}\right)\hat{\boldsymbol{\rho}}\cdot\left(\frac{\rho}{z^2}\hat{\boldsymbol{\rho}}\right) &= \hat{\boldsymbol{\rho}}\hat{\boldsymbol{\rho}}\cdot\frac{\partial}{\partial\rho}\left(\frac{\rho}{z^2}\hat{\boldsymbol{\rho}}\right)+\hat{\boldsymbol{\varphi}}\cdot\frac{1}{\rho}\frac{\partial}{\partial\varphi}\left(\frac{\rho}{z^2}\hat{\boldsymbol{\rho}}\right)+\hat{\mathbf{k}}\cdot\frac{\partial}{\partial z}\left(\frac{\rho}{z^2}\hat{\boldsymbol{\rho}}\right) \\ &= \hat{\boldsymbol{\rho}}\cdot\frac{1}{z^2}\hat{\boldsymbol{\rho}}+\hat{\boldsymbol{\varphi}}\cdot\frac{1}{\rho}\frac{\rho}{z^2}\hat{\boldsymbol{\varphi}}+\hat{\mathbf{k}}\cdot\left(-\frac{2\rho}{z^3}\hat{\boldsymbol{\rho}}\right) \\ &= \frac{2}{z^2} \end{aligned}$$

4. What is the divergence of $f\boldsymbol{\nabla}g$? Look at other simple identities involving the del operator as well. ANSWER:

$$\boldsymbol{\nabla}\cdot(f\boldsymbol{\nabla}g)=\boldsymbol{\nabla}f\cdot\boldsymbol{\nabla}g+f\nabla^2g$$

5. Use the divergence theorem to evaluate the volume integral over a sphere of radius R,

$$I = \int_0^R\int_0^{\frac{\pi}{2}}\int_0^{2\pi}\boldsymbol{\nabla}\cdot\mathbf{w}\,r^2\sin\theta drd\theta d\varphi$$

if the vector field $\mathbf{w}$ is given by

$$\mathbf{w}=\mathbf{w}(\rho,\varphi,z)=r\sin\theta\hat{\mathbf{r}}+\frac{r\sin\theta}{\cos^2\varphi}\hat{\boldsymbol{\theta}}+r\cos\varphi\hat{\boldsymbol{\varphi}}$$

ANSWER: I went through this in class.

6. Use separation of variables to write ordinary differential equations in *spherical coordinates* for

$$\mathbf{v}\cdot\boldsymbol{\nabla}\psi=0$$

where $\mathbf{v}=v_0\hat{\mathbf{i}}$ is a constant vector field in the y-direction. Do not try to solve the equations during the test. ANSWER: I went through this in class.

7. Spherical harmonics, $Y_l^m(\theta,\varphi)$. The solution to the Laplace equation in spherical coordinates may be written as

$$\psi(r,\theta,\varphi)=\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\left(A_{lm}r^l+\frac{B_{lm}}{r^{l+1}}\right)Y_{lm}(\theta,\varphi)$$

Find the solution to the Laplace equation inside a sphere of radius R if

$$\psi(R,\theta)=V_0\cos\theta=V_0P_1(\theta)$$

(Hint: it makes it very easy if we can write $\psi(R,\theta)$ in terms of P_l *or* $Y_l^m(x)$ like this!) The final exam might have any simple value on the right, expressed in terms of $P_l(\cos\theta)$ or $Y_l^m(\theta,\varphi)$. The technique is the same.

8. If you did not already know the solutions to

$$\frac{d^2f}{dx^2}+f=0$$

then you might try a power series solution. Find the solution using a power series of the form

$$f(x)=\sum_{n=0}^{\infty}a_nx^n$$

and find the recursion relation for the coefficients a_n. ANSWER: First, DO you know the solutions?! It's just the harmonic equation we've been working with all semester, so the general solution is

$$f(x)=A\cos x+B\sin x$$

Notice that the wave vector here is just 1. To find the recursion, substitute the *whole* series into the equation and take the derivatives,

$$\begin{aligned}\frac{d^2}{dx^2}\left(\sum_{n=0}^{\infty}a_nx^n\right)+\sum_{n=0}^{\infty}a_nx^n &= 0\\ \sum_{n=0}^{\infty}a_nn(n-1)x^{n-2}+\sum_{n=0}^{\infty}a_nx^n &= 0\end{aligned}$$

Notice that anything that depends on n must stay inside the summation. Next, rewrite the first sum so that there is a simple index for the powers. Here, let $m=n-2$. Then we replace all the ns by $n=m+2$,

$$\sum_{m=0}^{\infty}a_{m+2}(m+2)(m+1)x^m$$

Don't forget to change the index on a_n! We can start the sum at $m=0$ since the values $m=-1,-2$ give zero anyway. Write the second sum with the same letter, $\sum_{n=0}^{\infty}a_nx^n=\sum_{m=0}^{\infty}a_mx^m$ so we can collect it all in one sum:

$$\begin{aligned}\sum_{m=0}^{\infty}a_{m+2}(m+2)(m+1)x^m+\sum_{m=0}^{\infty}a_mx^m &= 0\\ \sum_{m=0}^{\infty}\left(a_{m+2}(m+2)(m+1)+a_m\right)x^m &= 0\end{aligned}$$

Now (and only now) we use the independence of different powers of x to say that each coefficient must vanish separately,

$$\left(a_{m+2}(m+2)(m+1)+a_m\right)=0$$

This gives a *recursion relation*, that is, a relation giving one of the a_m in terms of the previous one(s). In this case, we solve for a_{m+2} in terms of a_m:

$$a_{m+2} = -\frac{1}{(m+1)(m+2)} a_m$$

To see what function this describes, compute some coefficients. If we start with a_0 we set $m = 0$ and find

$$\begin{aligned} a_2 &= -\frac{1}{(0+1)(0+2)} a_0 \\ &= -\frac{1}{2} a_0 \end{aligned}$$

Then, (since we have $a_1 = 0$) we go up by 2. With $m = 2$,

$$\begin{aligned} a_4 &= -\frac{1}{(2+1)(2+2)} a_2 \\ &= -\frac{1}{3 \cdot 4} a_2 \\ &= \frac{1}{2 \cdot 3 \cdot 4} a_0 \\ &= \frac{1}{4!} a_0 \end{aligned}$$

Pretty soon, you spot the pattern and guess that

$$a_{2k} = \frac{(-1)^k}{(2k)!} a_0$$

To *prove* this (this part will not be on the Exam) we can use induction. Suppose $a_{2k} = \frac{(-1)^k}{(2k)!} a_0$. Then with $m = 2k$,

$$\begin{aligned} a_{2k+2} &= -\frac{1}{(2k+1)(2k+2)} a_{2k} \\ &= -\frac{1}{(2k+1)(2k+2)} \frac{(-1)^k}{(2k)!} a_0 \\ &= \frac{(-1)^{k+1}}{(2k+2)!} a_0 \end{aligned}$$

and this proves the general case. Therefore,

$$\begin{aligned} f(x) &= \sum_{n\,even}^{\infty} a_n x^n \\ &= \sum_{k=0}^{\infty} a_{2k} x^{2k} \\ &= a_0 \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} x^{2k} \end{aligned}$$

and we recognize the Taylor series for the cosine,

$$f(x) = a_0 \cos x$$

The odd terms give the sine and we recover the general solution.

9. Fourier series

(a) Find the Fourier series for the step function

$$f(x) = \begin{cases} -1 & -L < x < 0 \\ +1 & 0 < x < L \end{cases}$$

on the interval $[-L, L]$. ANSWER:

$$f(x) = b_0 + \sum_{n=1}^{\infty} \left(a_n \sin \frac{n\pi x}{L} + b_n \cos \frac{n\pi x}{L} \right)$$

Since $f(x)$ is odd, $f(-x) = -f(x)$ we need $b_n = 0$ for all n. Then multiply by $\sin \frac{k\pi x}{L}$ and integrate,

$$\int_{-L}^{L} f(x) \sin \frac{k\pi x}{L} dx = b_0 + \sum_{n=1}^{\infty} \left(a_n \int_{-L}^{L} \sin \frac{n\pi x}{L} \sin \frac{k\pi x}{L} \right)$$

We know that the right side vanishes unless $n = k$ so

$$\begin{aligned}
\int_{-L}^{0} (-1) \sin \frac{k\pi x}{L} dx + \int_{0}^{L} (+1) \sin \frac{k\pi x}{L} dx &= a_k \int_{-L}^{L} \sin^2 \frac{k\pi x}{L} dx \\
\frac{L}{k\pi} \left(1 - \cos \frac{k\pi(-L)}{L} \right) - \frac{L}{k\pi} \left(\cos \frac{k\pi L}{L} - 1 \right) &= \frac{1}{2} a_k 2L \\
\frac{L}{k\pi} \left(1 - (-1)^k \right) - \frac{L}{k\pi} \left((-1)^k - 1 \right) &= \frac{1}{2} a_k 2L \\
\frac{2L}{k\pi} \left(1 - (-1)^k \right) &= a_k L \\
a_k &= \begin{cases} \frac{4}{k\pi} & k \text{ odd} \\ 0 & k \text{ even} \end{cases}
\end{aligned}$$

Therefore,

$$f(x) = \frac{4}{\pi} \sum_{m=1}^{\infty} \frac{1}{2m+1} \sin \frac{(2m+1)\pi x}{L}$$

(b) Let $q(x,t)$ satisfy the 2-dimensional wave equation $-\frac{1}{c^2} \frac{\partial^2 q(x,t)}{\partial t^2} + \frac{\partial^2 q(x,t)}{\partial x^2} = 0$, and suppose that at time $t = 0$

$$\begin{aligned}
q(x,0) &= f(x) \\
\dot{q}(x,0) &= 0
\end{aligned}$$

with $f(x)$ as given in part (a). Using your answer to part (a), write $q(x,t)$.

$$q(x,t) = \frac{4}{\pi} \sum_{m=1}^{\infty} \frac{1}{2m+1} \sin \frac{(2m+1)\pi x}{L} (a \cos \omega t + b \sin \omega t)$$

The initial velocity will vanish if $b = 0$; the initial position will be $f(x)$ if $a = 1$, so

$$q(x,t) = \frac{4}{\pi} \sum_{m=1}^{\infty} \frac{1}{2m+1} \sin \frac{(2m+1)\pi x}{L} \cos \omega t$$

(c) This can be written as a sum of right and left moving waves each with half the original shape.

10. One possibility to test your understanding of the continuity equation: The continuity equation is

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot \mathbf{J} = 0$$

Use the divergence theorem to prove that the time rate of change in the quantity S, given by

$$S = \int\limits_{V} \rho \, d^3x$$

equals the rate at which the current flows out of the volume V.