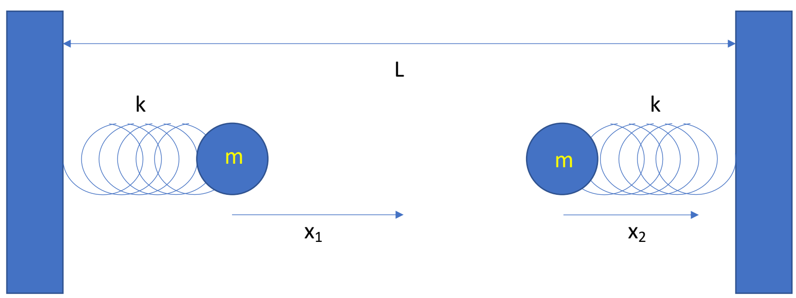
Coupled Oscillations

September 12, 2019

1 Coupled oscillators

Consider two mass-spring systems:

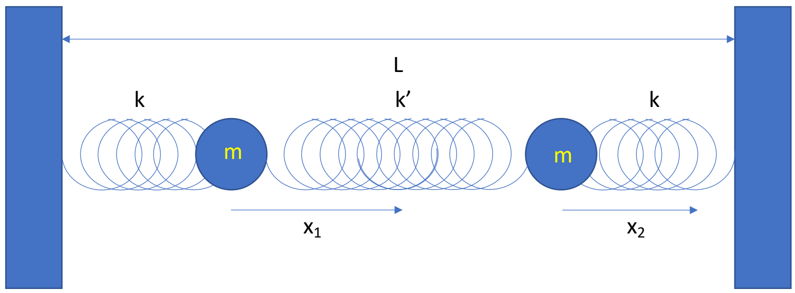


Suppose we have two identical mass-spring oscillators, each with mass m and spring constant k. (The masses and spring constants don't have to be the same, it just makes the example simpler) The oscillators satisfy

$$\begin{aligned} m\ddot{x}_1 + kx_1 &= 0 \\ m\ddot{x}_2 + kx_2 &= 0 \end{aligned}$$

with solutions that oscillate with frequency $\omega = \sqrt{\frac{k}{m}}$. Here we choose x_1 and x_2 to be the displacement of each respective mass to the right. Therefore, for positive x_1 or x_2 the force produced is to the left.

Now connect the two masses with a third spring of constant k' as shown in the next picture.



Each mass now feels two forces. We count forces to the left with a minus sign and to the right with a plus. On the left mass, positive x_1 produces a force to the left, $-kx_1$, and when a force from the middle spring of $k'(x_2 - x_1)$. This second force is to the right when $x_2 > x_1$ and to the left when $x_2 < x_1$. The equation of motion for the leftmost mass is therefore:

$$\begin{aligned} \sum_{all\ forces} F &= m\frac{d^2x_1}{dt^2} \\ -kx_1 + k'(x_2 - x_1) &= m\frac{d^2x_1}{dt^2} \end{aligned}$$

For the rightmost mass we have a similar equation,

$$-kx_2 - k'(x_2 - x_1) = m\frac{d^2x_2}{dt^2}$$

This gives us a pair of *coupled* oscillators. The motion of each mass depends on the position of both masses. The resulting motion may look complicated, but we'll see that there's a simple way to understand it. The trick is to take combinations of the two equations that *decouple* them so that we once again have equations for simple harmonic motion.

The simplest route to a solution is to notice that the coupling cancels if we add the equations:

$$\begin{aligned} -kx_1 - kx_2 &= m\frac{d^2x_1}{dt^2} + m\frac{d^2x_2}{dt^2} \\ -k(x_1 + x_2) &= m\frac{d^2(x_1 + x_2)}{dt^2} \end{aligned}$$

The combined length $Q = x_1 + x_2$ oscillates with frequency $\omega = \sqrt{\frac{k}{m}}$ according to

$$Q = A\sin(\omega t + \theta_0)$$

This doesn't yet solve both equations. However, if we subtract the two, we get a second simple harmonic motion:

$$\begin{aligned}
-k\left(x_1-x_2\right)+2k'\left(x_2-x_1\right) &= m\frac{d^2\left(x_1-x_2\right)}{dt^2} \\
-\left(k+2k'\right)\left(x_1-x_2\right) &= m\frac{d^2\left(x_1-x_2\right)}{dt^2}
\end{aligned}$$

Defining

$$\begin{aligned}
q &\equiv x_1-x_2 \\
\tilde{\omega} &\equiv \sqrt{\frac{k+2k'}{m}} \\
&= \sqrt{\omega^2+2\omega'^2}
\end{aligned}$$

where we set $\omega'=\sqrt{\frac{k'}{m}}$. The equation for a is now simply

$$\frac{d^2q}{dt^2}+\tilde{\omega}^2 q = 0$$

so that the solution is

$$q = B\sin\left(\tilde{\omega}t+\varphi_0\right)$$

Since

$$\begin{aligned}
Q &= x_1+x_2 \\
q &= x_1-x_2
\end{aligned}$$

we have

$$\begin{aligned}
x_1 &= \frac{1}{2}\left(Q+q\right) \\
&= \frac{1}{2}\left(A\sin\left(\omega t+\theta_0\right)+B\sin\left(\tilde{\omega}t+\varphi_0\right)\right) \\
x_2 &= \frac{1}{2}\left(Q-q\right) \\
&= \frac{1}{2}\left(A\sin\left(\omega t+\theta_0\right)-B\sin\left(\tilde{\omega}t+\varphi_0\right)\right)
\end{aligned}$$

The position of each mass is a linear combination of two simple harmonic oscillations. Notice that the initial position and initial velocity for each of the two masses exactly determine the four constants $A, B, \theta_0, \varphi_0$ in our solution.

2 A systematic approach

There is a more systematic way to go about this. We can write the original pair of equations as a single *matrix* equation. Since the equations are linear, we can let the two positions form a vector and extract the spring constants and derivatives as linear operators.

$$\begin{aligned}
m\frac{d^2x_1}{dt^2}+kx_1-k'\left(x_2-x_1\right) &= 0 \\
m\frac{d^2x_2}{dt^2}+kx_2+k'\left(x_2-x_1\right) &= 0
\end{aligned}$$

The force terms take the form

$$\begin{pmatrix} (k+k')\,x_1 - k'x_2 \\ -k'x_1 + (k+k')\,x_2 \end{pmatrix} = \begin{pmatrix} k+k' & -k' \\ -k' & k+k' \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

Then accelerations may be written as $\frac{d^2}{dt^2}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ and the pair of coupled equtions becomes

$$\left[\frac{d^2}{dt^2} + \frac{1}{m}\begin{pmatrix} k+k' & -k' \\ -k' & k+k' \end{pmatrix}\right]\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$$

This has the general form[1]

$$\left(\frac{d^2}{dt^2} + M\right)\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$$

with

$$\begin{aligned} M &= \frac{1}{m}\begin{pmatrix} k+k' & -k' \\ -k' & k+k' \end{pmatrix} \\ &= \frac{1}{m}\begin{pmatrix} k+k' & -k' \\ -k' & k+k' \end{pmatrix} \end{aligned}$$

2.1 Change of basis

Suppose we change the vector of position values $X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ by taking two linear combinations,

$$\begin{pmatrix} Q \\ q \end{pmatrix} \equiv AX = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

where A is now some invertible matrix. Then by inserting $1 = A^{-1}A$, our original equation may be changed to

$$A\left(\frac{d^2}{dt^2} + M\right)A^{-1}A\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$$

Since A is constant, the derivative term is unaltered, $A\left(\frac{d^2}{dt^2}\right)A^{-1} = \frac{d^2}{dt^2}$. Defining the transformed force matrix,

$$\tilde{M} \equiv AMA^{-1}$$

the equation of motion has the same form as before, but in the new variables:

$$\left(\frac{d^2}{dt^2} + \tilde{M}\right)\begin{pmatrix} Q \\ q \end{pmatrix} = 0$$

2.2 Diagonalization

The advantage of the change of basis is that it allows us to choose the transformation A so that $\tilde{M}$ is *diagonal*. Then, with

$$\tilde{M} = \begin{pmatrix} \omega_1^2 & 0 \\ 0 & \omega_2^2 \end{pmatrix}$$

[1] Strictly speaking, we should write the acceleration as $1\frac{d^2}{dt^2}$, where 1 is the 2×2 identity matrix. Then $\left(1\frac{d^2}{dt^2} + M\right)$ is a matrix operator.

the equations for Q and q take the simple harmonic form,

$$\begin{aligned}\frac{d^2Q}{dt^2}+\omega_1^2 Q &= 0\\ \frac{d^2q}{dt^2}+\omega_2^2 q &= 0\end{aligned}$$

and we have an immediate solution.

We have reduced the problem of solving the coupled pair of differential equations to the easier problem of diagonalizing a matrix.

The new dynamical variables Q, q oscillate with frequencies ω_1, ω_2 called the *normal mode frequencies.*

Diagonalizing a matrix amounts to solving the equation

$$\begin{aligned}AMA^{-1} &= \tilde{M}\\ \tilde{M}\begin{pmatrix} Q\\ q\end{pmatrix} &= \begin{pmatrix}\omega_1^2 & 0\\ 0 & \omega_2^2\end{pmatrix}\begin{pmatrix} Q\\ q\end{pmatrix}\\ &= \begin{pmatrix}\omega_1^2 Q\\ \omega_2^2 q\end{pmatrix}\end{aligned}$$

We may write this as two equations

$$\begin{aligned}\tilde{M}\begin{pmatrix} Q\\ 0\end{pmatrix} &= \omega_1^2\begin{pmatrix} Q\\ 0\end{pmatrix}\\ \tilde{M}\begin{pmatrix} 0\\ q\end{pmatrix} &= \omega_2^2\begin{pmatrix} 0\\ q\end{pmatrix}\end{aligned}$$

If we write

$$\begin{aligned}\begin{pmatrix} Q\\ 0\end{pmatrix} &= Q\begin{pmatrix} 1\\ 0\end{pmatrix}\\ \begin{pmatrix} 0\\ q\end{pmatrix} &= q\begin{pmatrix} 0\\ 1\end{pmatrix}\end{aligned}$$

we see that the magnitudes Q, q cancel, so $\tilde{M}$ also satisfies

$$\begin{aligned}\tilde{M}\begin{pmatrix} 1\\ 0\end{pmatrix} &= \omega_1^2\begin{pmatrix} 1\\ 0\end{pmatrix}\\ \tilde{M}\begin{pmatrix} 0\\ 1\end{pmatrix} &= \omega_2^2\begin{pmatrix} 0\\ 1\end{pmatrix}\end{aligned}$$

This shows the existence of two *eigenvectors*, $\begin{pmatrix} Q\\ 0\end{pmatrix}$ and $\begin{pmatrix} 0\\ q\end{pmatrix}$ such that the action of $\tilde{M}$ reproduces a multiple of the same vector. The multipliers, ω_1^2 and ω_2^2, are called the *eigenvalues.* The useful fact to notice is that this remains true in the original basis, since, acting with A^{-1} on the first equation and using the definition $\tilde{M} = AMA^{-1}$ gives

$$\begin{aligned}A^{-1}\tilde{M}\begin{pmatrix} 1\\ 0\end{pmatrix} &= \omega_1^2 A^{-1}\begin{pmatrix} 1\\ 0\end{pmatrix}\\ A^{-1}AMA^{-1}\begin{pmatrix} 1\\ 0\end{pmatrix} &= \omega_1^2 A^{-1}\begin{pmatrix} 1\\ 0\end{pmatrix}\\ MA^{-1}\begin{pmatrix} 1\\ 0\end{pmatrix} &= \omega_1^2 A^{-1}\begin{pmatrix} 1\\ 0\end{pmatrix}\end{aligned}$$

and similarly

$$MA^{-1}\begin{pmatrix}0\\1\end{pmatrix}=\omega_2^2A^{-1}\begin{pmatrix}0\\1\end{pmatrix}$$

Define

$$\begin{aligned}\mathbf{u} &= \begin{pmatrix}u_1\\u_2\end{pmatrix}\equiv A^{-1}\begin{pmatrix}1\\0\end{pmatrix}\\ \mathbf{v} &= \begin{pmatrix}v_1\\v_2\end{pmatrix}\equiv A^{-1}\begin{pmatrix}0\\1\end{pmatrix}\end{aligned}$$

Notice that $\mathbf{u}$ is the first column of the matrix A^{-1}, and $\mathbf{v}$ is the second column. Therefore, if we can find $\mathbf{u},\mathbf{v}$, we know the transformation A^{-1}, and inverting, A.

In terms of the *original* matrix M we now have

$$\begin{aligned}M\mathbf{u} &= \omega_1^2\mathbf{u}\\ M\mathbf{v} &= \omega_2^2\mathbf{v}\end{aligned}$$

These are called *eigenvector equations.* The eigenvectors $\mathbf{u},\mathbf{v}$ determine the transformation A^{-1} , and the *eigenvalues* ω_1,ω_2 are the normal mode frequencies. Since M is symmetric, we can show that A is orthogonal, and therefore has unit determinant. This means that $\mathbf{u}$ and $\mathbf{v}$ will be unit vectors, since $\begin{pmatrix}1\\0\end{pmatrix}$ and $\begin{pmatrix}0\\1\end{pmatrix}$ are unit vectors and orthogonal transformations preserve lengths of vectors.

2.3 Normal mode frequencies

If all we desire is to find the normal mode *frequencies*, the problem is even easier. The general form of the eigenvalue equation is

$$M\mathbf{u}=\lambda\mathbf{u}$$

for some number λ, with a similar equation for $\mathbf{v}$. Rewrite this as

$$(M-\lambda 1)\,\mathbf{u}=0$$

so the matrix $M-\lambda 1$ annihilates $\mathbf{u}$.

Now suppose $(M-\lambda 1)$ has and inverse N. Then $N\,(M-\lambda 1)=1$ so that the eigenvalue equation implies

$$0=N\,(M-\lambda 1)\,\mathbf{u}=\mathbf{u}$$

and there is no nontrivial eigenvector. Therefore, we must demand that $M-\lambda 1$ is *not* invertible. This is the case if and only if its determinant vanishes

$$\det\,(M-\lambda 1)=0$$

In our example, this equation is quadratic,

$$\begin{aligned}\det\left(\frac{1}{m}\begin{pmatrix}k+k' & -k'\\ -k' & k+k'\end{pmatrix}-\lambda 1\right) &= \det\frac{1}{m}\begin{pmatrix}k+k'-\lambda & -k'\\ -k' & k+k'-\lambda\end{pmatrix}\\ &= \left(\left(\frac{1}{m}(k+k')-\lambda\right)\left(\frac{1}{m}(k+k')-\lambda\right)-\left(\frac{k'}{m}\right)^2\right)\\ &= \lambda^2-\frac{2}{m}\lambda\,(k+k')+\frac{1}{m^2}\,(k+k')^2-\left(\frac{k'}{m}\right)^2\\ &= \lambda^2-\frac{2}{m}\lambda\,(k+k')+\frac{1}{m^2}\left(k^2+2kk'\right)\end{aligned}$$

Setting this to zero,

$$\lambda^2 - \frac{2}{m}\lambda\left(k+k'\right) + \frac{1}{m^2}\left(k^2+2kk'\right) = 0$$

we write the solutions using the quadratic formula,

$$\begin{aligned}
\lambda_\pm &= \frac{1}{2}\left(\frac{2}{m}\left(k+k'\right) \pm \sqrt{\frac{4}{m^2}\left(k+k'\right)^2 - 4\frac{4}{m^2}\left(k^2+2kk'\right)}\right) \\
&= \frac{1}{m}\left[\left(k+k'\right) \pm \sqrt{k^2+2kk'+k'^2-k^2-2kk'}\right] \\
&= \frac{1}{m}\left[\left(k+k'\right) \pm \sqrt{k'^2}\right] \\
\lambda_+ &= \frac{k+2k'}{m} = \tilde{\omega}^2 \\
\lambda_- &= \frac{k}{m} = \omega^2
\end{aligned}$$

We see that the eigenvalues of the force matrix are the squares of the normal mode frequencies.

The same techniques work if we couple together more oscillators. The only difference is that if we have n oscillators, M will be an $n \times n$ matrix so the vanishing determinant is an n^{th} order polynomial equation for the eigenvalues. Therefore, there will be n normal modes and n normal mode frequencies.

2.4 Normal modes

Now that we know the normal mode frequencies, we find the normal modes themselves. These are described by the eigenvectors, so we must solve each of the equations

$$\begin{aligned}
M\mathbf{u} &= \lambda_-\mathbf{u} \\
M\mathbf{v} &= \lambda_+\mathbf{v}
\end{aligned}$$

Letting

$$\mathbf{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

and substituting in M, the first becomes

$$\frac{1}{m}\begin{pmatrix} k+k' & -k' \\ -k' & k+k' \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \omega^2\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

This vector equation is really two coupled equations, on for each component of the vector. Multiplying out the matrix,

$$\begin{aligned}
\frac{1}{m}\left(k+k'\right)u_1 - \frac{k'}{m}u_2 &= \omega^2 u_1 \\
-\frac{k'}{m}u_1 + \frac{1}{m}\left(k+k'\right)u_2 &= \omega^2 u_2
\end{aligned}$$

Solving the first for u_2,

$$\begin{aligned}
\frac{1}{m}\left(k+k'\right)u_1 - \frac{k'}{m}u_2 &= \frac{k}{m}u_1 \\
\frac{k'}{m}u_2 &= \frac{1}{m}\left(k+k'\right)u_1 - \frac{k}{m}u_1 \\
k'u_2 &= \left(k+k'-k\right)u_1 \\
u_2 &= u_1
\end{aligned}$$

If we put this result into the second equation, it reduces to an identity:

$$\begin{aligned}
-\frac{k'}{m}u_1 + \frac{1}{m}(k+k')\,u_2 &= \omega^2 u_2 \\
-\frac{k'}{m}u_1 + \frac{1}{m}(k+k')\,u_1 &= \frac{k}{m}u_1 \\
-\frac{k'}{m}u_1 + \frac{1}{m}(k+k')\,u_1 &= \frac{k'}{m}u_1 \\
u_1 &= u_1
\end{aligned}$$

All we can say is that the first eigenvector has the form

$$\mathbf{u} = u_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

but this is expected because the eigenvector equation does not determine the overall magnitude of the eigenvectors. We may choose u_1 any way we please.

For the second eigenvalue equation we have

$$\frac{1}{m}\begin{pmatrix} k+k' & -k' \\ -k' & k+k' \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \tilde{\omega}^2 \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

and therefore, with $\tilde{\omega}^2 = \frac{k+2k'}{m}$,

$$\begin{aligned}
\frac{1}{m}(k+k')\,v_1 - \frac{k'}{m}v_2 &= \frac{k+2k'}{m}v_1 \\
-\frac{k'}{m}v_1 + \frac{1}{m}(k+k')\,v_2 &= \frac{k+2k'}{m}v_2
\end{aligned}$$

From the first,

$$\begin{aligned}
\frac{1}{m}(k+k')\,v_1 - \frac{k'}{m}v_2 &= \frac{k+2k'}{m}v_1 \\
(k+k')\,v_1 - (k+2k')\,v_1 &= k'v_2 \\
(k+k'-k-2k')\,v_1 &= k'v_2 \\
-v_1 &= v_2
\end{aligned}$$

and therefore

$$\mathbf{v} = v_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

Finally, choose $u_1 = v_1 = \frac{1}{\sqrt{2}}$ so that

$$\begin{aligned}
\mathbf{u} &= \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \\
\mathbf{v} &= \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}
\end{aligned}$$

are unit eigenvectors.

The two unit eigenvectors, $\mathbf{u}, \mathbf{v}$ are the columns of the inverse transformation matrix, A^{-1}, so

$$A^{-1} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

This is the matrix that connects our original oscillator positions, $\mathbf{x} = (x_1, x_2)$ to the normal mode positions (Q, q) according to

$$\begin{pmatrix} Q \\ q \end{pmatrix} = A \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

or

$$\begin{aligned} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} &= A^{-1} \begin{pmatrix} Q \\ q \end{pmatrix} \\ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} &= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} Q \\ q \end{pmatrix} \end{aligned}$$

Carrying out the matrix multiplication,

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} Q + q \\ Q - q \end{pmatrix}$$

or simply,

$$\begin{aligned} x_1 &= \frac{1}{\sqrt{2}} (Q + q) \\ x_2 &= \frac{1}{\sqrt{2}} (Q - q) \end{aligned}$$

This relationship holds at all times, and we know the time dependence of $Q(t)$ and $q(t)$ is simple harmonic with the normal frequencies. We may write the solutions as:

$$\begin{aligned} Q(t) &= A \sin(\omega t + \varphi_0) \\ q(t) &= B \sin(\tilde{\omega} t + \theta_0) \end{aligned}$$

Therefore, the general solution for the positions of the two oscillators is

$$\begin{aligned} x_1(t) &= \frac{1}{\sqrt{2}} (A \sin(\omega t + \varphi_0) + B \sin(\tilde{\omega} t + \theta_0)) \\ x_2(t) &= \frac{1}{\sqrt{2}} (A \sin(\omega t + \varphi_0) - B \sin(\tilde{\omega} t + \theta_0)) \end{aligned}$$

where the four constants $A, B, \varphi_0, \theta_0$ are determined by the initial position and initial velocity for each of the two oscillators.

To find the motions that characterize the normal modes of the spring system, we may look at what happens when $B = 0$ or when $A = 0$. With $B = 0$, the motion is purely of frequency ω,

$$\begin{aligned} x_1(t) &= \frac{A}{\sqrt{2}} \sin(\omega t + \varphi_0) \\ x_2(t) &= \frac{A}{\sqrt{2}} \sin(\omega t + \varphi_0) \end{aligned}$$

This describes a situation where both masses move back and forth together at the natural frequency, $\omega = \sqrt{\frac{k}{m}}$. Notice that if the two move together there is no stretching or contraction of the middle spring, so k' does not enter the answer. For the motion of the second normal mode, we set $A = 0$ to find

$$\begin{aligned} x_1(t) &= \frac{B}{\sqrt{2}} \sin(\tilde{\omega} t + \theta_0) \\ x_2(t) &= -\frac{B}{\sqrt{2}} \sin(\tilde{\omega} t + \theta_0) \end{aligned}$$

Now one mass moves right while the other moves left, and vice versa. This mode maximally stretches and compresses the middle spring, while also stretching or compressing the original springs. This is why the frequency $\tilde{\omega}^2 = \frac{k+2k'}{m}$ depends on both spring constants.

3 N oscillators

Suppose we couple a large number, N, of oscillators. Then each mass is affected by two springs. Let all masses and spring constants be equal, and let the displacement from equilibrium of the i^{th} mass be q_i. Then the equation of motion for the i^{th} mass is

$$m\frac{d^2q_i}{dt^2} - k\left(q_{i+1} - q_i\right) + k\left(q_i - q_{i-1}\right) = 0$$

To find the normal mode frequencies of this system, we assume that, for each normal mode, each mass oscillates as

$$q_i = A_i e^{i\Omega t}$$

where $A_i = a_i e^{i\varphi_i}$. This means that in that normal mode, each mass oscillates with the same frequency but possible different phase, φ_i. Substituting into each equation of motion, the $e^{i\Omega t}$ factor cancels, leaving

$$-m\Omega^2 A_i - k\left(A_{i+1} - A_i\right) + k\left(A_i - A_{i-1}\right) = 0$$

This is a recursion relation, giving A_{i+1} in terms of A_i and A_{i-1}. Letting $\omega^2 = \frac{k}{m}$,

$$\begin{aligned} \omega^2 A_{i+1} &= -\Omega^2 A_i + \omega^2 A_i + \omega^2 A_i - \omega^2 A_{i-1} \\ A_{i+1} &= -\frac{\Omega^2}{\omega^2} A_i + 2A_i - A_{i-1} \end{aligned}$$

Let $\eta = \frac{\Omega^2}{\omega^2}$ so that

$$A_{i+1} = \left(2 - \eta\right) A_i - A_{i-1}$$

we define $\alpha = 2 - \eta$ to get the simple form

$$A_{i+1} = \alpha A_i - A_{i-1}$$

If we start with initial conditions $q_0 = q_{N+1} = 0$, so the endpoints are fixed, then $A_0 = 0$ and setting $i = 1$,

$$A_2 = \alpha A_1$$

Continuing to $i = 2$,

$$\begin{aligned} A_3 &= \alpha A_2 - A_1 \\ &= \left(\alpha^2 - 1\right) A_1 \end{aligned}$$

then

$$\begin{aligned} A_4 &= \alpha A_3 - A_2 \\ &= \alpha\left(\alpha^2 - 1\right) A_1 - \alpha A_1 \\ &= \left[\left(\alpha^3 - \alpha\right) - \alpha\right] A_1 \\ &= \left(\alpha^3 - 2\alpha\right) A_1 \end{aligned}$$

It's clear that we get a series of alternating even and odd polynomials, and that the final equation for $A_{N+1} = 0$ will take the form

$$A_{N+1} = 0 = \alpha^N + \cdots$$

This is an N^{th} order polynomial equation giving the eigenvalues–the same equation we would have gotten by taking the determinant of $M - \lambda 1$.

However, we need a still more systematic approach. For a long chain of masses, we might guess that the amplitudes of displacement are periodic, so that A_i may be written as

$$A_i = a \sin i\varphi$$

for some numbers a and φ. Alternatively, we might notice that using sine or cosine in this form allows us to use the addition formula,

$$a\sin(i\pm 1)\varphi = \sin i\varphi\cos\varphi \pm \sin\varphi\cos i\varphi$$

so we can write the recursion as an equation for A_i alone.

Either way, making the substitution and using the addition formula puts the recursion formula in the form

$$\begin{aligned}
a\sin((i+1)\varphi) &= (2-\eta)\,a\sin(i\varphi) - a\sin((i-1)\varphi) \\
\sin i\varphi\cos\varphi + \sin\varphi\cos i\varphi &= (2-\eta)\sin(i\varphi) - \sin i\varphi\cos\varphi + \sin\varphi\cos i\varphi \\
\sin i\varphi\cos\varphi &= (2-\eta)\sin(i\varphi) - \sin i\varphi\cos\varphi
\end{aligned}$$

Solving for the sine

$$\sin i\varphi\,(2\cos\varphi - (2-\eta)) = 0$$

As long as $\sin i\varphi$ does not vanish for some i, this requires

$$\begin{aligned}
0 &= (2\cos\varphi - (2-\eta)) \\
&= 2(\cos\varphi - 1) + \frac{\Omega^2}{\omega^2} \\
\Omega^2 &= 2\omega^2(1-\cos\varphi) \\
&= 4\omega^2\sin^2\frac{\varphi}{2} \\
\Omega &= 2\omega\sin\frac{\varphi}{2}
\end{aligned}$$

We still need the second boundary condition. We have $A_0 = a\sin 0 = 0$ but we still need

$$\begin{aligned}
0 &= A_{N+1} \\
&= a\sin(N+1)\varphi
\end{aligned}$$

and therefore,

$$(N+1)\varphi = n\pi$$

Since $n=0$ and $n=N+1$ give zero, we have n modes given by

$$\begin{aligned}
\varphi_n &= \frac{n\pi}{N+1}, \quad n = 1,\cdots,n \\
\Omega_n &= 2\omega\sin\frac{n\pi}{2(N+1)}
\end{aligned}$$

4 The normal modes

We have written the solution for the k^{th} mass in *some* normal mode as

$$q_k = A_k e^{i\Omega t}$$

or, taking the imaginary part,

$$q_k = A_k\sin\Omega t$$

Next, we assumed that the amplitude A_k is given by $A_k = a\sin k\varphi$ and we found that Ω and φ satisfy

$$\Omega = 2\omega\sin\frac{\varphi}{2}$$

Imposing boundary conditions leads to N possible solutions, for Ω and A_k,

$$\begin{aligned}
\varphi_n &= \frac{n\pi}{N+1} \\
\Omega_n &= 2\omega \sin \frac{\varphi_n}{2} \\
&= 2\omega \sin \frac{n\pi}{2\left(N+1\right)} \\
A_{k,n} &= a \sin k\varphi_n \\
&= a \sin \frac{kn\pi}{N+1}
\end{aligned}$$

where n may take any value from 1 to N.

Together, these give the motion of the k^{th} particle in the n^{th} mode,

$$\begin{aligned}
q_k\left(n^{th} mode\right) &= A_i \sin \Omega t \\
&= a \sin\left[\frac{kn\pi}{N+1}\right] \sin\left[2\left(\sin\frac{n\pi}{2\left(N+1\right)}\right)\omega t\right]
\end{aligned}$$

This gives us the motion of each mass for any normal mode. The general motion of any one mass (the k^{th}) in the system is a superposition of these normal mode oscillations over all n,

$$q_k\left(t\right) = \sum_{n=1}^{N} a_n \sin\left[\frac{kn\pi}{N+1}\right] \sin\left[2\left(\sin\frac{n\pi}{2\left(N+1\right)}\right)\omega t\right]$$