Continuum Limit

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Consider again the case of N coupled oscillators. We found that Newton's second law applied to the i^{th} oscillator gives

$$m\frac{d^2q_i}{dt^2} - k\left(q_{i+1} - q_i\right) + k\left(q_i - q_{i-1}\right) = 0$$

We wish to take the limit of an infinite number of oscillators as their separation shrinks to zero. To begin, let the i^{th} position coordinate q_i be written as a function of its equilibrium position

$$\begin{aligned} q_i\left(t\right) &\rightarrow q\left(x,t\right) \\ x = &= id \end{aligned}$$

Then the force term becomes

$$k\left(q_{i+1} - q_i\right) \rightarrow k\left(q\left(x+d,t\right) - q\left(x,t\right)\right)$$

and the equation of motion is now

$$m\frac{\partial^2 q\left(x,t\right)}{\partial t^2} - k\left(q\left(x+d,t\right) - q\left(x,t\right)\right) + k\left(q\left(x,t\right) - q\left(x-d,t\right)\right) = 0$$

Notice that since q is now a function of two variables, we change the total derivatives to partial derivatives. We may simplify this by expanding $q\left(x+d\right)$ and $q\left(x-d\right)$ around $q\left(x\right)$:

$$\begin{aligned} q\left(x+d,t\right) &= q\left(x,t\right) + \left.\frac{\partial q}{\partial x}\right|_{x,t} d + \frac{1}{2!}\left.\frac{\partial^2 q}{\partial x^2}\right|_{x,t} d^2 + \cdots \\ q\left(x-d,t\right) &= q\left(x,t\right) - \left.\frac{\partial q}{\partial x}\right|_{x,t} d + \frac{1}{2!}\left.\frac{\partial^2 q}{\partial x^2}\right|_{x,t} d^2 - \cdots \end{aligned}$$

Substituting into the equation of motion,

$$m\frac{\partial^2 q\left(x,t\right)}{\partial t^2} - k\left(\left.\frac{\partial q}{\partial x}\right|_{x,t} d + \frac{1}{2!}\left.\frac{\partial^2 q}{\partial x^2}\right|_{x,t} d^2 + \cdots\right) + k\left(-\left.\frac{\partial q}{\partial x}\right|_{x,t} d + \frac{1}{2!}\left.\frac{\partial^2 q}{\partial x^2}\right|_{x,t} d^2 - \cdots\right) = 0$$

The linear terms in d cancel. Dividing by m leaves us with\;

$$\frac{\partial^2 q\left(x,t\right)}{\partial t^2} - \frac{kd^2}{m}\left.\frac{\partial^2 q}{\partial x^2}\right|_{x,t} + terms\ of\ order\ d^3\ and\ higher = 0$$

We now take the limit as $d \rightarrow 0$. This requires us to be precise about the limit of the constant, $\frac{kd^2}{m}$.

Springs in series

Suppose we have two springs with spring constants k_1 and k_2, connected end to end, and then to a mass m. If we stretch the system by a length x, the force each spring exerts on the other and on the mass must be the same, so with the stretch of each spring being x_1 and x_2 respectively, we must have

$$\begin{aligned} x &= x_1 + x_2 \\ kx = k_2 x_2 &= k_1 x_1 \end{aligned}$$

Therefore,

$$\begin{aligned} x_1 &= \frac{kx}{k_1} \\ x_2 &= \frac{kx}{k_2} \end{aligned}$$

and $x = x_1 + x_2$ becomes

$$\begin{aligned} x &= x_1 + x_2 \\ &= \frac{kx}{k_1} + \frac{kx}{k_2} \end{aligned}$$

Cancelling the overall factor of x, the effective spring constant is given by

$$\frac{1}{k} = \frac{1}{k_1} + \frac{1}{k_2}$$

Finally, if $k_1 = k_2$,

$$\begin{aligned} \frac{1}{k} &= \frac{2}{k_1} \\ k_1 &= 2k \end{aligned}$$

The $d \to 0$ limit

Applied to our oscillators, this means that if we double the number of masses, the spring constant between mass pairs doubles. More generally, k is inversely proportional to the separation of masses,

$$k = \frac{\kappa}{d}$$

where κ_0 is the spring constant per unit length. Also, with each doubling of the number of masses, we cut each mass in half, so that

$$m = \mu d$$

with μ_0 givng the mass per unit length. We hold κ and μ constant. Putting these together, the constant in the wave equation is

$$\frac{kd^2}{m} = \frac{\frac{\kappa}{d} d^2}{\mu d} = \frac{\kappa}{\mu}$$

This quantity does not change as $d \to 0$,

$$\lim_{d \to 0} \left(\frac{kd^2}{m} \right) = \frac{\kappa}{\mu}$$

Since terms of cubic order and higher vanish as $d \to 0$, therefore, the continuum limit is

$$\frac{\partial^2 q\left(x,t\right)}{\partial t^2} - \frac{\kappa}{\mu} \frac{\partial^2 q\left(x,t\right)}{\partial x^2} = 0$$

This is once again the 2-dimensional wave equation. Since the x dependence of $q(x,t)$ spans all of the former q_i, this equation combines the full couple set of $N \to \infty$ equations. As we shall see, there are now infinitely many normal modes of oscillation.

The constant has units of velocity squared:

$$\begin{aligned} \left[\frac{\kappa}{\mu}\right] &= \left[\frac{kd^2}{m}\right] \\ &= \frac{[F/x]\cdot m^2}{kg} \\ &= \frac{kg\cdot m^2}{kg\cdot s^2} \\ &= \left(\frac{m}{s}\right)^2 \end{aligned}$$

We will see that this velocity, $c \equiv \sqrt{\frac{\kappa}{\mu}}$, is the speed of waves in the continuous medium. We may write the wave equation in the final form:

$$-\frac{1}{c^2}\frac{\partial^2 q}{\partial t^2} + \frac{\partial^2 q}{\partial x^2} = 0$$