Continuity, conservation, Stokes, and Maxwell

November 21, 2019

1 The continuity equation

Suppose the wave equation is satisfied by a displacement from equilibrium ψ

$$-\frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2}+\nabla^2\psi = 0$$

in a medium of density μ, and consider the definitions

$$\begin{aligned}\rho &= \frac{\mu}{2}\left[\left(\frac{\partial\psi}{\partial t}\right)^2+c^2\boldsymbol{\nabla}\psi\cdot\boldsymbol{\nabla}\psi\right]\\ \mathbf{J} &= -\mu c^2\frac{\partial\psi}{\partial t}\boldsymbol{\nabla}\psi\end{aligned}$$

These have units of

$$\begin{aligned}[\rho] &= [\mu]\left[\frac{\partial\psi}{\partial t}\right]^2\\ &= \frac{kg}{m^3}\frac{m^2}{s^2}\\ &= \frac{E}{m^3}\end{aligned}$$

which we may think of as energy density and

$$\begin{aligned}[\mathbf{J}] &= [\mu]\left[c^2\right]\left[\frac{\partial\psi}{\partial t}\right][\boldsymbol{\nabla}\psi]\\ &= \frac{kg}{m^3}\frac{m^2}{s^2}\frac{m}{s}\frac{m}{m}\\ &= \frac{Ec}{m^3}\\ &= \frac{E}{m^2 s}\end{aligned}$$

which an energy flux (energy per unit area per second).

Then (magically),

$$\begin{aligned}\frac{\partial\rho}{\partial t} &= \mu\frac{\partial\psi}{\partial t}\frac{\partial^2\psi}{\partial t^2}+\mu c^2\boldsymbol{\nabla}\psi\cdot\boldsymbol{\nabla}\left(\frac{\partial\psi}{\partial t}\right)\\ \boldsymbol{\nabla}\cdot\mathbf{J} &= -\mu c^2\left(\boldsymbol{\nabla}\frac{\partial\psi}{\partial t}\right)\cdot\boldsymbol{\nabla}\psi-\mu c^2\frac{\partial\psi}{\partial t}\boldsymbol{\nabla}\cdot\boldsymbol{\nabla}\psi\end{aligned}$$

If we add these together,

$$\begin{aligned}
\frac{\partial \rho}{\partial t}+\boldsymbol{\nabla}\cdot\mathbf{J} &= \mu\frac{\partial \psi}{\partial t}\frac{\partial^2 \psi}{\partial t^2}+\cancel{\mu c^2\boldsymbol{\nabla}\psi\cdot\boldsymbol{\nabla}\left(\frac{\partial \psi}{\partial t}\right)}-\cancel{\mu c^2\left(\boldsymbol{\nabla}\frac{\partial \psi}{\partial t}\right)\cdot\boldsymbol{\nabla}\psi}-\mu c^2\frac{\partial \psi}{\partial t}\boldsymbol{\nabla}\cdot\boldsymbol{\nabla}\psi \\
&= -\mu c^2\frac{\partial \psi}{\partial t}\left(-\frac{1}{c^2}\frac{\partial^2 \psi}{\partial t^2}+\nabla^2\psi\right) \\
&= 0
\end{aligned}$$

where we use the wave equation in the last step.

There is a systematic way to find quantities for which this equation holds, but it depends on the Lagrangian, which we will not look into now. However, the *continuity equation*

$$\frac{\partial \rho}{\partial t}+\boldsymbol{\nabla}\cdot\mathbf{J} = 0 \tag{1}$$

is important because it tells us we have a conserved quantity.

1.1 Conservation of energy

In the example above, the units are chosen to correspond to conservation of energy. To see that the total energy actually is conserved, we integrate the energy density over a volume V and use the divergence theorem. Let

$$E=\int\limits_V \rho d^3x$$

Then the time derivative of E is

$$\begin{aligned}
\frac{dE}{dt} &= \frac{d}{dt}\int\limits_V \rho\left(\mathbf{x},t\right)d^3x \\
&= \int\limits_V \frac{\partial \rho}{\partial t}d^3x
\end{aligned}$$

Using the continuity equation, Eq.(1), this becomes

$$\frac{dE}{dt} = -\int\limits_V \boldsymbol{\nabla}\cdot\mathbf{J}d^3x$$

Now, recall that the divergence theorem tells us that the volume integral of a divergence of a vector field equals the flux of that vector field across the boundary,

$$\int\limits_V \boldsymbol{\nabla}\cdot\mathbf{v}d^3x=\oint\limits_S \hat{\mathbf{n}}\cdot\mathbf{v}d^2x$$

Here V is any volume and S is the closed boundary of that volume; $\hat{\mathbf{n}}$ is the unit outward normal to S, so that at any point on the boundary, $\hat{\mathbf{n}}\cdot\mathbf{v}$ is the part of $\mathbf{v}$ crossing the boundary.

Returning to the time rate of change of E and applying the divergence theorem,

$$\frac{dE}{dt} = -\oint\limits_S \hat{\mathbf{n}}\cdot\mathbf{J}d^2x$$

This means that the only change in energy E in the volume V is that decrease due to a flow of energy outward across the boundary of the region. If no energy crosses the boundary (or the boundary is taken at infinity) then energy is conserved.

1.2 Probability and the Schrödinger equation

This sort of conservation law applies to most field theories. For example, starting with the Schrödinger equation,

$$-\frac{\hbar^2}{2m}\nabla^2\psi + V\psi = i\hbar\frac{\partial\psi}{\partial t}$$

we multiply by the complex conjugate of the wave function, ψ^*,

$$-\frac{\hbar^2}{2m}\psi^*\nabla^2\psi + V\psi^*\psi = i\hbar\psi^*\frac{\partial\psi}{\partial t}$$

Now take the complex conjugate of the whole equation,

$$-\frac{\hbar^2}{2m}\psi\nabla^2\psi^* + V\psi\psi^* = -i\hbar\psi\frac{\partial\psi^*}{\partial t}$$

These must both be true. Now subtract the two equations,

$$-\frac{\hbar^2}{2m}\left(\psi^*\nabla^2\psi - \psi\nabla^2\psi^*\right) = i\hbar\left(\psi^*\frac{\partial\psi}{\partial t} + \psi\frac{\partial\psi^*}{\partial t}\right) \tag{2}$$

Exercise: Show that

$$\psi^*\nabla^2\psi - \psi\nabla^2\psi^* = \boldsymbol{\nabla}\cdot\left(\psi\boldsymbol{\nabla}\psi^* - \psi^*\boldsymbol{\nabla}\psi\right)$$

and that

$$\psi^*\frac{\partial\psi}{\partial t} + \psi\frac{\partial\psi^*}{\partial t} = \frac{\partial}{\partial t}\left(\psi\psi^*\right)$$

Rewriting Eq.(2) with the results of the exercise,

$$-\frac{\hbar^2}{2m}\boldsymbol{\nabla}\cdot\left(\psi\boldsymbol{\nabla}\psi^* - \psi^*\boldsymbol{\nabla}\psi\right) = i\hbar\frac{\partial}{\partial t}\left(\psi\psi^*\right)$$

we define

$$\begin{aligned}\rho &\equiv \psi^*\psi \\ \mathbf{J} &= \frac{i\hbar}{2m}\left(\psi^*\boldsymbol{\nabla}\psi - \psi\boldsymbol{\nabla}\psi^*\right)\end{aligned}$$

Then after we cancel an overall factor of $i\hbar$, the equation takes the form of the continuity equation,

$$\frac{\partial\rho}{\partial t} + \boldsymbol{\nabla}\cdot\mathbf{J} = 0$$

From this form, we immediately know that the integral of ρ is a conserved quantity,

$$\frac{d}{dt}\left[\int_V \psi^*\psi d^3x\right] = -\oint_S \hat{\mathbf{n}}\cdot\mathbf{J}d^2x$$

In quantum mechanics, $\psi^*\psi$ is given the interpretation of a *probability density*, and its integral is the probability of finding the particle in the volume V. The current $\mathbf{J}$ is a probability flux, giving the flow of probability from one place to another. If we take V to be all space, so that no current can flow out, the probability of finding the particle must be 1. Therefore, we normalize the wave function so that its integral over all space is 1,

$$\int_V \psi^*\psi d^3x = 1$$

and this quantity must be conserved.

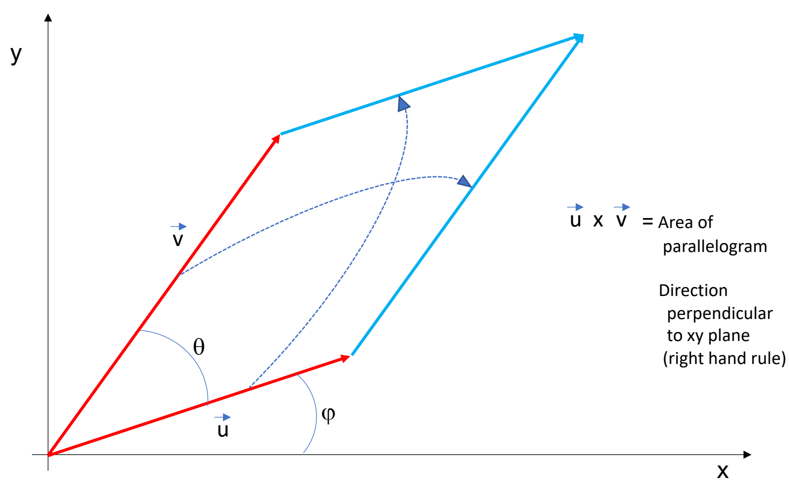
1.3 Conservation of electric charge

We will look at Maxwell's equations in more detail shortly. In particular, we will show that it is a consequence of Maxwell's equations that the electric charge density ρ and the current density $\mathbf{J}$ satisfy the continuity equation. Therefore, total electric charge is conserved, and the charge in any volume V changes only if current flows across the boundary of V.

2 The cross product, the curl, and Stokes' theorem

2.1 Cross product

We have used the dot product to form a scalar from two vectors. The *cross product* is another geometrically significant way to combine a pair of vectors, this time to get a third vector. The cross product is an *oriented area*. Given any two vectors, we may form a parallelogram by placing the tail of one of the vectors to the tip of the other.



The two vectors shown lie in the xy-plane. The area of the parallelogram is the base times the height, which we may take as the projection of $\vec{\mathbf{v}}$ perpendicular to $\vec{\mathbf{u}}$, times the length of $\vec{\mathbf{u}}$. The direction is the direction perpendicular to both, given by the right hand rule. Here it is the z-direction, so

$$\vec{\mathbf{u}} \times \vec{\mathbf{v}} = uv \sin\theta\, \hat{\mathbf{k}}$$

where θ is the angle between the two vectors.

To compute the cross product in terms of the components of $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$, rotate both vectors through an angle $-\varphi$, so that the new vector $\vec{\mathbf{u}}$ - call it $\tilde{\mathbf{u}}$ - lies along the x axis. Then the area of the parallelogram is just the length of $\tilde{\mathbf{u}}$, which is its x-component, $\tilde{u} = \tilde{u}_x$, times the component of $\tilde{\mathbf{v}}$ perpendicular to $\tilde{\mathbf{u}}$. This is just $\tilde{v}_y$, and since the area is the same before and after rotation,

$$\left|\vec{\mathbf{u}} \times \vec{\mathbf{v}}\right| = uv \sin\theta = \tilde{u}_x \tilde{v}_y$$

But the components of $\tilde{\mathbf{u}}$ and $\tilde{\mathbf{v}}$ are just found by rotation through the angle $-\varphi$ between $\tilde{\mathbf{u}}$ and the x-axis,

$$\begin{aligned}\begin{pmatrix}\tilde{u}_x\\0\end{pmatrix} &= \begin{pmatrix}\cos\varphi & \sin\varphi\\ -\sin\varphi & \cos\varphi\end{pmatrix}\begin{pmatrix}u_x\\u_y\end{pmatrix}\\ &= \begin{pmatrix}u_x\cos\varphi+u_y\sin\varphi\\ -u_x\sin\varphi+u_y\cos\varphi\end{pmatrix}\end{aligned}$$

and

$$\begin{aligned}\begin{pmatrix}\tilde{v}_x\\\tilde{v}_y\end{pmatrix} &= \begin{pmatrix}\cos\varphi & \sin\varphi\\ -\sin\varphi & \cos\varphi\end{pmatrix}\begin{pmatrix}v_x\\v_y\end{pmatrix}\\ &= \begin{pmatrix}v_x\cos\varphi+v_y\sin\varphi\\ -v_x\sin\varphi+v_y\cos\varphi\end{pmatrix}\end{aligned}$$

Therefore, we know that

$$\begin{aligned}\tilde{u}_x &= u_x\cos\varphi+u_y\sin\varphi\\ \tilde{v}_y &= -v_x\sin\varphi+v_y\cos\varphi\end{aligned}$$

and the condition that $\tilde{u}_y=0$ shows that

$$-u_x\sin\varphi+u_y\cos\varphi = 0$$

We can also express the sine and cosine in terms of the original components,

$$\begin{aligned}\cos\varphi &= \frac{u_x}{u}\\ \sin\varphi &= \frac{u_y}{u}\end{aligned}$$

where $u^2=u_x^2+u_y^2$.

Now we have

$$\begin{aligned}\left|\vec{\mathbf{u}}\times\vec{\mathbf{v}}\right| &= uv\sin\theta\\ &= \tilde{u}_x\tilde{v}_y\\ &= (u_x\cos\varphi+u_y\sin\varphi)(-v_x\sin\varphi+v_y\cos\varphi)\\ &= -v_xu_x\cos\varphi\sin\varphi+v_yu_x\cos^2\varphi-v_xu_y\sin\varphi\sin\varphi+v_yu_y\sin\varphi\cos\varphi\\ &= -v_xu_x\cos\varphi\sin\varphi+v_yu_x\cos^2\varphi-v_xu_y\sin^2\varphi+v_yu_y\sin\varphi\cos\varphi\\ &= -v_xu_x\frac{u_x}{u}\frac{u_y}{u}+v_yu_x\frac{u_x^2}{u^2}-v_xu_y\frac{u_y^2}{u^2}+v_yu_y\frac{u_y}{u}\frac{u_x}{u}\\ &= \frac{1}{u^2}\left(-v_xu_yu_x^2+v_yu_xu_x^2-v_xu_yu_y^2+u_xv_yu_y^2\right)\\ &= \frac{1}{u^2}\left((-v_xu_y+v_yu_x)u_x^2+(-v_xu_y+u_xv_y)u_y^2\right)\\ &= \frac{1}{u^2}(-v_xu_y+v_yu_x)\left(u_x^2+u_y^2\right)\\ &= u_xv_y-u_yv_x\end{aligned}$$

Therefore, the curl is

$$\vec{\mathbf{u}}\times\vec{\mathbf{v}} = (u_xv_y-u_yv_x)\,\hat{\mathbf{k}}$$

We may carry out the same sort of calculation for all three components of a more general pair of vectors to show that for arbitrary $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$, the cross product is given by

$$\vec{\mathbf{u}}\times\vec{\mathbf{v}} = (u_yv_z-u_zv_y)\,\hat{\mathbf{j}}+(u_zv_x-u_xv_z)\,\hat{\mathbf{j}}+(u_xv_y-u_yv_x)\,\hat{\mathbf{k}} \tag{3}$$

2.2 The curl

An important variant of the cross product occurs if we replace the first vector with the gradient,

$$\boldsymbol{\nabla} = \hat{\mathbf{i}}\frac{\partial}{\partial x} + \hat{\mathbf{j}}\frac{\partial}{\partial y} + \hat{\mathbf{k}}\frac{\partial}{\partial z}$$

and let the derivatives act on the second vector. Let $\mathbf{v}(\mathbf{x})$ be a vector field with Cartesian components

$$\mathbf{v} = v_x\hat{\mathbf{i}} + v_y\hat{\mathbf{j}} + v_z\hat{\mathbf{k}}$$

Then the *curl* of $\mathbf{v}$ is written as $\boldsymbol{\nabla} \times \mathbf{v}$ and substituting the gradient for $\overrightarrow{\mathbf{u}}$ in Eq.(3),

$$\boldsymbol{\nabla} \times \mathbf{v} = \hat{\mathbf{i}}\left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right) + \hat{\mathbf{j}}\left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}\right) + \hat{\mathbf{k}}\left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)$$

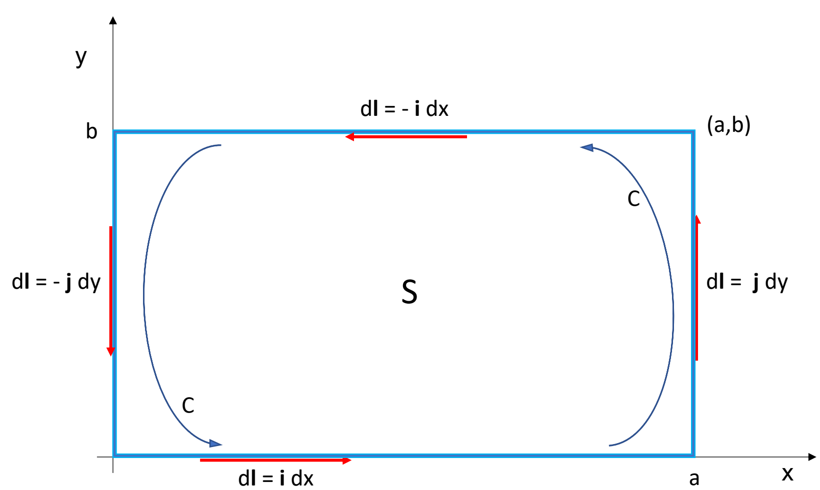
2.3 Stokes' theorem

The meaning of the curl is given by *Stokes' theorem.*

Stokes' theorem Let S be any 2-dimensional surface with closed boundary curve C, and let the normal at any point of S be $\hat{\mathbf{n}}$. Then the integral of the normal component of the curl over S equals the line integral of $\mathbf{v}$ around C:

$$\iint\limits_S \hat{\mathbf{n}} \cdot (\boldsymbol{\nabla} \times \mathbf{v}) \quad = \quad \oint\limits_C \mathbf{v} \cdot d\mathbf{l}$$

Proof: We start with a small rectangular surface with sides a and b. We may choose our coordinates in any way we please, so let the x and y directions be taken along the sides of the rectangle, with the normal in the z-direction.



Then the surface integral of the curl is

$$\begin{aligned}
\iint\limits_S \hat{\mathbf{n}}\cdot(\boldsymbol{\nabla}\times\mathbf{v}) &= \iint\limits_S \hat{\mathbf{k}}\cdot\hat{\mathbf{k}}\left(\frac{\partial v_y(x,y)}{\partial x}-\frac{\partial v_x(x,y)}{\partial y}\right)\left(\frac{\partial v_x}{\partial y}-\frac{\partial v_y(x,y)}{\partial x}\right) \\
&= \int_0^a dx\int_0^b dy\left(\frac{\partial v_y(x,y)}{\partial x}-\frac{\partial v_x(x,y)}{\partial y}\right) \\
&= \int_0^b dy\left(\int_0^a dx\frac{\partial v_y(x,y)}{\partial x}\right)-\int_0^a dx\left(\int_0^b dy\frac{\partial v_x(x,y)}{\partial y}\right) \\
&= \int_0^b dy\left(v_y(a,y)-v_y(0,y)\right)-\int_0^a dx\left(v_x(x,b)-v_x(x,0)\right)
\end{aligned}$$

and re-ordering the integrals progressively around the rectangle,

$$\iint\limits_S \hat{\mathbf{n}}\cdot(\boldsymbol{\nabla}\times\mathbf{v}) = \int_0^a dx v_x(x,0)+\int_0^b dy v_y(a,y)-\int_0^a dx v_x(x,b)-\int_0^b dy v_y(0,y)$$

Now let $d\mathbf{l}$ be an infinitesimal vector along the counterclockwise boundary of the rectangle. That is, along $y=0$, $d\mathbf{l}=\hat{\mathbf{i}}dx$, then along the side at $x=a$ we have $d\mathbf{l}=\hat{\mathbf{j}}dy$. Back across the top at $y=b$ the displacement is along $d\mathbf{l}=-\hat{\mathbf{i}}dx$, and returning down the y axis to the origin $d\mathbf{l}=-\hat{\mathbf{j}}dy$. The remaining integrals become

$$\begin{aligned}
\iint\limits_S \hat{\mathbf{n}}\cdot(\boldsymbol{\nabla}\times\mathbf{v}) &= \int_0^a dx v_x(x,0)+\int_0^b dy v_y(a,y)-\int_0^a dx v_x(x,b)-\int_0^b dy v_y(0,y) \\
&= \int_0^a\left(\hat{\mathbf{i}}dx\right)\cdot\mathbf{v}(x,0)\,dx+\int_0^b\left(\hat{\mathbf{j}}dy\right)\cdot\mathbf{v}(a,y)+\int_0^a\left(-\hat{\mathbf{i}}dx\right)\cdot\mathbf{v}(x,b)+\int_0^b\left(-\hat{\mathbf{j}}dy\right)\cdot\mathbf{v}(0,y) \\
&= \int_0^a d\mathbf{l}\cdot\mathbf{v}(x,0)\,dx+\int_0^b d\mathbf{l}\cdot\mathbf{v}(a,y)+\int_0^a d\mathbf{l}\cdot\mathbf{v}(x,b)+\int_0^b d\mathbf{l}\cdot\mathbf{v}(0,y) \\
&= \oint\limits_{boundary}\mathbf{v}\cdot d\mathbf{l}
\end{aligned}$$

To complete the proof, we cover a general surface S with infinitesimal rectangles and apply this result to each one. Where the rectangles have sides in common, the counterclockwise paths cancel so the net effect a line integral around the perimeter of the entire surface.

From Stokes' theorem we see that the curl tells us how much the vector field $\mathbf{v}$ tends to circle around any given closed curve.

3 Maxwell's equations

We are now in a position to discuss the Maxwell equations for electromagnetism. Maxwell's equations may be written as either integral equations or differential equations. The two forms are connected by the divergence theorem and Stokes' theorem.

3.1 Gauss's law

In integral form, the first equation expresses Gauss's law that the integral of the electric field over any closed surface is proportional to the total enclosed charge,

$$\oiint \mathbf{E} \cdot \hat{\mathbf{n}}\, d^2x = 4\pi Q_{enclosed}$$

If we write the charge enclosed as a volume integral over the region inside the closed Gaussian surface,

$$Q_{enclosed} = \iiint\limits_V \rho\, d^3x$$

and use the divergence theorem to write the surface integral as an integral over the same volume,

$$\oiint \mathbf{E} \cdot \hat{\mathbf{n}}\, d^2x = \iiint\limits_V \boldsymbol{\nabla} \cdot \mathbf{E}\, d^3x$$

then, combining the integrals, Gauss's law becomes

$$\iiint\limits_V \left(\boldsymbol{\nabla} \cdot \mathbf{E} - 4\pi\rho \right) d^3x = 0$$

Since this holds for any volume V we may take the limit as V approaches any point, and the integrand must vanish at that point. Therefore,

$$\boldsymbol{\nabla} \cdot \mathbf{E} = 4\pi\rho \qquad (4)$$

is the differential form of Gauss's law.

3.2 Gauss's law for magnetism

The differential form for Gauss's law for magnetism follows in exactly the same way, but since there are no separate magnetic charges, the right side of the equation is zero. We immediately have the vanishing divergence of the magnetic field,

$$\boldsymbol{\nabla} \cdot \mathbf{B} = 0 \qquad (5)$$

3.3 Faraday's law

Faraday's law of magnetic induction states that a changing magnetic flux through a loop produces a potential around the loop. Magnetic flux is given by the surface integral of the normal component,

$$\iint\limits_S \mathbf{B} \cdot \hat{\mathbf{n}}\, d^2x$$

while the potential around the loop ("electromotive force") is given by the integral of the electric field around the boundary C of S, so the law is

$$\oint\limits_C \mathbf{E} \cdot d\mathbf{l} = -\frac{1}{c}\frac{d}{dt} \iint\limits_S \mathbf{B} \cdot \hat{\mathbf{n}}\, d^2x$$

The relative negative sign follows from Lenz's law. Using Stokes theorem to write the electric integral as a curl,

$$\oint\limits_C \mathbf{E} \cdot d\mathbf{l} = \iint\limits_S \hat{\mathbf{n}} \cdot \left(\boldsymbol{\nabla} \times \mathbf{E} \right) d^2x$$

both the two integrals and the dot products with the normal combine,

$$\iint\limits_S \hat{\mathbf{n}} \cdot \left(\boldsymbol{\nabla} \times \mathbf{E} + \frac{1}{c} \frac{\partial}{\partial t} \mathbf{B} \right) d^2x \quad = \quad 0$$

The surface is arbitrary, so we may shrink it to any point to find that the integrand must vanish. In fact, since we may tip the surface in any direction at the point in question, the direction of the normal is also arbitrary and we conclude that

$$\boldsymbol{\nabla} \times \mathbf{E} + \frac{1}{c} \frac{\partial \mathbf{B}}{\partial t} = 0$$

at every point.

3.4 Ampère's law

Ampère's law states that the flux of a current through any surface equals the line integral of the magnetic field around boundary of the surface. Maxwell noted the additional need for any time rate of change of electric flux through the surface, so (including units) the integral form of Ampère's law takes the form

$$\frac{4\pi}{c} \iint\limits_S \mathbf{J} \cdot \hat{\mathbf{n}}\, d^2x \quad = \quad \oint\limits_C \mathbf{B} \cdot d\mathbf{l} - \frac{1}{c} \frac{d}{dt} \iint\limits_S \mathbf{E} \cdot \hat{\mathbf{n}}\, d^2x$$

Stokes' theorem lets us write all three integrals as a single surface integral by setting

$$\oint\limits_C \mathbf{B} \cdot d\mathbf{l} \quad = \quad \iint\limits_S \hat{\mathbf{n}} \cdot (\boldsymbol{\nabla} \times \mathbf{B})\, d^2x$$

The combined equations all involve the normal component of a vector,

$$0 \quad = \quad \iint\limits_S \hat{\mathbf{n}} \cdot \left[(\boldsymbol{\nabla} \times \mathbf{B}) - \frac{1}{c} \frac{\partial \mathbf{E}}{\partial t} - \frac{4\pi}{c} \mathbf{J} \right] d^2x$$

and once again, since we may orient the surface in any direction as we shrink it to any point, we get the differential form of Ampère's law,

$$\boldsymbol{\nabla} \times \mathbf{B} - \frac{1}{c} \frac{\partial \mathbf{E}}{\partial t} \quad = \quad \frac{4\pi}{c} \mathbf{J}$$

3.5 Maxwell's equations

Collecting the differential forms, we have the full set of Maxwell equations,

$$\begin{aligned}
\boldsymbol{\nabla} \cdot \mathbf{E} &= 4\pi\rho && (6)\\
\boldsymbol{\nabla} \cdot \mathbf{B} &= 0 && (7)\\
\boldsymbol{\nabla} \times \mathbf{E} + \frac{1}{c} \frac{\partial \mathbf{B}}{\partial t} &= 0 && (8)\\
\boldsymbol{\nabla} \times \mathbf{B} - \frac{1}{c} \frac{\partial \mathbf{E}}{\partial t} &= \frac{4\pi}{c} \mathbf{J} && (9)
\end{aligned}$$

With one more result, we can show that the sources on the right hand side of Maxwell's equations satisfy the continuity equation. The result we need is

Exercise: Show that the divergence of a curl always vanishes,

$$\boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} \times \mathbf{v}) = 0$$

Exercise: Show that the curl of a gradient always vanishes,

$$\boldsymbol{\nabla} \times \boldsymbol{\nabla} f = 0$$

Exercise: Add the time derivative of Eq.(6) to c times the divergence of Eq.(9) to show that ρ and $\mathbf{J}$ satisfy the continuity equation. What is the conserved charge?

3.5.1 One more identity: the curl of a curl

When the sources vanish, the electric and magnetic fields satisfy wave equations. To see this we require one further identity, involving the curl of a curl:

$$\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \mathbf{v})$$

There are much easier ways to prove this once we have a bit more notation, but it is not too bad if we just take it a step at a time. Let $\mathbf{w}$ be the curl of $\mathbf{v}$,

$$\begin{aligned}
\mathbf{w} &= \boldsymbol{\nabla} \times \mathbf{v} \\
&= \hat{\mathbf{i}} \left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z} \right) + \hat{\mathbf{j}} \left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x} \right) + \hat{\mathbf{k}} \left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y} \right)
\end{aligned}$$

so the components of $\mathbf{w}$ are

$$\begin{aligned}
w_x &= \frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z} \\
w_y &= \frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x} \\
w_z &= \frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}
\end{aligned}$$

Then

$$\begin{aligned}
\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \mathbf{v}) &= \boldsymbol{\nabla} \times \mathbf{w} \\
&= \hat{\mathbf{i}} \left(\frac{\partial w_z}{\partial y} - \frac{\partial w_y}{\partial z} \right) + \hat{\mathbf{j}} \left(\frac{\partial w_x}{\partial z} - \frac{\partial w_z}{\partial x} \right) + \hat{\mathbf{k}} \left(\frac{\partial w_y}{\partial x} - \frac{\partial w_x}{\partial y} \right) \\
&= \hat{\mathbf{i}} \left(\frac{\partial}{\partial y} \left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y} \right) - \frac{\partial}{\partial z} \left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x} \right) \right) \\
&\quad + \hat{\mathbf{j}} \left(\frac{\partial}{\partial z} \left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z} \right) - \frac{\partial}{\partial x} \left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y} \right) \right) \\
&\quad + \hat{\mathbf{k}} \left(\frac{\partial}{\partial x} \left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x} \right) - \frac{\partial}{\partial y} \left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z} \right) \right) \\
&= \hat{\mathbf{i}} \left(\frac{\partial^2 v_y}{\partial y \partial x} - \frac{\partial^2 v_x}{\partial y^2} - \frac{\partial^2 v_x}{\partial z^2} + \frac{\partial^2 v_z}{\partial z \partial x} \right) \\
&\quad + \hat{\mathbf{j}} \left(\frac{\partial^2 v_z}{\partial z \partial y} - \frac{\partial^2 v_y}{\partial z^2} - \frac{\partial^2 v_y}{\partial x^2} + \frac{\partial^2 v_x}{\partial x \partial y} \right) \\
&\quad + \hat{\mathbf{k}} \left(\frac{\partial^2 v_x}{\partial x \partial z} - \frac{\partial^2 v_z}{\partial x^2} - \frac{\partial^2 v_z}{\partial y^2} + \frac{\partial^2 v_y}{\partial y \partial z} \right)
\end{aligned}$$

Within each component we replace the unmixed second derivatives with the Laplacian,

$$-\frac{\partial^2 v_x}{\partial y^2} - \frac{\partial^2 v_x}{\partial z^2} = \frac{\partial^2 v_x}{\partial x^2} - \nabla^2 v_x$$

and similarly for the other two components. This gives

$$\begin{aligned}
\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \mathbf{v}) &= \hat{\mathbf{i}} \left(\frac{\partial^2 v_y}{\partial y \partial x} + \frac{\partial^2 v_x}{\partial x^2} + \frac{\partial^2 v_z}{\partial z \partial x} \right) - \nabla^2 v_x \hat{\mathbf{i}} \\
&\quad + \hat{\mathbf{j}} \left(\frac{\partial^2 v_z}{\partial z \partial y} + \frac{\partial^2 v_y}{\partial y^2} + \frac{\partial^2 v_x}{\partial x \partial y} \right) - \nabla^2 v_y \hat{\mathbf{j}} \\
&\quad + \hat{\mathbf{k}} \left(\frac{\partial^2 v_x}{\partial x \partial z} + \frac{\partial^2 v_z}{\partial z^2} + \frac{\partial^2 v_y}{\partial y \partial z} \right) - \nabla^2 v_z \hat{\mathbf{k}}
\end{aligned}$$

The remaining terms all have a common derivative. Pulling it out, what remains is a divergence,

$$\begin{aligned}
\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \mathbf{v}) &= \hat{\mathbf{i}} \frac{\partial}{\partial x} \left(\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} \right) - \nabla^2 v_x \hat{\mathbf{i}} \\
&\quad + \hat{\mathbf{j}} \frac{\partial}{\partial y} \left(\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} \right) - \nabla^2 v_y \hat{\mathbf{j}} \\
&\quad + \hat{\mathbf{k}} \frac{\partial}{\partial z} \left(\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} \right) - \nabla^2 v_z \hat{\mathbf{k}}
\end{aligned}$$

We can recombine the gradient with the Laplacian terms combining as $\nabla^2 \mathbf{v}$:

$$\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \mathbf{v}) = \boldsymbol{\nabla} (\boldsymbol{\nabla} \cdot \mathbf{v}) - \nabla^2 \mathbf{v}$$

Now we blink twice in amazement and smile.

3.6 Wave equations

Suppose we are in a region away from sources so that

$$\begin{aligned}
\rho &= 0 \\
\mathbf{J} &= 0
\end{aligned}$$

Then Gauss's law (6) and Ampère's law (9) simplify to

$$\boldsymbol{\nabla} \cdot \mathbf{E} = 0 \tag{10}$$

$$\boldsymbol{\nabla} \times \mathbf{B} - \frac{1}{c} \frac{\partial \mathbf{E}}{\partial t} = 0 \tag{11}$$

Now look at the curl of Faraday's law,

$$\boldsymbol{\nabla} \times \left(\boldsymbol{\nabla} \times \mathbf{E} + \frac{1}{c} \frac{\partial \mathbf{B}}{\partial t} \right) = 0$$

Using our new identity,

$$\boldsymbol{\nabla} (\boldsymbol{\nabla} \cdot \mathbf{E}) - \nabla^2 \mathbf{E} + \boldsymbol{\nabla} \times \left(\frac{1}{c} \frac{\partial \mathbf{B}}{\partial t} \right) = 0$$

Interchanging the order of differentiation of the magnetic field, we can replace $\boldsymbol{\nabla} \times \mathbf{B}$ using Eq.(11). Then, dropping $\boldsymbol{\nabla} \cdot \mathbf{E}$ we see that the electric field satisfies the wave equation,

$$-\frac{1}{c^2} \frac{\partial \mathbf{E}}{\partial t} + \nabla^2 \mathbf{E} = 0$$

Exercise: Show that the magnetic field satisfies the wave equation.

3.7 Electric and magnetic potentials

Two of Maxwell's equations–Gauss's law for magnetism, Eq.(7) and Faraday's law, Eq.(8),

$$\begin{aligned} \boldsymbol{\nabla}\cdot\mathbf{B} &= 0 \\ \boldsymbol{\nabla}\times\mathbf{E}+\frac{1}{c}\frac{\partial\mathbf{B}}{\partial t} &= 0 \end{aligned}$$

allow us to define potentials. We have seen that the divergence of a curl vanishes, and there is a converse: if the divergence of a vector field vanishes then (in a suitable region) the vector field may be written as the curl of another vector. Therefore, Gauss's law for magnetism implies the existence of another vector $\mathbf{A}$ such that

$$\mathbf{B} = \boldsymbol{\nabla}\times\mathbf{A}$$

The new vector $\mathbf{A}$ is called the vector potential. If we substitute this into Faraday's law, the entire law becomes a vanishing curl,

$$\begin{aligned} \boldsymbol{\nabla}\times\mathbf{E}+\frac{1}{c}\frac{\partial}{\partial t}(\boldsymbol{\nabla}\times\mathbf{A}) &= 0 \\ \boldsymbol{\nabla}\times\left(\mathbf{E}+\frac{1}{c}\frac{\partial\mathbf{A}}{\partial t}\right) &= 0 \end{aligned}$$

Recall that potential energy $V=-\int\mathbf{F}\cdot d\mathbf{x}$ exists when the integral for the work

$$W=\int\mathbf{F}\cdot d\mathbf{x}$$

is independent of path. Path independence means that if we go up any one path and back another we get zero, so that

$$\oint\mathbf{F}\cdot d\mathbf{x}=0$$

for any closed curve. By Stokes' theorem, this means that the curl of the force vanishes.

The situation is the same here. Because the curl of $\mathbf{E}+\frac{1}{c}\frac{\partial\mathbf{A}}{\partial t}$ vanishes, it may be written as the gradient of a potential,

$$\mathbf{E}+\frac{1}{c}\frac{\partial\mathbf{A}}{\partial t}=-\boldsymbol{\nabla}\phi$$

Therefore, given the scalar and vector potentials ϕ and $\mathbf{A}$, we may find both the electric and magnetic fields,

$$\begin{aligned} \mathbf{E} &= -\boldsymbol{\nabla}\phi-\frac{1}{c}\frac{\partial\mathbf{A}}{\partial t} \\ \mathbf{B} &= \boldsymbol{\nabla}\times\mathbf{A} \end{aligned} \tag{12}$$

3.8 Wave equations for the potentials

Now, if we use the source-free Gauss law, Eq.(10), the first of these becomes

$$\begin{aligned} 0 &= \boldsymbol{\nabla}\cdot\mathbf{E} \\ &= \boldsymbol{\nabla}\cdot\left(-\boldsymbol{\nabla}\phi-\frac{1}{c}\frac{\partial\mathbf{A}}{\partial t}\right) \\ &= -\nabla^2\phi-\frac{1}{c}\frac{\partial}{\partial t}(\boldsymbol{\nabla}\cdot\mathbf{A}) \end{aligned}$$

Reorganizing this into a wave equation by adding and subtracting a second time derivative, there is a bit extra:

$$\begin{aligned}
-\frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2}+\nabla^2\phi+\frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2}+\frac{1}{c}\frac{\partial}{\partial t}(\boldsymbol{\nabla}\cdot\mathbf{A}) &= 0\\
-\frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2}+\nabla^2\phi+\frac{1}{c}\frac{\partial}{\partial t}\left(\frac{1}{c}\frac{\partial\phi}{\partial t}+\boldsymbol{\nabla}\cdot\mathbf{A}\right) &= 0 \qquad (13)
\end{aligned}$$

Before dealing with the extra terms, we consider a second equation. Substituting the potentials for the fields in the source-free Ampère law,

$$\begin{aligned}
\boldsymbol{\nabla}\times\mathbf{B}-\frac{1}{c}\frac{\partial\mathbf{E}}{\partial t} &= 0\\
\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\mathbf{A})-\frac{1}{c}\frac{\partial}{\partial t}\left(-\boldsymbol{\nabla}\phi-\frac{1}{c}\frac{\partial\mathbf{A}}{\partial t}\right) &= 0\\
\boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\mathbf{A})-\nabla^2\mathbf{A}+\frac{1}{c}\boldsymbol{\nabla}\frac{\partial\phi}{\partial t}+\frac{1}{c^2}\frac{\partial^2\mathbf{A}}{\partial t^2} &= 0
\end{aligned}$$

Once again, we have a wave equation plus extra terms

$$-\frac{1}{c^2}\frac{\partial^2\mathbf{A}}{\partial t^2}+\nabla^2\mathbf{A}-\boldsymbol{\nabla}\left(\frac{1}{c}\frac{\partial\phi}{\partial t}+\boldsymbol{\nabla}\cdot\mathbf{A}\right) = 0 \qquad (14)$$

It would be very convenient if the extra term $\frac{1}{c}\frac{\partial\phi}{\partial t}+\boldsymbol{\nabla}\cdot\mathbf{A}$, which occurs in both wave equations, were to vanish. It turns out that we can require this, because there is some freedom in the choice of the scalar and potential. This freedom is called *gauge invariance*.

3.9 Gauge invariance

The specification of ϕ and $\mathbf{A}$ is not unique. Given the expressions for the fields, Eqs.(12),

$$\begin{aligned}
\mathbf{E} &= -\boldsymbol{\nabla}\phi-\frac{1}{c}\frac{\partial\mathbf{A}}{\partial t}\\
\mathbf{B} &= \boldsymbol{\nabla}\times\mathbf{A}
\end{aligned}$$

we notice that the magnetic field is unchanged if we add the gradient of any function to the vector potential, because the curl of a gradient vanishes:

$$\mathbf{B}=\boldsymbol{\nabla}\times\mathbf{A}=\boldsymbol{\nabla}\times(\mathbf{A}+\boldsymbol{\nabla}f)$$

Of course, such a change also changes the expression for the electric field,

$$\begin{aligned}
\mathbf{E}' &= -\boldsymbol{\nabla}\phi-\frac{1}{c}\frac{\partial(\mathbf{A}+\boldsymbol{\nabla}f)}{\partial t}\\
&= -\boldsymbol{\nabla}\phi-\frac{1}{c}\frac{\partial\mathbf{A}}{\partial t}-\boldsymbol{\nabla}\left(\frac{1}{c}\frac{\partial f}{\partial t}\right)
\end{aligned}$$

However, if we change ϕ at the same time by the time derivative of f, everything works out. That is, make two replacements at once:

$$\begin{aligned}
\mathbf{A} &\Rightarrow \tilde{\mathbf{A}}=\mathbf{A}+\boldsymbol{\nabla}f\\
\phi &\Rightarrow \tilde{\phi}=\phi-\frac{1}{c}\frac{\partial f}{\partial t} \qquad (15)
\end{aligned}$$

Together these leave the electric and magnetic fields unchanged. Eqs.(15) are called a gauge transformation.

Exercise: Substitute the gauge transformation, Eqs.(15) into the expressions for the fields, Eqs.(12), just to check that the fields are unchanged.

3.10 Back to the wave equation

Using our freedom to choose the function f in the gauge transformation, we now ask if it is possible to choose f so that

$$\frac{1}{c}\frac{\partial\tilde{\phi}}{\partial t}+\boldsymbol{\nabla}\cdot\tilde{\mathbf{A}}=0$$

Suppose we start with a potential and vector potential such that this expression does not vanish, but instead gives some function, g

$$\frac{1}{c}\frac{\partial\phi}{\partial t}+\boldsymbol{\nabla}\cdot\mathbf{A}=g$$

Now compute the same combination of derivatives in a different gauge choice,

$$\begin{aligned}\frac{1}{c}\frac{\partial\tilde{\phi}}{\partial t}+\boldsymbol{\nabla}\cdot\tilde{\mathbf{A}} &= \frac{1}{c}\frac{\partial}{\partial t}\left(\phi-\frac{1}{c}\frac{\partial f}{\partial t}\right)+\boldsymbol{\nabla}\cdot\left(\mathbf{A}+\boldsymbol{\nabla}f\right)\\ &= \frac{1}{c}\frac{\partial\phi}{\partial t}-\frac{1}{c^2}\frac{\partial^2 f}{\partial t^2}+\boldsymbol{\nabla}\cdot\mathbf{A}+\boldsymbol{\nabla}\cdot\boldsymbol{\nabla}f\\ &= \left(\frac{1}{c}\frac{\partial\phi}{\partial t}+\boldsymbol{\nabla}\cdot\mathbf{A}\right)-\frac{1}{c^2}\frac{\partial^2 f}{\partial t^2}+\nabla^2 f\\ &= g-\frac{1}{c^2}\frac{\partial^2 f}{\partial t^2}+\nabla^2 f\end{aligned}$$

We would like to choose f so that this new expression vanishes. But we *can*, because it just involves solving

$$-\frac{1}{c^2}\frac{\partial^2 f}{\partial t^2}+\nabla^2 f=-g$$

This is just another wave equation with g as a source. Once we solve this, we may make a gauge transformation from $(\phi,\mathbf{A})$ to $\left(\tilde{\phi},\tilde{\mathbf{A}}\right)$, and the new potentials will satisfy

$$\frac{1}{c}\frac{\partial\tilde{\phi}}{\partial t}+\boldsymbol{\nabla}\cdot\tilde{\mathbf{A}}=0 \tag{16}$$

This choice is called the Lorentz gauge.

With the new potentials satisfying the Lorentz gauge condition, the wave equations, Eqs.(13) and (14) reduce to

$$\begin{aligned}-\frac{1}{c^2}\frac{\partial^2\tilde{\phi}}{\partial t^2}+\nabla^2\tilde{\phi} &= 0\\ -\frac{1}{c^2}\frac{\partial^2\tilde{\mathbf{A}}}{\partial t^2}+\nabla^2\tilde{\mathbf{A}} &= 0\end{aligned}$$

Therefore, the potentials may be chosen to satisfy the wave equation.

3.11 Electromagnetic energy

In addition to conservation of charge, the Maxwell equations also imply conservation of energy and momentum. To identify these, begin with Faraday and Ampère's laws with vanishing current, $\mathbf{J}=0$,

$$\begin{aligned}\boldsymbol{\nabla}\times\mathbf{E}+\frac{1}{c}\frac{\partial\mathbf{B}}{\partial t} &= 0\\ \boldsymbol{\nabla}\times\mathbf{B}-\frac{1}{c}\frac{\partial\mathbf{E}}{\partial t} &= 0\end{aligned}$$

Consider the dot product of the electric field with Faraday's law,

$$\mathbf{E}\cdot(\boldsymbol{\nabla}\times\mathbf{B})-\frac{1}{c}\mathbf{E}\cdot\frac{\partial\mathbf{E}}{\partial t} = 0$$

and a similar expression from the dot product of the magnetic field with Ampère's law,

$$\mathbf{B}\cdot(\boldsymbol{\nabla}\times\mathbf{E})+\frac{1}{c}\mathbf{B}\cdot\frac{\partial\mathbf{B}}{\partial t} = 0$$

Subtracting,

$$\begin{aligned}\mathbf{B}\cdot(\boldsymbol{\nabla}\times\mathbf{E})+\frac{1}{c}\mathbf{B}\cdot\frac{\partial\mathbf{B}}{\partial t}-\left(\mathbf{E}\cdot(\boldsymbol{\nabla}\times\mathbf{B})-\frac{1}{c}\mathbf{E}\cdot\frac{\partial\mathbf{E}}{\partial t}\right) &= 0\\ \frac{1}{c}\mathbf{E}\cdot\frac{\partial\mathbf{E}}{\partial t}\frac{1}{c}\mathbf{B}\cdot\frac{\partial\mathbf{B}}{\partial t}+\mathbf{B}\cdot(\boldsymbol{\nabla}\times\mathbf{E})-\mathbf{E}\cdot(\boldsymbol{\nabla}\times\mathbf{B}) &= 0 \qquad (17)\end{aligned}$$

The first two terms may each be written as half the time derivative of $\mathbf{E}\cdot\mathbf{E}$ and $\mathbf{B}\cdot\mathbf{B}$,

$$\begin{aligned}\frac{1}{c}\mathbf{E}\cdot\frac{\partial\mathbf{E}}{\partial t} &= \frac{1}{2c}\frac{\partial}{\partial t}(\mathbf{E}\cdot\mathbf{E})\\ \frac{1}{c}\mathbf{B}\cdot\frac{\partial\mathbf{B}}{\partial t} &= \frac{1}{2c}\frac{\partial}{\partial t}(\mathbf{B}\cdot\mathbf{B})\end{aligned}$$

To simplify the remaining two terms, we need the triple product of three vectors, $\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C})$. The triple product is the volume of a parallelepiped: the cross product gives the area of the base and the dot product gives the height perpendicular to the base. Since we may regard any two of the vectors as defining the base, we may cyclically permute the order of the vectors,

$$\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C})=\mathbf{B}\cdot(\mathbf{C}\times\mathbf{A})=\mathbf{C}\cdot(\mathbf{A}\times\mathbf{B})$$

A similar result holds if one of the vectors is the gradient operator. Taking into account the sign because of the odd permutation, we can show that

$$\mathbf{B}\cdot(\boldsymbol{\nabla}\times\mathbf{E})-\mathbf{E}\cdot(\boldsymbol{\nabla}\times\mathbf{B})=\boldsymbol{\nabla}\cdot(\mathbf{E}\times\mathbf{B})$$

Combining these results, Eq.(17) simplifies to

$$\frac{1}{2c}\frac{\partial}{\partial t}\left(\mathbf{E}^2+\mathbf{B}^2\right)+\boldsymbol{\nabla}\cdot(\mathbf{E}\times\mathbf{B})=0$$

and identifying

$$\begin{aligned}u &\equiv \frac{1}{2c}\left(\mathbf{E}^2+\mathbf{B}^2\right)\\ \mathbf{S} &\equiv \mathbf{E}\times\mathbf{B}\end{aligned}$$

we recognize the continuity equation,

$$\frac{\partial u}{\partial t}+\boldsymbol{\nabla}\cdot\mathbf{S}=0$$

The density $u=\frac{1}{2c}\left(\mathbf{E}^2+\mathbf{B}^2\right)$ is the *energy density* of an electromagnetic wave, while the *Poynting vector*,

$$\mathbf{S}=\mathbf{E}\times\mathbf{B}$$

gives the energy flux (*joules per square meter per second*) of an electromagnetic wave. The Poynting vector points in the direction of propagation of the wave. The integral of the continuity equation shows that

$$\begin{aligned}\frac{dE}{dt} &= \frac{d}{dt}\int\limits_V u\,d^3x\\ &= -\int\limits_S \hat{\mathbf{n}}\cdot\mathbf{S}\,d^2x\end{aligned}$$

so that the total energy $E = \int_V u d^3x$ in any volume V changes only by the Poynting flux out across the boundary S of V.