Complex analysis

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1 Complex numbers

The real numbers may be extended by defining the imaginary unit,

$$i = \sqrt{-1}$$

The set of numbers of the form

$$z = x + iy$$

then form a field, with addition and multiplication defined by

$$\begin{aligned}
z_1 + z_2 &= (x_1 + iy_1) + (x_2 + iy_2) \\
&= (x_1 + x_2) + i(y_1 + y_2) \\
z_1 z_2 &= (x_1 + iy_1)(x_2 + iy_2) \\
&= x_1 x_2 + i x_1 y_2 + i y_1 x_2 + i^2 y_1 y_2 \\
&= (x_1 x_2 - y_1 y_2) + i(x_1 y_2 + x_2 y_1)
\end{aligned}$$

Clearly, there is exactly one complex number for each point of the plane. This gives rise to the polar representation

$$z = re^{i\varphi}$$

where $r = \sqrt{x^2 + y^2}$ and $\tan\varphi = \frac{y}{x}$. Expanding in a Taylor series it is straightforward to show that

$$e^{i\varphi} = \cos\varphi + i\sin\varphi$$

which includes the amusing Euler relation, $e^{i\pi} = -1$. In polar form, multiplying by a phase $e^{i\theta}$ simply rotates z, i.e.,

$$ze^{i\theta} = re^{i(\varphi+\theta)}$$

Any function of on the plane, $f(x, y)$ may equally well be written as a function $f(z, \bar{z})$ where $\bar{z}$ is the *complex conjugate* of z, found by replacing i by $-i$,

$$\bar{z} = x - iy$$

The squared norm of z is the product

$$z\bar{z} = x^2 + y^2$$

Using this, the inverse of any nonzero complex number is

$$\begin{aligned}
\frac{1}{z} &= \frac{\bar{z}}{z\bar{z}} \\
&= \frac{x - iy}{x^2 + y^2}
\end{aligned}$$

Every n-degree polynomial equation over the complex numbers,

$$a_n z^n + a_{n-1} z^{n-1} + \cdots + a_0 = 0$$

has exactly n complex solutions.

2 Cauchy Riemann equations

We would like to differentiate and integrate using complex numbers. However, the complex numbers lie in a plane, so there are two independent directions (and any linear combination of these) from which we may take limits. These different limits must agree.

Specifically, we define

$$\frac{df}{dz} = \lim_{\varepsilon \to 0} \frac{f(z+\varepsilon) - f(z)}{\varepsilon}$$

where ε is an arbitrary complex number. If we let $\varepsilon = (a+bi)\,\delta = w\delta$ and set

$$\begin{aligned} z &= x + iy \\ f(z) &= u(x,y) + iv(x,y) \end{aligned}$$

then we have

$$\begin{aligned} \frac{df}{dz} &= \lim_{\delta \to 0} \frac{u(x+a\delta, y+b\delta) + iv(x+a\delta, y+b\delta) - u(x,y) - iv(x,y)}{(a+bi)\,\delta} \\ &= \frac{1}{w} \lim_{\delta \to 0} \frac{u(x+a\delta, y+b\delta) + iv(x+a\delta, y+b\delta) - u(x,y) - iv(x,y)}{\delta} \end{aligned}$$

To take these limits, add and subtract a term. For the real part,

$$\begin{aligned} \lim_{\delta \to 0} \frac{u(x+a\delta, y+b\delta) - u(x,y)}{\delta} &= \lim_{\delta \to 0} \frac{u(x+a\delta, y+b\delta) - u(x, y+b\delta) + u(x, y+b\delta) - u(x,y)}{\delta} \\ &= \lim_{\delta \to 0} \frac{u(x+a\delta, y+b\delta) - u(x, y+b\delta)}{\delta} + \lim_{\delta \to 0} \frac{u(x, y+b\delta) - u(x,y)}{\delta} \\ &= a \lim_{a\delta \to 0} \frac{u(x+a\delta, y+b\delta) - u(x, y+b\delta)}{a\delta} + b \lim_{b\delta \to 0} \frac{u(x, y+b\delta) - u(x,y)}{b\delta} \\ &= a \lim_{a\delta \to 0} \frac{\partial u(x, y+b\delta)}{\partial x} + b \frac{\partial u(x,y)}{\partial y} \\ &= a \frac{\partial u(x,y)}{\partial x} + b \frac{\partial u(x,y)}{\partial y} \end{aligned}$$

Similarly, the imaginary part becomes

$$\lim_{\delta \to 0} \frac{v(x+a\delta, y+b\delta) - v(x,y)}{\delta} = a \frac{\partial v(x,y)}{\partial x} + b \frac{\partial v(x,y)}{\partial y}$$

Therefore, if we define $w = a + bi$,

$$\frac{df}{dz} = \frac{1}{w} \left(a\frac{\partial u}{\partial x} + b\frac{\partial u}{\partial y} + ia\frac{\partial v}{\partial x} + ib\frac{\partial v}{\partial y} \right)$$

The derivative of f exists if and only if this holds for all $a$and b, which happens if and only if

$$a\frac{\partial u}{\partial x} + b\frac{\partial u}{\partial y} + ia\frac{\partial v}{\partial x} + ib\frac{\partial v}{\partial y}$$

is a multiple of w. This requires, for some functions g and h,

$$\begin{aligned} a\frac{\partial u}{\partial x} + b\frac{\partial u}{\partial y} + ia\frac{\partial v}{\partial x} + ib\frac{\partial v}{\partial y} &= (g+ih)(a+bi) \\ a\frac{\partial u}{\partial x} + b\frac{\partial u}{\partial y} + ia\frac{\partial v}{\partial x} + ib\frac{\partial v}{\partial y} &= ag - bh + iah + igb \end{aligned}$$

and therefore, since a and b are arbitrary,

$$\begin{aligned} a\frac{\partial u}{\partial x}+b\frac{\partial u}{\partial y} &= ag-bh \\ ia\frac{\partial v}{\partial x}+ib\frac{\partial v}{\partial y} &= iah+igb \end{aligned}$$

and therefore

$$\begin{aligned} g &= \frac{\partial u}{\partial x}=\frac{\partial v}{\partial y} \\ h &= -\frac{\partial u}{\partial y}=\frac{\partial v}{\partial x} \end{aligned}$$

The necessary and sufficient conditions for a well-defined derivative are therefore,

$$\begin{aligned} \frac{\partial u}{\partial y} &= -\frac{\partial v}{\partial x} \\ \frac{\partial v}{\partial y} &= \frac{\partial u}{\partial x} \end{aligned} \tag{1}$$

and these are called the *Cauchy-Riemann conditions*.

Now, we have several ways to write the derivative,

$$\begin{aligned} \frac{df}{dz} &= \frac{\partial u}{\partial x}-i\frac{\partial u}{\partial y} \\ &= \frac{\partial v}{\partial y}+i\frac{\partial v}{\partial x} \end{aligned}$$

2.1 Derivatives with respect to $\bar{z}$

Suppose we have a complex valued function of two variables, which we may write as either $f(x,y)$ or as $f(z,\bar{z})$. Consider the consequences of the Cauchy-Riemann conditions for the derivatives $\frac{df}{dz}$ and $\frac{df}{d\bar{z}}$.

We may write $f(z,\bar{z}) = u(z,\bar{z}) + iv(z,\bar{z})$ where u and v take only real values. Furthermore, since $z = x+iy$ and $\bar{z} = x-iy$ we have

$$\begin{aligned} x &= \frac{1}{2}(z+\bar{z}) \\ y &= \frac{1}{2i}(z-\bar{z}) \end{aligned}$$

so that with the chain rule we may evaluate

$$\begin{aligned} \frac{df}{dz} &= \frac{\partial f}{\partial x}\frac{\partial x}{\partial z}+\frac{\partial f}{\partial y}\frac{\partial y}{\partial z} \\ &= \frac{1}{2}\frac{\partial f}{\partial x}+\frac{1}{2i}\frac{\partial f}{\partial y} \\ &= \frac{1}{2}\frac{\partial u}{\partial x}+\frac{i}{2}\frac{\partial v}{\partial x}+\frac{1}{2i}\frac{\partial u}{\partial y}+\frac{1}{2}\frac{\partial v}{\partial y} \end{aligned}$$

and

$$\begin{aligned} \frac{df}{d\bar{z}} &= \frac{\partial f}{\partial x}\frac{\partial x}{\partial \bar{z}}+\frac{\partial f}{\partial y}\frac{\partial y}{\partial \bar{z}} \\ &= \frac{1}{2}\frac{\partial f}{\partial x}-\frac{1}{2i}\frac{\partial f}{\partial y} \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2}\frac{\partial(u+iv)}{\partial x} - \frac{1}{2i}\frac{\partial(u+iv)}{\partial y} \\
&= \frac{1}{2}\frac{\partial u}{\partial x} + \frac{i}{2}\frac{\partial v}{\partial x} - \frac{1}{2i}\frac{\partial u}{\partial y} - \frac{1}{2}\frac{\partial v}{\partial y}
\end{aligned}
$$

Using the Cauchy-Riemann conditions Eqs.(1) to replace the y-derivatives these become

$$
\begin{aligned}
\frac{df}{dz} &= \frac{1}{2}\frac{\partial u}{\partial x} + \frac{i}{2}\frac{\partial v}{\partial x} + \frac{1}{2i}\frac{\partial u}{\partial y} + \frac{1}{2}\frac{\partial v}{\partial y} \\
&= \frac{1}{2}\frac{\partial u}{\partial x} + \frac{i}{2}\frac{\partial v}{\partial x} - \frac{1}{2i}\frac{\partial v}{\partial x} + \frac{1}{2}\frac{\partial u}{\partial x} \\
&= \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} \\
&= \frac{\partial}{\partial x}(u+iv)
\end{aligned}
$$

and

$$
\begin{aligned}
\frac{\partial u}{\partial y} &= -\frac{\partial v}{\partial x} \\
\frac{\partial v}{\partial y} &= \frac{\partial u}{\partial x}
\end{aligned}
$$

$$
\begin{aligned}
\frac{df}{d\bar{z}} &= \frac{1}{2}\frac{\partial u}{\partial x} + \frac{i}{2}\frac{\partial v}{\partial x} - \frac{1}{2i}\frac{\partial u}{\partial y} - \frac{1}{2}\frac{\partial v}{\partial y} \\
&= \frac{1}{2}\frac{\partial u}{\partial x} + \frac{i}{2}\frac{\partial v}{\partial x} + \frac{1}{2i}\frac{\partial v}{\partial x} - \frac{1}{2}\frac{\partial u}{\partial x} \\
&= 0
\end{aligned}
$$

Essentially, a differentiable complex function depends in either coordinate system on only one of the variables. In particular, a function $f(z,\bar{z})$ is differentiable if it is independent of $\bar{z}$, $f(z,\bar{z}) = f(z)$.

3 Higher derivatives

Suppose F is the complex derivative of $f(z)$,

$$
\begin{aligned}
F(x) &= \frac{df}{dz} \\
&= \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y} \\
&= \frac{\partial v}{\partial y} + i\frac{\partial v}{\partial x}
\end{aligned}
$$

With $F = U + iV$ we have

$$
\begin{aligned}
U &= \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \\
V &= -\frac{\partial u}{\partial y} = \frac{\partial v}{\partial x}
\end{aligned}
$$

Now consider the Cauchy-Riemann conditions for the existence of $\frac{dF}{dz} = \frac{d^2 f}{dz^2}$,

$$
\begin{aligned}
\frac{\partial U}{\partial x} &= \frac{\partial V}{\partial y} \\
\frac{\partial U}{\partial y} &= -\frac{\partial V}{\partial x}
\end{aligned}
$$

Substituting for U and V in the first condition,

$$\begin{aligned}
\frac{\partial U}{\partial x} &= \frac{\partial V}{\partial y} \\
\frac{\partial^2 v}{\partial x \partial y} &= \frac{\partial^2 v}{\partial y \partial x}
\end{aligned}$$

and in the second,

$$\begin{aligned}
\frac{\partial U}{\partial y} &= -\frac{\partial V}{\partial x} \\
\frac{\partial^2 u}{\partial y \partial x} &= \frac{\partial^2 u}{\partial x \partial y}
\end{aligned}$$

we see that the Cauchy-Riemann conditions for F are identically satisfied by the equality of mixed partials. Therefore, the second derivative of $f(z)$ exists whenever the first derivative exists, provided only that the separate real or imaginary parts, u, v are similarly differentiable. Since we may repeat this arguement ad infinitum, $f(z)$ satisfies the Cauchy-Riemann conditions at all orders if and only if u and v are C^∞ functions, in which case, all derivatives of $f(z)$ exist.

4 Analytic extension

Consider any real values function with a convergent Taylor series for all x,

$$f(x) = \sum_{n=0}^{\infty} \frac{1}{n!} a_n x^n$$

Then we define the *analytic extension* of f to be the complex valued function

$$f(z) = \sum_{n=0}^{\infty} \frac{1}{n!} a_n z^n$$

To see that this exists for all z, write $z = re^{i\phi}$ and $e^{in\phi} = \cos n\phi + i \sin n\phi$, then expand

$$\begin{aligned}
f(z) &= \sum_{n=0}^{\infty} \frac{1}{n!} a_n r^n e^{in\phi} \\
&= \sum_{n=0}^{\infty} \frac{1}{n!} a_n r^n (\cos n\phi + i \sin n\phi)
\end{aligned}$$

Now, each of the series,

$$\begin{aligned}
u(r, \phi) &= \sum_{n=0}^{\infty} \frac{1}{n!} a_n r^n \cos n\phi \\
v(r, \phi) &= \sum_{n=0}^{\infty} \frac{1}{n!} a_n r^n \sin n\phi
\end{aligned}$$

converges since,

$$u(r, \phi) = \sum_{n=0}^{\infty} \frac{1}{n!} a_n r^n \cos n\phi$$

$$
\begin{aligned}
&\leq \sum_{n=0}^{\infty} \frac{1}{n!} a_n r^n \left|\cos n\phi\right| \\
&\leq \sum_{n=0}^{\infty} \frac{1}{n!} a_n r^n = f\left(r\right) \\
v\left(r,\phi\right) &\leq \sum_{n=0}^{\infty} \frac{1}{n!} a_n r^n \left|\sin n\phi\right| \\
&\leq f\left(r\right)
\end{aligned}
$$

and therefore both converge.

Furthermore, $f\left(z\right)$ satisfies the Cauchy-Riemann conditions. To check, we first observe that with $f\left(z\right) = u\left(r,\phi\right) + iv\left(r,\phi\right)$ in polar coordinates we can immediately write the real and imaginary parts,

$$
\begin{aligned}
u &= \sum_{n=0}^{\infty} \frac{1}{n!} a_n r^n \cos n\phi \\
v &= \sum_{n=0}^{\infty} \frac{1}{n!} a_n r^n \sin n\phi
\end{aligned}
$$

To perform the derivatives we need to express the derivatives in polar coordinates. Using the chain rule,

$$
\begin{aligned}
\frac{\partial}{\partial x} &= \frac{\partial r}{\partial x}\frac{\partial}{\partial r} + \frac{\partial \phi}{\partial x}\frac{\partial}{\partial \phi} \\
\frac{\partial}{\partial y} &= \frac{\partial r}{\partial y}\frac{\partial}{\partial r} + \frac{\partial \phi}{\partial y}\frac{\partial}{\partial \phi} \qquad (2)
\end{aligned}
$$

We can find the partials we need from the coordinate transformation,

$$
\begin{aligned}
\tan\phi &= \frac{y}{x} \\
\frac{1}{\cos^2\phi} d\phi &= \frac{xdy - ydx}{x^2} \\
\frac{x^2}{\cos^2\phi} d\phi &= xdy - ydx \\
r^2 d\phi &= xdy - ydx \\
d\phi &= \frac{x}{r^2} dy - \frac{y}{r^2} dx \\
d\phi &= \frac{\cos\phi}{r} dy - \frac{\sin\phi}{r} dx
\end{aligned}
$$

and

$$
\begin{aligned}
r &= \sqrt{x^2+y^2} \\
dr &= \frac{1}{2\sqrt{x^2+y^2}}\left(2xdx + 2ydy\right) \\
&= \frac{1}{r}\left(r\cos\phi\, dx + r\sin\phi\, dy\right) \\
&= \cos\phi\, dx + \sin\phi\, dy
\end{aligned}
$$

so we see that

$$
\begin{aligned}
\frac{\partial \phi}{\partial x} &= -\frac{\sin\phi}{r} \\
\frac{\partial \phi}{\partial y} &= \frac{\cos\phi}{r}
\end{aligned}
$$

and

$$\begin{aligned}\frac{\partial r}{\partial x} &= \cos\phi \\ \frac{\partial r}{\partial y} &= \sin\phi\end{aligned}$$

Therefore, substituting the partials into Eqs.(2),

$$\begin{aligned}\frac{\partial}{\partial x} &= \cos\phi\frac{\partial}{\partial r} - \frac{1}{r}\sin\phi\frac{\partial}{\partial\phi} \\ \frac{\partial}{\partial y} &= \sin\phi\frac{\partial}{\partial r} + \frac{1}{r}\cos\phi\frac{\partial}{\partial\phi}\end{aligned}$$

and the partial derivatives required for the first Cauchy-Riemann conditions become

$$\begin{aligned}\frac{\partial u}{\partial x} &= \cos\phi\frac{\partial u}{\partial r} - \frac{1}{r}\sin\phi\frac{\partial u}{\partial\phi} \\ &= \left(\cos\phi\frac{\partial}{\partial r} - \frac{1}{r}\sin\phi\frac{\partial}{\partial\phi}\right)\left(\sum_{n=0}^{\infty}\frac{1}{n!}a_n r^n\cos n\phi\right) \\ &= \sum_{n=0}^{\infty}\frac{1}{n!}a_n\left(\cos\phi\frac{\partial}{\partial r} - \frac{1}{r}\sin\phi\frac{\partial}{\partial\phi}\right)r^n\cos n\phi \\ &= \sum_{n=0}^{\infty}\frac{1}{n!}a_n\left(nr^{n-1}\cos\phi\cos n\phi + nr^{n-1}\sin\phi\sin n\phi\right) \\ &= \sum_{n=0}^{\infty}\frac{1}{(n-1)!}a_n r^{n-1}\cos(n-1)\phi\end{aligned}$$

while

$$\begin{aligned}\frac{\partial v}{\partial y} &= \sin\phi\frac{\partial v}{\partial r} + \frac{1}{r}\cos\phi\frac{\partial v}{\partial\phi} \\ &= \left(\sin\phi\frac{\partial}{\partial r} + \frac{1}{r}\cos\phi\frac{\partial}{\partial\phi}\right)\left(\sum_{n=0}^{\infty}\frac{1}{n!}a_n r^n\sin n\phi\right) \\ &= \sum_{n=0}^{\infty}\frac{1}{n!}a_n\left(nr^{n-1}\sin\phi\sin n\phi + nr^{n-1}\cos\phi\cos n\phi\right) \\ &= \sum_{n=0}^{\infty}\frac{1}{(n-1)!}a_n r^{n-1}\cos(n-1)\phi\end{aligned}$$

so that

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$

is identically satisfied.

For the second condition, we need

$$\begin{aligned}\frac{\partial u}{\partial y} &= \sin\phi\frac{\partial u}{\partial r} + \frac{1}{r}\cos\phi\frac{\partial u}{\partial\phi} \\ &= \left(\cos\phi\frac{\partial}{\partial r} - \frac{1}{r}\sin\phi\frac{\partial}{\partial\phi}\right)\left(\sum_{n=0}^{\infty}\frac{1}{n!}a_n r^n\sin n\phi\right)\end{aligned}$$

$$
\begin{aligned}
&= \sum_{n=0}^{\infty} \frac{1}{n!} a_n n r^{n-1} (\cos\phi \sin n\phi - \sin\phi \cos n\phi) \\
\frac{\partial v}{\partial x} &= \cos\phi \frac{\partial v}{\partial r} - \frac{1}{r} \sin\phi \frac{\partial v}{\partial \phi} \\
&= \left(\sin\phi \frac{\partial}{\partial r} + \frac{1}{r} \cos\phi \frac{\partial}{\partial \phi} \right) \left(\sum_{n=0}^{\infty} \frac{1}{n!} a_n r^n \cos n\phi \right) \\
&= \sum_{n=0}^{\infty} \frac{1}{n!} a_n n r^{n-1} (\sin\phi \cos n\phi - \cos\phi \sin n\phi)
\end{aligned}
$$

and these are negatives of one another

$$
\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}
$$

as required for analyticity.

Therefore, analytic extension of any real Taylor series gives an analytic function $f(z)$. Conversely, any complex Taylor series gives an analytic function.

Exercise: Show that the composition of two analytic functions is analytic. That is, if $f(z)$ and $g(w)$ both satisfy the Cauchy-Riemann conditions, show that $g(f(z))$ also satisfies the Cauchy-Riemann conditions.

5 Contour Integrals

5.1 Integral of an analytic function using the Cauchy-Riemann conditions

Now consider a function $f(z)$ with derivatives of all orders in some region of the complex plane, and consider the integral of $f(z)$ around a closed curve, C,

$$
\oint_C f(z)\, dz
$$

We may expand this as a pair of functions of two variables,

$$
\begin{aligned}
\oint_C f(z)\, dz &= \oint_C (u(x,y) + iv(x,y))(dx + idy) \\
&= \oint_C (udx + iudy + ivdx - vdy) \\
&= \oint_C (udx - vdy) + i \oint_C (udy + vdx)
\end{aligned}
$$

The final integrals are expressed as real integrals along curves in the plane. Define a vector field in R^3

$$
\vec{\mathbf{u}} := (u(x,y), -v(x,y), 0)
$$

and write $d\vec{\mathbf{x}} = (dx, dy, dz)$ as an infinitesimal vector displacement. Then we can use Stoke's theorem. The first integral becomes

$$
\begin{aligned}
\oint_C (udx - vdy) &= \oint_C \vec{\mathbf{u}} \cdot d\vec{\mathbf{x}} \\
&= \iint_S (\nabla \times \vec{\mathbf{u}}) \cdot \hat{\mathbf{n}} d^2 x
\end{aligned}
$$

where S is any region bounded by C and the normal $\hat{\mathbf{n}}$ is in the z-direction. The z-component of the curl of $\vec{\mathbf{u}}$, however, is

$$\begin{aligned}\hat{\mathbf{k}}\cdot(\nabla\times\vec{\mathbf{u}}) &= \frac{\partial u_x}{\partial y}-\frac{\partial u_y}{\partial x}\\ &= \frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\\ &= 0\end{aligned}$$

where the result vanishes by the Cauchy-Riemann conditions. Therefore,

$$\oint\limits_C (udx - vdy) = 0$$

For the second integral, let $\vec{\mathbf{w}} = (v, u, 0)$ be a vector field, so that

$$\begin{aligned}\oint\limits_C (udy + vdx) &= \oint\limits_C \vec{\mathbf{w}}\cdot d\vec{\mathbf{x}}\\ &= \iint\limits_S (\nabla\times\vec{\mathbf{w}})\cdot\hat{\mathbf{n}}d^2x\end{aligned}$$

Again we need the z-component of the curl, which is

$$\begin{aligned}\hat{\mathbf{k}}\cdot(\nabla\times\vec{\mathbf{w}}) &= \frac{\partial w_x}{\partial y}-\frac{\partial w_y}{\partial x}\\ &= \frac{\partial v}{\partial y}-\frac{\partial u}{\partial x}\\ &= 0\end{aligned}$$

using the second Riemann-Cauchy condition. We conclude that $\oint_C (udy + vdx) = 0$ as well.

Therefore, for any analytic function f, we have

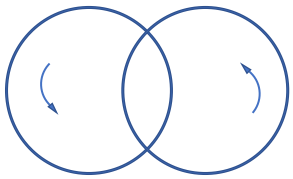
$$\oint\limits_C f(z)\, dz = 0$$

around any closed curve, C.

5.2 Deforming the curve

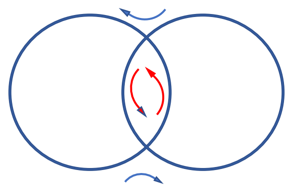
If two closed curves share a common segment, then we can add the curves together to get a larger curve equal to the outer boundary of both curves. Starting with a given curve, we can therefore imagine adding a small second loop in such a way that the combined contour is slightly altered from the first. This is called a deformation of the contour, and it will not change the value of the integral as long as the small loop we add lies entirely within a region where f is analytic. In any analytic region we may therefore deform the path of integration in any way we like without changing the value of the integral.

For the alternative proof where we have a small circle with vanishing integral, we can imagine a pair of such circles centered on nearby points. The overlap region is surrounded by an interior closed curve with vanishing integral, and we may disregard it in evaluating the net contour integral. In this way, we may build up contours around finite regions.

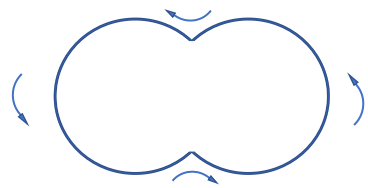


Two overlapping circles, each integrated clockwise

The central portion makes no contribution, since integrating up one side and down the other forms a closed curve.



The net integral is around the outer curve.



6 The Residue Theorem

We can use this result to simplify integrals where the function is not analytic in the entire complex plane. Suppose a function is analytic everywhere except a single point, z_0. Then in addition to a Taylor series for the function, there may be a *Laurent expansion* which includes poles at z_0,

$$\frac{1}{(z-z_0)^n}$$

Such terms are fine away from the point z_0, so they do not affect analticity elsewhere. Consider the class of functions which have a Laurent series, i.e., for some finite number N, the function may be expressed as

$$f(z) = \sum_{n=-N}^{\infty} a_n (z-z_0)^n$$

This has poles of orders $1, 2, \ldots, N$. Since the mapping $w = g(z) = z - z_0$ is analytic, we might as well write this as

$$f(w) = \sum_{n=-N}^{\infty} a_n w^n$$

where the poles are now at $w = 0$. Now consider a contour integral of the form

$$\oint\limits_C f(w)\, dw$$

for any closed curve C. Since f is analytic everywhere except the origin, the integral vanishes if C does not enclose the origin – we may deform the curve down to a single point. If C does include the origin, we may deform C until it is a circle of radius R about the origin, and the deformation does not affect the value of the integral. Then on the circle

$$dw = d\left(Re^{i\phi}\right) = iRe^{i\phi}d\phi$$

so we have

$$\begin{aligned}
\oint_C f(w)\,dw &= \int_0^{2\pi} \sum_{n=-N}^{\infty} a_n R^n e^{in\phi}\left(iRe^{i\phi}d\phi\right) \\
&= i\sum_{n=-N}^{\infty} a_n R^{n+1} \int_0^{2\pi} e^{i(n+1)\phi}d\phi \\
&= i\sum_{n\neq -1}^{\infty} a_n R^{n+1} \left.\frac{e^{i(n+1)\phi}}{i(n+1)}\right|_0^{2\pi} + ia_{-1}\int_0^{2\pi} d\phi \\
&= i\sum_{n=-N}^{\infty} a_n R^n \frac{1}{i(n+1)}\left(e^{2(n+1)\pi i}-1\right) + ia_{-1}\int_0^{2\pi} d\phi \\
&= 2\pi i a_{-1}
\end{aligned}$$

We see that the integral depends only on the coefficient of the *simple pole* (i.e., the pole of order 1). This coefficient is called the *residue* of f at z_0, and we write

$$\begin{aligned}
Res\left(f(z)\right) &= Res\left(\sum_{n=-N}^{\infty} a_n (z-z_0)^n\right) \\
&= a_{-1}
\end{aligned}$$

The residue theorem now states that the integral of a complex function about a pole equals $2\pi i$ times the residue of the function at the pole. If there are multiple poles, the result is the sum of the residues at all poles included within the contour C. Thus, the residue theorem becomes

$$\oint_C f(w)\,dw = 2\pi i \sum Res(f)$$

where the sum is over all poles included within C.

6.1 Example: Completeness relation for Fourier integrals

Suppose we can expand a function $f(\mathbf{x})$ as

$$f(\mathbf{x}) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(\mathbf{k})\,e^{-i\mathbf{k}\cdot\mathbf{x}}d^3k$$

We would like to show that this transformation is invertible, and this requires the completeness relation for Fourier transformations. To see this, consider inverting the transformation. Multiply both sides by $e^{i\mathbf{k}'\cdot\mathbf{x}}$ and integrate over all $\mathbf{x}$,

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(\mathbf{x})\,e^{i\mathbf{k}'\cdot\mathbf{x}}d^3x = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\mathbf{k}'\cdot\mathbf{x}}d^3x \int_{-\infty}^{\infty} g(\mathbf{k})\,e^{-i\mathbf{k}\cdot\mathbf{x}}d^3k$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(\mathbf{k})\, e^{-i\left(\mathbf{k}-\mathbf{k}'\right)\cdot\mathbf{x}} d^3k\, d^3x$$
$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} g(\mathbf{k})\, d^3k \int_{-\infty}^{\infty} e^{-i\left(\mathbf{k}-\mathbf{k}'\right)\cdot\mathbf{x}}\, d^3x$$

We desire the result of this integration to be the transform, $g(\mathbf{k})$, and this will be true if and only if

$$\delta^3(\mathbf{k}-\mathbf{k}') = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\left(\mathbf{k}-\mathbf{k}'\right)\cdot\mathbf{x}}\, d^3x$$

or equivalently

$$\delta^3(\mathbf{k}) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\mathbf{k}\cdot\mathbf{x}}\, d^3x$$

Using Cartesian coordinates, this breaks into three identical integrals of the form

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ikx}\, dx$$

which we may use contour integration to evaluate.

Our goal is to show that this integral is a Dirac delta function, which means that for any test function $g(k)$ (i.e., $g(k)$ is bounded, as differentiable as we like, and vanishes outside a compact region),

$$g(0) = \int_{-\infty}^{\infty} g(k)\left[\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ikx}\, dx\right] dk$$

Replace the infinite limit on the inner integral by R. We will let $R \to \infty$ at the end of the calculation. Then, carrying out the integral of the exponential,

$$\lim_{R\to\infty}\int_{-\infty}^{\infty} g(k)\left[\frac{1}{2\pi}\int_{-R}^{R} e^{-ikx}\, dx\right] dk = \lim_{R\to\infty}\int_{-\infty}^{\infty} g(k)\left[-\frac{1}{2\pi ik}\left(e^{-ikR}-e^{ikR}\right)\right] dk$$

We can carry this out using contour integration.

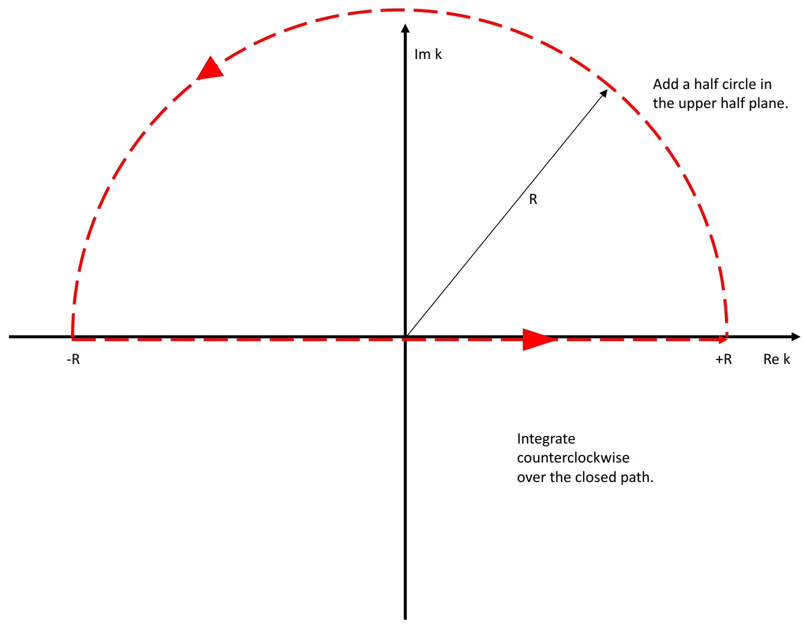
In order to use contour integration, we need to enclose the simple pole at $k=0$ with a curve. There are two problems. First, the pole here lies directly on the path of integration. We solve this difficulty with a trick: displace the pole slightly, then do the integral, then take the limit as the displacement vanishes. Specifically, let ε be an arbitrary positive real number and write the integral as

$$\begin{aligned}\int_{-\infty}^{\infty} g(k)\left[-\frac{1}{2\pi ik}\left(e^{-ikR}-e^{ikR}\right)\right] dk &= \lim_{\varepsilon\to 0}\int_{-\infty}^{\infty} g(k)\left[\frac{1}{2\pi i(k-i\varepsilon)}\left(e^{ikR}-e^{-ikR}\right)\right] dk \\ &= \lim_{\varepsilon\to 0}\int_{-\infty}^{\infty}\frac{g(k)\, e^{ikR}}{2\pi i(k-i\varepsilon)} dk - \lim_{\varepsilon\to 0}\int_{-\infty}^{\infty}\frac{g(k)\, e^{-ikR}}{2\pi i(k-i\varepsilon)} dk\end{aligned}$$

The second problem is to complete a closed curve without changing the value of the integral. We begin by analytically extending the integration variable k to a complex variable, $k = k_R + ik_I$. The first integral is then

$$\lim_{\varepsilon\to 0}\int_{-\infty}^{\infty}\frac{g(k)\, e^{ikR}}{2\pi i(k-i\varepsilon)} dk = \lim_{\varepsilon\to 0}\int_{-\infty}^{\infty}\frac{g(k_R+ik_I)\, e^{ik_RR}e^{-k_IR}}{2\pi i(k_R+ik_I-i\varepsilon)} dk$$

and we see that if $k_I > 0$ the integrand is suppressed by $e^{-k_I R}$. If we close the contour by adding a semicircle of radius R in the upper half plane:



At any angle ϕ on the semicircle the imaginary component k_I is given by $k_I = R\sin\phi$. As the radius tends to infinity, $R \to \infty$, so $k_I R$ diverges and the exponential factor $e^{-k_I R}$ tends to zero. This means that the integrand vanishes on this upper semicircle and we can integrate over a closed curve C which runs along the entire real k axis and returns on the semicircle, without changing the value of the integral,

$$\lim_{\varepsilon\to 0}\lim_{R\to\infty}\int_{-\infty}^{\infty}\frac{g\left(k_R + ik_I\right)e^{i(k_R+ik_I)R}}{2\pi i\left(k_R + ik_I - i\varepsilon\right)} = \lim_{\varepsilon\to 0}\lim_{R\to\infty}\oint_C \frac{g\left(k\right)e^{ikR}}{2\pi i\left(k - i\varepsilon\right)}dk$$

We may now apply the Residue Theorem. The contour is integrated in the positive sense, i.e., counterclockwise, and encloses the simple pole at $k = i\varepsilon$, so the residue is taken there

$$\begin{aligned}
\lim_{\varepsilon\to 0}\oint_C \frac{g\left(k\right)e^{ikR}}{2\pi i\left(k - i\varepsilon\right)}dk &= \lim_{\varepsilon\to 0} 2\pi i Res\left(\frac{g\left(k\right)e^{ikR}}{2\pi i\left(k - i\varepsilon\right)}\right) \\
&= \lim_{\varepsilon\to 0} 2\pi i\left(\frac{g\left(k\right)e^{ikR}}{2\pi i}\right)\Bigg|_{k=i\varepsilon} \\
&= \lim_{\varepsilon\to 0} 2\pi i\left(\frac{g\left(i\varepsilon\right)e^{\varepsilon R}}{2\pi i}\right) \\
&= \lim_{\varepsilon\to 0}\left(g\left(i\varepsilon\right)e^{-\varepsilon R}\right) \\
&= g\left(0\right)
\end{aligned}$$

The second integral is handled in the same way,

$$\lim_{\varepsilon\to 0}\int_{-\infty}^{\infty}\frac{g\left(k\right)e^{-ikR}}{2\pi i\left(k - i\varepsilon\right)}dk = \lim_{\varepsilon\to 0}\int_{-\infty}^{\infty}\frac{g\left(k_R + ik_I\right)e^{-ik_R R}e^{+k_I R}}{2\pi i\left(k_R + ik_I - i\varepsilon\right)}$$

There is one important difference. The exponential factor is now $e^{k_I R}$, which converges only when $k_I < 0$. This means that we must close the contour, C', in the lower half plane. We now have a clockwise contour, running along the entire real k axis then circling back along a semicircle in the lower half plane. We pick up a minus sign because of the direction of the contour, but more importantly, the shifted pole no longer lies inside the contour. Since the integrand lies in a region containing no poles it is analytic and the second integral vanishes.

Returning to the original problem, we have

$$\begin{aligned}
\lim_{R\to\infty} \int_{-\infty}^{\infty} g(k)\left[-\frac{1}{2\pi i k}\left(e^{-ikR} - e^{ikR}\right)\right] dk &= \lim_{R\to\infty} \int_{-\infty}^{\infty} g(k)\,\frac{1}{2\pi i k} e^{ikR} dk - \lim_{R\to\infty} \int_{-\infty}^{\infty} g(k)\,\frac{1}{2\pi i k} e^{-ikR} dk \\
&= \lim_{R\to\infty} g(0) \\
&= g(0)
\end{aligned}$$

and we have established that

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ikx}\, dx = \delta(k)$$

This shows the completeness of Fourier integrals.