

Complex analysis

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1 Complex numbers

The real numbers may be extended by defining the imaginary unit,

$$i = \sqrt{-1}$$

The set of numbers of the form

$$z = x + iy$$

then form a field, with addition and multiplication defined by

$$\begin{aligned} z_1 + z_2 &= (x_1 + iy_1) + (x_2 + iy_2) \\ &= (x_1 + x_2) + i(y_1 + y_2) \\ z_1 z_2 &= (x_1 + iy_1)(x_2 + iy_2) \\ &= x_1 x_2 + ix_1 y_2 + iy_1 x_2 + i^2 y_1 y_2 \\ &= (x_1 x_2 - y_1 y_2) + i(x_1 y_2 + x_2 y_1) \end{aligned}$$

Clearly, there is exactly one complex number for each point of the plane. This gives rise to the polar representation

$$z = r e^{i\varphi}$$

where $r = \sqrt{x^2 + y^2}$ and $\tan \varphi = \frac{y}{x}$. Expanding in a Taylor series it is straightforward to show that

$$e^{i\varphi} = \cos \varphi + i \sin \varphi$$

which includes the amusing Euler relation, $e^{i\pi} = -1$. In polar form, multiplying by a phase $e^{i\theta}$ simply rotates z , i.e.,

$$z e^{i\theta} = r e^{i(\varphi+\theta)}$$

Any function of on the plane, $f(x, y)$ may equally well be written as a function $f(z, \bar{z})$ where \bar{z} is the *complex conjugate* of z , found by replacing i by $-i$,

$$\bar{z} = x - iy$$

The squared norm of z is the product

$$z \bar{z} = x^2 + y^2$$

Using this, the inverse of any nonzero complex number is

$$\begin{aligned} \frac{1}{z} &= \frac{\bar{z}}{z \bar{z}} \\ &= \frac{x - iy}{x^2 + y^2} \end{aligned}$$

Every n -degree polynomial equation over the complex numbers,

$$a_n z^n + a_{n-1} z^{n-1} + \cdots + a_0 = 0$$

has exactly n complex solutions.

2 Cauchy Riemann equations

We would like to differentiate and integrate using complex numbers. However, the complex numbers lie in a plane, so there are two independent directions (and any linear combination of these) from which we may take limits. These different limits must agree.

Specifically, we define

$$\frac{df}{dz} = \lim_{\varepsilon \rightarrow 0} \frac{f(z + \varepsilon) - f(z)}{\varepsilon}$$

where ε is an arbitrary complex number. If we let $\varepsilon = (a + bi)\delta$ and set

$$\begin{aligned} z &= x + iy \\ f(z) &= u(x, y) + iv(x, y) \end{aligned}$$

then we have

$$\begin{aligned} \frac{df}{dz} &= \lim_{\delta \rightarrow 0} \frac{u(x + a\delta, y + b\delta) + iv(x + a\delta, y + b\delta) - u(x, y) - iv(x, y)}{(a + bi)\delta} \\ &= \frac{1}{a^2 + b^2} \lim_{\delta \rightarrow 0} (a - bi) \frac{u(x + a\delta, y + b\delta) + iv(x + a\delta, y + b\delta) - u(x, y) - iv(x, y)}{\delta} \\ &= \frac{1}{a^2 + b^2} \lim_{\delta \rightarrow 0} \frac{au(x + a\delta, y + b\delta) + bv(x + a\delta, y + b\delta) - au(x, y) - bv(x, y)}{\delta} \\ &\quad + \frac{i}{a^2 + b^2} \lim_{\delta \rightarrow 0} \frac{-bu(x + a\delta, y + b\delta) + av(x + a\delta, y + b\delta) + bu(x, y) - av(x, y)}{\delta} \\ &= \frac{1}{a^2 + b^2} \left(a \lim_{\delta \rightarrow 0} \frac{u(x + a\delta, y + b\delta) - u(x, y)}{\delta} + b \lim_{\delta \rightarrow 0} \frac{v(x + a\delta, y + b\delta) - v(x, y)}{\delta} \right) \\ &\quad + \frac{i}{a^2 + b^2} \left(a \lim_{\delta \rightarrow 0} \frac{v(x + a\delta, y + b\delta) - v(x, y)}{\delta} - b \lim_{\delta \rightarrow 0} \frac{u(x + a\delta, y + b\delta) - u(x, y)}{\delta} \right) \end{aligned}$$

To take these limits, add and subtract a term,

$$\begin{aligned} \lim_{\delta \rightarrow 0} \frac{u(x + a\delta, y + b\delta) - u(x, y)}{\delta} &= \lim_{\delta \rightarrow 0} \frac{u(x + a\delta, y + b\delta) - u(x, y + b\delta) + u(x, y + b\delta) - u(x, y)}{\delta} \\ &= \lim_{\delta \rightarrow 0} \frac{u(x + a\delta, y + b\delta) - u(x, y + b\delta)}{\delta} + \lim_{\delta \rightarrow 0} \frac{u(x, y + b\delta) - u(x, y)}{\delta} \\ &= a \lim_{a\delta \rightarrow 0} \frac{u(x + a\delta, y + b\delta) - u(x, y + b\delta)}{a\delta} + b \lim_{b\delta \rightarrow 0} \frac{u(x, y + b\delta) - u(x, y)}{b\delta} \\ &= a \lim_{a\delta \rightarrow 0} \frac{\partial u(x, y + b\delta)}{\partial x} + b \frac{\partial u(x, y)}{\partial y} \\ &= a \frac{\partial u(x, y)}{\partial x} + b \frac{\partial u(x, y)}{\partial y} \end{aligned}$$

Therefore, if we define $w = a + bi$,

$$\begin{aligned} \frac{df}{dz} &= \frac{1}{a^2 + b^2} \left(a^2 \frac{\partial u}{\partial x} + ab \frac{\partial u}{\partial y} + ab \frac{\partial v}{\partial x} + b^2 \frac{\partial v}{\partial y} \right) \\ &\quad + \frac{i}{a^2 + b^2} \left(a^2 \frac{\partial v}{\partial x} + ab \frac{\partial v}{\partial y} - ab \frac{\partial u}{\partial x} - b^2 \frac{\partial u}{\partial y} \right) \\ &= \frac{1}{w\bar{w}} \left(\left(a\bar{w} \frac{\partial u}{\partial x} + ib\bar{w} \frac{\partial v}{\partial y} \right) + \left(b\bar{w} \frac{\partial u}{\partial y} + ia\bar{w} \frac{\partial v}{\partial x} \right) \right) \\ &= \frac{1}{w} \left(a \frac{\partial u}{\partial x} + b \frac{\partial u}{\partial y} + ia \frac{\partial v}{\partial x} + ib \frac{\partial v}{\partial y} \right) \end{aligned}$$

The derivative of f exists if and only if this holds for all a and b , which happens if and only if

$$a \frac{\partial u}{\partial x} + b \frac{\partial u}{\partial y} + ia \frac{\partial v}{\partial x} + ib \frac{\partial v}{\partial y}$$

is a multiple of w . This requires, for some functions g and h ,

$$\begin{aligned} a \frac{\partial u}{\partial x} + b \frac{\partial u}{\partial y} + ia \frac{\partial v}{\partial x} + ib \frac{\partial v}{\partial y} &= (g + ih)(a + bi) \\ a \frac{\partial u}{\partial x} + b \frac{\partial u}{\partial y} + ia \frac{\partial v}{\partial x} + ib \frac{\partial v}{\partial y} &= ag - bh + iah + igb \end{aligned}$$

and therefore,

$$\begin{aligned} a \frac{\partial u}{\partial x} + b \frac{\partial u}{\partial y} &= ag - bh \\ ia \frac{\partial v}{\partial x} + ib \frac{\partial v}{\partial y} &= iah + igb \end{aligned}$$

and therefore

$$\begin{aligned} g &= \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \\ h &= -\frac{\partial u}{\partial y} = \frac{\partial v}{\partial x} \end{aligned}$$

The necessary and sufficient conditions for a well-defined derivative are therefore,

$$\begin{aligned} \frac{\partial u}{\partial y} &= -\frac{\partial v}{\partial x} \\ \frac{\partial v}{\partial y} &= \frac{\partial u}{\partial x} \end{aligned}$$

and these are called the *Cauchy-Riemann conditions*.

Now, we have several ways to write the derivative,

$$\begin{aligned} \frac{df}{dz} &= \frac{\partial u}{\partial x} - i \frac{\partial u}{\partial y} \\ &= \frac{\partial v}{\partial y} + i \frac{\partial v}{\partial x} \end{aligned}$$

3 Higher derivatives

Suppose F is the complex derivative of $f(z)$,

$$\begin{aligned} F(x) &= \frac{df}{dz} \\ &= \frac{\partial u}{\partial x} - i \frac{\partial u}{\partial y} \\ &= \frac{\partial v}{\partial y} + i \frac{\partial v}{\partial x} \end{aligned}$$

With $F = U + iV$ we have

$$\begin{aligned} U &= \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \\ V &= -\frac{\partial u}{\partial y} = \frac{\partial v}{\partial x} \end{aligned}$$

Now consider the Cauchy-Riemann conditions for the existence of $\frac{dF}{dz} = \frac{d^2 f}{dz^2}$,

$$\begin{aligned}\frac{\partial U}{\partial x} &= \frac{\partial V}{\partial y} \\ \frac{\partial U}{\partial y} &= -\frac{\partial V}{\partial x}\end{aligned}$$

Substituting for U and V in the first,

$$\begin{aligned}\frac{\partial U}{\partial x} &= \frac{\partial V}{\partial y} \\ \frac{\partial^2 v}{\partial x \partial y} &= \frac{\partial^2 v}{\partial y \partial x}\end{aligned}$$

and in the second,

$$\begin{aligned}\frac{\partial U}{\partial y} &= -\frac{\partial V}{\partial x} \\ \frac{\partial^2 u}{\partial y \partial x} &= \frac{\partial^2 u}{\partial x \partial y}\end{aligned}$$

we see that the Cauchy-Riemann conditions for F are identically satisfied by the equality of mixed partials. Therefore, the second derivative of $f(z)$ exists whenever the first derivative exists, provided only that the separate real or imaginary parts, u, v are similarly differentiable. Since we may repeat this argument ad infinitum, $f(z)$ satisfies the Cauchy-Riemann conditions at all orders if and only if u and v are C^∞ functions, in which case, all derivatives of $f(z)$ exist.

4 Analytic extension

Consider any real values function with a convergent Taylor series for all x ,

$$f(x) = \sum_{n=0}^{\infty} \frac{1}{n!} a_n x^n$$

Then we define the *analytic extension* of f to be the complex valued function

$$f(z) = \sum_{n=0}^{\infty} \frac{1}{n!} a_n z^n$$

This exists for all z .

Furthermore, $f(z)$ we show that satisfies the Cauchy-Riemann conditions, since with $f(z) = u(r, \phi) + iv(r, \phi)$

$$\begin{aligned}u &= \sum_{n=0}^{\infty} \frac{1}{n!} a_n r^n \cos n\phi \\ v &= \sum_{n=0}^{\infty} \frac{1}{n!} a_n r^n \sin n\phi\end{aligned}$$

In polar coordinates,

$$\begin{aligned}\frac{\partial}{\partial x} &= \frac{\partial r}{\partial x} \frac{\partial}{\partial r} + \frac{\partial \phi}{\partial x} \frac{\partial}{\partial \phi} \\ &= \frac{x}{r} \frac{\partial}{\partial r} + \frac{\partial \phi}{\partial x} \frac{\partial}{\partial \phi}\end{aligned}$$

We can find the partials from the coordinate transformation,

$$\begin{aligned}\tan \phi &= \frac{y}{x} \\ \frac{1}{\cos^2 \phi} d\phi &= \frac{xdy - ydx}{x^2} \\ \frac{x^2}{\cos^2 \phi} d\phi &= xdy - ydx \\ r^2 d\phi &= xdy - ydx \\ d\phi &= \frac{x}{r^2} dy - \frac{y}{r^2} dx \\ d\phi &= \frac{\cos \phi}{r} dy - \frac{\sin \phi}{r} dx\end{aligned}$$

so we see that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= -\frac{\sin \phi}{r} \\ \frac{\partial \phi}{\partial y} &= \frac{\cos \phi}{r}\end{aligned}$$

Therefore,

$$\begin{aligned}\frac{\partial}{\partial x} &= \cos \phi \frac{\partial}{\partial r} - \frac{1}{r} \sin \phi \frac{\partial}{\partial \phi} \\ \frac{\partial}{\partial y} &= \sin \phi \frac{\partial}{\partial r} + \frac{1}{r} \cos \phi \frac{\partial}{\partial \phi}\end{aligned}$$

and the Cauchy-Riemann conditions become

$$\begin{aligned}\frac{\partial u}{\partial x} &= \frac{\partial v}{\partial y} \\ \cos \phi \frac{\partial u}{\partial r} - \frac{1}{r} \sin \phi \frac{\partial u}{\partial \phi} &= \sin \phi \frac{\partial v}{\partial r} + \frac{1}{r} \cos \phi \frac{\partial v}{\partial \phi}\end{aligned}$$

and

$$\begin{aligned}\frac{\partial u}{\partial y} &= -\frac{\partial v}{\partial x} \\ \sin \phi \frac{\partial u}{\partial r} + \frac{1}{r} \cos \phi \frac{\partial u}{\partial \phi} &= -\cos \phi \frac{\partial v}{\partial r} + \frac{1}{r} \sin \phi \frac{\partial v}{\partial \phi}\end{aligned}$$

Now substitute for u and v . In the first,

$$\begin{aligned}\cos \phi \frac{\partial u}{\partial r} - \frac{1}{r} \sin \phi \frac{\partial u}{\partial \phi} &= \left(\cos \phi \frac{\partial}{\partial r} - \frac{1}{r} \sin \phi \frac{\partial}{\partial \phi} \right) \sum_{n=0}^{\infty} \frac{1}{n!} a_n r^n \cos n\phi \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} a_n \left(\cos \phi \frac{\partial}{\partial r} - \frac{1}{r} \sin \phi \frac{\partial}{\partial \phi} \right) r^n \cos n\phi \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} a_n (nr^{n-1} \cos \phi \cos n\phi + nr^{n-1} \sin \phi \sin n\phi) \\ &= \sum_{n=0}^{\infty} \frac{1}{(n-1)!} a_n r^{n-1} \cos (n-1)\phi\end{aligned}$$

$$\begin{aligned}
\sin \phi \frac{\partial v}{\partial r} + \frac{1}{r} \cos \phi \frac{\partial v}{\partial \phi} &= \left(\sin \phi \frac{\partial}{\partial r} + \frac{1}{r} \cos \phi \frac{\partial}{\partial \phi} \right) \sum_{n=0}^{\infty} \frac{1}{n!} a_n r^n \sin n\phi \\
&= \sum_{n=0}^{\infty} \frac{1}{n!} a_n (nr^{n-1} \sin \phi \sin n\phi + nr^{n-1} \cos \phi \cos n\phi) \\
&= \sum_{n=0}^{\infty} \frac{1}{(n-1)!} a_n r^{n-1} \cos(n-1)\phi
\end{aligned}$$

and these are indeed equal. For the second,

$$\begin{aligned}
\cos \phi \frac{\partial v}{\partial r} - \frac{1}{r} \sin \phi \frac{\partial v}{\partial \phi} &= \left(\cos \phi \frac{\partial}{\partial r} - \frac{1}{r} \sin \phi \frac{\partial}{\partial \phi} \right) \sum_{n=0}^{\infty} \frac{1}{n!} a_n r^n \sin n\phi \\
&= \sum_{n=0}^{\infty} \frac{1}{n!} a_n nr^{n-1} (\cos \phi \sin n\phi - \sin \phi \cos n\phi) \\
\sin \phi \frac{\partial u}{\partial r} + \frac{1}{r} \cos \phi \frac{\partial u}{\partial \phi} &= \left(\sin \phi \frac{\partial}{\partial r} + \frac{1}{r} \cos \phi \frac{\partial}{\partial \phi} \right) \sum_{n=0}^{\infty} \frac{1}{n!} a_n r^n \cos n\phi \\
&= \sum_{n=0}^{\infty} \frac{1}{n!} a_n nr^{n-1} (\sin \phi \cos n\phi - \cos \phi \sin n\phi)
\end{aligned}$$

and these are negatives of one another as expected.

Therefore, analytic extension of any real Taylor series gives an analytic function $f(z)$. Conversely, any complex Taylor series gives an analytic function.

Exercise: Show that the composition of two analytic functions is analytic. That is, if $f(z)$ and $g(w)$ both satisfy the Cauchy-Riemann conditions, show that $g(f(z))$ also satisfies the Cauchy-Riemann conditions.

5 Contour Integrals

Now consider a function $f(z)$ with derivatives of all orders in some region of the complex plane, and consider the integral of $f(z)$ around a closed curve, C ,

$$\oint_C f(z) dz$$

We may expand this as a pair of functions of two variables,

$$\begin{aligned}
\oint_C f(z) dz &= \oint_C (u(x, y) + iv(x, y))(dx + idy) \\
&= \oint_C (udx + iudy + ivdx - vdy) \\
&= \oint_C ((u + iv) dx + i(u + iv) dy) \\
&= \oint_C (udx - vdy) + i \oint_C (udy + vdx)
\end{aligned}$$

Think of $\vec{u} = (u(x, y), -v(x, y), 0)$ as a vector field in R^3 and $d\vec{x} = (dx, dy, dz)$ as an infinitesimal displacement. Then we can use Stoke's theorem. The first integral becomes

$$\begin{aligned} \oint_C (u dx - v dy) &= \oint_C \vec{u} \cdot d\vec{x} \\ &= \oint_C (\nabla \times \vec{u}) \cdot \hat{n} d^2x \end{aligned}$$

where the normal is in the z -direction. The z -component of the curl of \vec{u} , however, is

$$\begin{aligned} \hat{k} \cdot (\nabla \times \vec{u}) &= \frac{\partial u_x}{\partial y} - \frac{\partial u_y}{\partial x} \\ &= \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \\ &= 0 \end{aligned}$$

where the result vanishes by the Cauchy-Riemann conditions. For the second integral, let $\vec{w} = (v, u, 0)$

$$\begin{aligned} \oint_C (u dy + v dx) &= \oint_C \vec{w} \cdot d\vec{x} \\ &= \oint_C (\nabla \times \vec{w}) \cdot \hat{n} d^2x \end{aligned}$$

and the z -component of the curl is

$$\begin{aligned} \hat{k} \cdot (\nabla \times \vec{w}) &= \frac{\partial w_x}{\partial y} - \frac{\partial w_y}{\partial x} \\ &= \frac{\partial v}{\partial y} - \frac{\partial u}{\partial x} \\ &= 0 \end{aligned}$$

using the other Riemann-Cauchy condition.

Therefore, for any analytic function f , we have

$$\oint_C f(z) dz = 0$$

If two curves share a common segment, then we can add the curves together to get a larger curve. Starting with a given curve, we can therefore imagine adding a small second loop in such a way that the combined contour is slightly altered from the first. This is called a deformation of the contour, and it will not change the value of the integral as long as the small loop we add lies entirely within a region where f is analytic.

6 The Residue Theorem

We can use this result to simplify integrals where the function is not analytic in the entire complex plane. Suppose a function is analytic everywhere except a single point, z_0 . Then in addition to a Taylor series for the function, there may be a *Laurent expansion* which includes poles at z_0 ,

$$\frac{1}{(z - z_0)^n}$$

Such terms are fine away from the point z_0 , so they do not affect analyticity elsewhere. Consider the class of functions which have a Laurent series, i.e., for some finite number N , the function may be expressed as

$$f(z) = \sum_{n=-N}^{\infty} a_n (z - z_0)^n$$

This has poles of orders $1, 2, \dots, N$. Since the mapping $w = g(z) = z - z_0$ is analytic, we might as well write this as

$$f(w) = \sum_{n=-N}^{\infty} a_n w^n$$

where the poles are now at $w = 0$. Now consider a contour integral of the form

$$\oint_C f(w) dw$$

for any closed curve C . Since f is analytic everywhere except the origin, the integral vanishes if C does not enclose the origin – we may deform the curve down to a single point. If C does include the origin, we may deform C until it is a circle of radius R about the origin, and the deformation does not affect the value of the integral. Then on the circle

$$dw = d(Re^{i\phi}) = iRe^{i\phi}d\phi$$

so we have

$$\begin{aligned} \oint_C f(w) dw &= \int_0^{2\pi} \sum_{n=-N}^{\infty} a_n R^n e^{in\phi} (iRe^{i\phi}d\phi) \\ &= i \sum_{n=-N}^{\infty} a_n R^{n+1} \int_0^{2\pi} e^{i(n+1)\phi} d\phi \\ &= i \sum_{n \neq -1}^{\infty} a_n R^{n+1} \left. \frac{e^{i(n+1)\phi}}{i(n+1)} \right|_0^{2\pi} + ia_{-1} \int_0^{2\pi} d\phi \\ &= i \sum_{n=-N}^{\infty} a_n R^{n+1} \frac{1}{i(n+1)} (e^{2(n+1)\pi i} - 1) + ia_{-1} \int_0^{2\pi} d\phi \\ &= 2\pi ia_{-1} \end{aligned}$$

We see that the integral depends only on the coefficient of the *simple pole* (i.e., the pole of order 1). This coefficient is called the *residue* of f at z_0 , and we write

$$\begin{aligned} Res(f(z)) &= Res\left(\sum_{n=-N}^{\infty} a_n (z - z_0)^n\right) \\ &= a_{-1} \end{aligned}$$

The residue theorem now states that the integral of a complex function about a pole equals $2\pi i$ times the residue of the function at the pole. If there are multiple poles, the result is the sum of the residues at all poles included within the contour C . Thus, the residue theorem becomes

$$\oint_C f(w) dw = 2\pi i \sum Res(f)$$

where the sum is over all poles included within C .

6.1 Example: Completeness relation for Fourier integrals

Suppose we can expand a function $f(\mathbf{x})$ as

$$f(\mathbf{x}) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(\mathbf{k}) e^{-i\mathbf{k}\cdot\mathbf{x}} d^3k$$

We would like to show that this transformation is invertible, and this requires the completeness relation for Fourier transformations. To see this, consider inverting the transformation. Multiply both sides by $e^{i\mathbf{k}'\cdot\mathbf{x}}$ and integrate over all \mathbf{x} ,

$$\begin{aligned} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(\mathbf{x}) e^{i\mathbf{k}'\cdot\mathbf{x}} d^3x &= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\mathbf{k}'\cdot\mathbf{x}} d^3x \int_{-\infty}^{\infty} g(\mathbf{k}) e^{-i\mathbf{k}\cdot\mathbf{x}} d^3k \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(\mathbf{k}) e^{-i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{x}} d^3k d^3x \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} g(\mathbf{k}) d^3k \int_{-\infty}^{\infty} e^{-i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{x}} d^3x \end{aligned}$$

We desire the result of this integration to be the transform, $g(\mathbf{k})$, and this will be true if and only if

$$\delta^3(\mathbf{k}-\mathbf{k}') = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{x}} d^3x$$

or equivalently

$$\delta^3(\mathbf{k}) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\mathbf{k}\cdot\mathbf{x}} d^3x$$

Using Cartesian coordinates, this breaks into three identical integrals of the form

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ikx} dx$$

which we may use contour integration to evaluate.

Our goal is to show that this integral is a Dirac delta function, which means that for any test function $g(k)$ (i.e., $g(k)$ is bounded, as differentiable as we like, and vanishes outside a compact region),

$$g(0) = \int_{-\infty}^{\infty} g(k) \left[\frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ikx} dx \right] dk$$

Replace the infinite limit on the inner integral by R . We will let $R \rightarrow \infty$ at the end of the calculation. Then, carrying out the integral of the exponential,

$$\lim_{R \rightarrow \infty} \int_{-\infty}^{\infty} g(k) \left[\frac{1}{2\pi} \int_{-R}^R e^{-ikx} dx \right] dk = \lim_{R \rightarrow \infty} \int_{-\infty}^{\infty} g(k) \left[-\frac{1}{2\pi ik} (e^{-ikR} - e^{ikR}) \right] dk$$

We can carry this out using contour integration.

In order to use contour integration, we need to enclose the simple pole at $k = 0$ with a curve. There are two problems. First, the pole here lies directly on the path of integration. We solve this difficulty with a trick: displace the pole slightly, then do the integral, then take the limit as the displacement vanishes. Specifically, let ε be an arbitrary positive real number and write the integral as

$$\begin{aligned} \int_{-\infty}^{\infty} g(k) \left[-\frac{1}{2\pi i k} (e^{-ikR} - e^{ikR}) \right] dk &= \lim_{\varepsilon \rightarrow 0} \int_{-\infty}^{\infty} g(k) \left[\frac{1}{2\pi i (k - i\varepsilon)} (e^{ikR} - e^{-ikR}) \right] dk \\ &= \lim_{\varepsilon \rightarrow 0} \int_{-\infty}^{\infty} \frac{g(k) e^{ikR}}{2\pi i (k - i\varepsilon)} dk - \lim_{\varepsilon \rightarrow 0} \int_{-\infty}^{\infty} \frac{g(k) e^{-ikR}}{2\pi i (k - i\varepsilon)} dk \end{aligned}$$

The second problem is to complete a closed curve without changing the value of the integral. We begin by analytically extending the integration variable k to a complex variable, $k = k_R + ik_I$. The first integral is then

$$\lim_{\varepsilon \rightarrow 0} \int_{-\infty}^{\infty} \frac{g(k) e^{ikR}}{2\pi i (k - i\varepsilon)} = \lim_{\varepsilon \rightarrow 0} \int_{-\infty}^{\infty} \frac{g(k_R + ik_I) e^{ik_R R} e^{-k_I R}}{2\pi i (k_R + ik_I - i\varepsilon)}$$

and we see that if $k_I > 0$ the integrand is suppressed by $e^{-k_I R}$. If we close the contour by adding a semicircle in the upper half plane of radius R , then at any angle ϕ on the semicircle the imaginary component k_I is given by $k_I = R \sin \phi$. As the radius tends to infinity, $R \rightarrow \infty$, so $k_I R$ diverges and the exponential factor $e^{-k_I R}$ tends to zero. This means that the integrand vanishes on this upper semicircle and we can integrate over a closed curve C which runs along the entire real k axis and returns on the semicircle, without changing the value of the integral,

$$\lim_{\varepsilon \rightarrow 0} \lim_{R \rightarrow \infty} \int_{-\infty}^{\infty} \frac{g(k_R + ik_I) e^{i(k_R + ik_I)R}}{2\pi i (k_R + ik_I - i\varepsilon)} = \lim_{\varepsilon \rightarrow 0} \lim_{R \rightarrow \infty} \oint_C \frac{g(k) e^{ikR}}{2\pi i (k - i\varepsilon)} dk$$

We may now apply the Residue Theorem. The contour is integrated in the positive sense, i.e., counterclockwise, and encloses the simple pole at $k = i\varepsilon$, so the residue is taken there

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} \oint_C \frac{g(k) e^{ikR}}{2\pi i (k - i\varepsilon)} dk &= \lim_{\varepsilon \rightarrow 0} 2\pi i \operatorname{Res} \left(\frac{g(k) e^{ikR}}{2\pi i (k - i\varepsilon)} \right) \\ &= \lim_{\varepsilon \rightarrow 0} 2\pi i \left(\frac{g(k) e^{ikR}}{2\pi i} \right) \Big|_{k=i\varepsilon} \\ &= \lim_{\varepsilon \rightarrow 0} 2\pi i \left(\frac{g(i\varepsilon) e^{\varepsilon R}}{2\pi i} \right) \\ &= \lim_{\varepsilon \rightarrow 0} (g(i\varepsilon) e^{-\varepsilon R}) \\ &= g(0) \end{aligned}$$

The second integral is handled in the same way,

$$\lim_{\varepsilon \rightarrow 0} \int_{-\infty}^{\infty} \frac{g(k) e^{-ikR}}{2\pi i (k - i\varepsilon)} dk = \lim_{\varepsilon \rightarrow 0} \int_{-\infty}^{\infty} \frac{g(k_R + ik_I) e^{-ik_R R} e^{+k_I R}}{2\pi i (k_R + ik_I - i\varepsilon)}$$

There is one important difference. The exponential factor is now $e^{k_I R}$, which converges only when $k_I < 0$. This means that we must close the contour, C' , in the lower half plane. We now have a clockwise contour, running along the entire real k axis then circling back along a semicircle in the lower half plane. We pick

up a minus sign because of the direction of the contour, but more importantly, the shifted pole no longer lies inside the contour. Since the integrand lies in a region containing no poles it is analytic and the second integral vanishes.

Returning to the original problem, we have

$$\begin{aligned} \lim_{R \rightarrow \infty} \int_{-\infty}^{\infty} g(k) \left[-\frac{1}{2\pi ik} (e^{-ikR} - e^{ikR}) \right] dk &= \lim_{R \rightarrow \infty} \int_{-\infty}^{\infty} g(k) \frac{1}{2\pi ik} e^{ikR} dk - \lim_{R \rightarrow \infty} \int_{-\infty}^{\infty} g(k) \frac{1}{2\pi ik} e^{-ikR} dk \\ &= \lim_{R \rightarrow \infty} g(0) \\ &= g(0) \end{aligned}$$

and we have established that

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ikx} dx = \delta(k)$$

This shows the completeness of Fourier integrals.