1 Overview

Lecture 1 Course / Syllabus Overview

1.1 Physical models, wave and other
1.2 Linear vs. nonlinear
1.3 Superposition

2 Simple harmonic oscillation

1. Mass-spring (see notes)
2. Simple pendulum: Taylor series
3. General potentials
4. Complex representations

Readings:
1. Torre, Introduction
2. Torre: Harmonic Oscillations, Complex numbers ... Ch 1
3. Riffe: Lecture 2

Problems: (due Thursday, Sept. 5)
- Torre, problems 1.2, 1.4, 1.5, 1.6, 1.8, 1.9, 1.11, 1.18

3 Coupled oscillators and normal modes

Readings:
1. Torre:
   - 2. Two Coupled Oscillators
   - 3. How to Find Normal Modes
   - 4. Linear Chain of Coupled Oscillators
2. Riffe:
Problems: (due Thursday, Sept. 12)

1. Torre, problems 2.1, 2.3, 2.5
2. Riffe, problems 2.6, 2.7, 3.1
3. Completely solve the 3 mass case, by solving all parts of Torre 2.6. Once you’ve found the eigenvectors and eigenvalues, describe the motions of each of the three masses in each of the normal modes. Write the general solution for the motion as a vector superposition of eigenmodes.

4 Two approaches to the wave equation

Approach 1: The continuum limit of coupled oscillators
Read:  Note that Torre and Riffe give slightly different approaches here.

1. Torre
   • 5. The Continuum Limit and the Wave Equation
   • 6. Elementary Solutions to the Wave Equation
2. Riffe
   • Lecture 7: Long Wavelength Limit / Normal Modes
3. Wheeler
   • Lecture Notes: The continuum limit

Approach 2: Waves on a string
Read my notes:
• Lecture Notes: Waves on a string

5 Solutions to the wave equation
Read:

1. Torre
   • 6. Elementary Solutions to the Wave Equation
   • 7. General Solution to the One Dimensional Wave Equation
2. Riffe
   • Lecture 9 Traveling Waves, Standing Waves, and the Dispersion Relation
   • Lecture 10 1D Wave Equation - General Solution / Gaussian Function
• Lecture 9 General Solution with Boundary Conditions.

3. Wheeler
• Lecture Notes: The continuum limit
• Lecture Notes: Solutions to the wave equation

Problems: (Due Thursday, September 19)

1. Torre: problems 3.1, 3.2, 3.3, 3.4

2. Riffe:
   (a) Lecture 4: problems 4.1, 4.3
   (b) Lecture 6: problem 6.3
   (c) Lecture 8: problems 8.1,

3. Find the eigenvalues and eigenvectors of the matrix

\[ M = \begin{pmatrix} 3 & 4 \\ 4 & 3 \end{pmatrix} \]

First Midterm Exam, Tuesday, September 24

The First Midterm Exam will cover material through our discussion of solutions to the wave equation. The exam will consist of 6-8 short-to-medium length problems similar to the problem assignments. Therefore the problems are your best study guide.

You may bring a 3 x 3 card with formulas; you may use both sides. Bring a pen/pencil. Paper will be provided.

Problems  Brief synopsis of problems assigned so far. Pay attention to the general topics covered in the problems, not the specific problems.

1.2 Potential energy, small oscillations
1.4 SHO with sign flip—exponential divergence
1.5 Absolute value and phase of complex numbers
1.6 \( z, z^2 \) in \( x + iy \) and polar form
1.8 When is \( q(t) = Ae^{i\omega t} + Be^{-i\omega t} \) real?
1.9 Relation between \( \text{Re}(q) \) and \( \text{Im}(q) \)
1.11 \( \cos(\alpha + \beta) \) and \( \sin(\alpha + \beta) \)
1.18 Derive Euler formula from Taylor series and by solving \( f'' + f = 0 \) two ways.

T 2.1 Two-dim SHO
T 2.3 Complex solutions to characteristic equation
T 2.5 Show that the \( N = 2 \) limit of the general \( N \) oscillator case reduces to our solution for two oscillators
T 2.6 Three coupled oscillators, complete solution
R 2.6 Write complex numbers in Cartesian form
R 2.7 Write complex numbers in polar form
R 3.1 Invert the Euler formula to derive expressions for \( \cos x \) and \( \sin x \)

T 3.1 Show that \( \cos(\omega t) \cos(kx) + i \sin(\omega t) \sin(kx) \) where \( \omega = kc \) solves the wave equation. With \( A = B \) show that \( q(t) = A \cos(k(x-ct)) \).

T 3.2 Change of variables and chain rule for \( q = te^x, u = x + ct \) and \( s = x - ct \).
T 3.3 Show that \( q(x,t) = f(x - ct) + g(x + ct) \) solves the wave equation
T 3.4 Find solutions to the wave equation with various initial conditions.
R 4.1 Apply the inverse transformation to the normal mode solution to get $x(t)$
R 4.3 With 0 initial position, find the initial velocities that lead to only the second mode
R 6.3 Show that the sum of $\cos(\omega t) \cos(kx)$ and $\sin(\omega t) \sin(kx)$ is a traveling wave. Which direction?
W 8.1 Partial derivatives directly and using chain rule
W Find the eigenvalues and eigenvectors of $M = \begin{pmatrix} 3 & 4 \\ 4 & 3 \end{pmatrix}$.

6 Vector spaces; function spaces

Read:
1. Torre
   - Appendix B. Vector Spaces
2. Riffe
   - Lecture 13 (Vector Spaces / Real Space)
   - Lecture 14 (A Vector Space of Functions)
3. Wheeler
   - Lecture Notes: Vector spaces

Problems: (Due Thursday, October 3)

1. Riffe:
   (a) Lecture 13: problems 13.2, 13.7
   (b) Lecture 14: problem 14.3

2. Wheeler:
   (a) Use Gram-Schmidt orthogonalization to find the third unit vector to complete the basis ($\mathbf{u}, \mathbf{v}_1$) found in the lecture notes.
   (b) Derive the cubic Legendre polynomial $P_3(x)$ by starting with
      \[ P_3(x) = ax^3 + bx^2 + cx + d \]
      and imposing the orthogonality conditions
      \[ \langle P_3(x), P_0(x) \rangle = 0 \]
      \[ \langle P_3(x), P_1(x) \rangle = 0 \]
      \[ \langle P_3(x), P_2(x) \rangle = 0 \]
      and finally requiring the normalization condition $P_3(1) = 1$.
   (c) Show that $x^3$ and $x^5$ are not orthogonal.
   (d) Show that the space of $2 \times 2$ matrices is a vector space by showing that they satisfy all the properties listed in Section 1.1 of the lecture notes.
7 Fourier series

Read:

1. Riffe
   - Lecture 11 (Introduction to Fourier Series)
   - Lecture 12 (Complex Fourier Series)

2. Wheeler
   - Lecture Notes: Fourier series

Problems: (Due Thursday, October 10) Work all exercises from my lecture notes. See the notes for additional context.

1. Complete the proof that \( \{ \sqrt{\frac{2}{L}} \sin \frac{n\pi y}{L}, \sqrt{\frac{2}{L}} \cos \frac{n\pi y}{L} \mid n = 0, 1, 2, \ldots \} \) is an orthonormal set when integrated over a full period. By “full period”, we mean a full oscillation of the fundamental \((n = 1)\) mode of the sine or cosine, given by

\[
\sqrt{\frac{2}{L}} \sin \frac{\pi y}{L} \Rightarrow \sqrt{\frac{2}{L}} \sin \left( \frac{\pi y}{L} + 2\pi \right) = \sqrt{\frac{2}{L}} \sin \left( \frac{\pi (y + 2L)}{L} \right) \]
\[
y \Rightarrow y + 2L
\]

Therefore, orthonormality means showing that for any \( a \),

\[
\frac{1}{L} \int_a^{a+2L} \sin \frac{n\pi x}{L} \cos \frac{m\pi x}{L} \, dx = 0
\]

\[
\frac{1}{L} \int_a^{a+2L} \cos \frac{n\pi x}{L} \cos \frac{m\pi x}{L} \, dx = \delta_{nm}
\]

2. Show that the complex Fourier modes, \( \left\{ \frac{1}{\sqrt{2\pi}} e^{ikx} \mid k = 0, \pm 1, \pm 2, \ldots \right\} \), are orthonormal on the interval \([-\pi, \pi]\), using the complex inner product \( \langle f, g \rangle = \int_{-\pi}^{\pi} f^* g \, dx \).

3. For the square wave solution in the lecture notes, let \( a = \frac{1}{3} \) and \( b = \frac{2}{3} \). Plot the partial sums,

\[
f_N (x) = \frac{1}{2} (b - a) + \frac{1}{\pi} \sum_{k=1}^{N} \frac{1}{k} ((\sin k\pi (x - a)) - \sin k\pi (x - b))
\]

for \( N = 2, 5, 10, 100, \text{ and } 1000 \) to see how the series approaches a unit step.

4. Prove that the only Fourier series that gives the zero function, \( f (x) = 0 \), has all zero coefficients.

5. Using the previous exercise, and without integrating to calculate any of the coefficients, prove that any symmetric function, \( f (-x) = f (x) \), on a symmetric interval (you may take \([-\pi, \pi]\)) may be written as a cosine series and that any odd function, \( f (-x) = -f (x) \) on the same interval may be written as a sine series.
6. We argued by symmetry that the even terms in the Fourier series of

\[ f(x) = \begin{cases} 
  x & 0 < x < \frac{L}{2} \\
  L - x & \frac{L}{2} < x < L \\
  0 & \text{elsewhere}
\end{cases} \]

on the interval \([0, L]\) must vanish. Compute the coefficients of the even terms

\[ a_{2m} = \sqrt{\frac{2}{L}} \int_{0}^{L} f(x) \sin \frac{2m\pi x}{L} dx \]

explicitly to show that they do indeed all vanish.

7. Compute the Fourier series of the function

\[ f(x) = \begin{cases} 
  A \left(x - \frac{L}{2}\right)^2 & 0 < x < L \\
  0 & \text{elsewhere}
\end{cases} \]

on the interval \([0, L]\)

8. Fourier transforms

Read:
1. Torre
   - 8. Fourier analysis
2. Riffe
   - Lecture 16 (Introduction to Fourier Transforms)
   - Lecture 17 (Fourier Transforms and the Wave Equation)
3. Wheeler
   - Lecture Notes: Fourier analysis

Problems: (Due Thursday, October 17)

1. Use contour integration to compute the integral

\[ \int_{-\infty}^{\infty} \frac{e^{ikx}}{x^2 - i\epsilon} dx \]

2. Dirac delta
   a) Choose the constants \( A_n \) to normalize each of the functions

\[ h_n(x) = \begin{cases} 
  A_n \left(\frac{\pi}{n}\right)^2 - x^2 & -\frac{\pi}{n} < x < \frac{\pi}{n} \\
  0 & \text{otherwise}
\end{cases} \]

to one, so that

\[ \int_{-\infty}^{\infty} h_n(x) dx = 1 \]
b) Show that the limit as \( n \to \infty \) is a Dirac delta function,

\[
\delta (x) = \lim_{n \to \infty} h_n (x)
\]

by proving that for any test function \( f(x) \),

\[
\lim_{n \to \infty} \int_{-\infty}^{\infty} h_n (x) f(x) \, dx = f(0)
\]

(Note: Because \( h_n (x) \) vanishes outside of a shrinking interval, you should be able to make a very rigorous proof.)

3. By studying \( \int \delta (g(x)) f(x) \, dx \) in a sufficiently small neighborhood of each zero, prove that for any smooth function \( g(x) \) with isolated simple zeros at points \( x_i, i = 1, 2, \ldots, n \),

\[
\delta (g(x)) = \sum_{i=1}^{n} \frac{1}{|g'(x_i)|} \delta (x - x_i)
\]

where \( g'(x_i) \) is the first derivative of \( g(x) \) evaluated at the \( i \)th pole.

4. Suppose we pluck a guitar string of length \( L \) and fixed ends by raising the center to form a triangle, then releasing it from rest. We have already seen that the initial triangle wave

\[
f(x) = \begin{cases} 
  x & 0 < x < \frac{L}{2} \\
  L - x & \frac{L}{2} < x < L \\
  0 & \text{elsewhere}
\end{cases}
\]

may be represented by a Fourier series,

\[
f(x) = \frac{4L}{\pi^2} \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m+1)^2} \sin \left( \frac{(2m+1) \pi x}{L} \right)
\]

(a) Find the time evolution of the guitar string if we release the it at rest from its stretched triangular position.

(b) Write the solution as a sum of right moving and left moving waves.

(c) Describe the resulting waves.

5. Find the Fourier transform of the function

\[
f(x) = \begin{cases} 
  e^{ax} & x < 0 \\
  e^{-ax} & x \geq 0
\end{cases}
\]

where \( a \) is a positive real number. Verify that \( \hat{f}(k) = \hat{f}^*(-k) \).

9 Waves in 3 dimensions: Cartesian

Read:

1. Torre
   - 9. The Wave Equation in 3 Dimensions
   - 10. Plane Waves
11. Separation of Variables

2. Riffe
   - Lecture 18 3D Wave Equation and Plane Waves / 3D Differential Operators
   - Lecture 19 Separation of Variables in Cartesian Coordinates

3. Wheeler
   - Lecture Notes: TBA

Problems: (TBA)

Second Midterm Exam, Thursday, October 24

The Second Midterm Exam will cover material through our discussion of 3 dimensional waves in Cartesian coordinates. The main topics includ:

1. Vector spaces and function spaces
2. Fourier series
3. Fourier transforms
4. Contour integration and the residue theorem
5. Dirac delta functions
6. Three dimensional waves (time permitting)
7. Separation of variables (time permitting)

The exam will consist of 6-8 short-to-medium length problems similar to the problem assignments. Therefore the problems are your best study guide. You may bring a $3 \times 3$ card with formulas; you may use both sides. Bring a pen/pencil. Paper will be provided.

10 Waves in 3 dimensions: Cylindrical

Read:

1. Torre
   - 12. Cylindrical Coordinates

2. Riffe
   - Lecture 21 Separation of Variables in Cylindrical Coordinates

3. Wheeler
   - Lecture Notes: TBA

Problems: (TBA)
11 Waves in 3 dimensions: Spherical

Read:

1. Torre
   - 13. Spherical Coordinates

2. Riffe
   - Lecture 22 Separation of Variables in Spherical Coordinates
   - Lecture 23 Spherical Coordinates II / A B. V. Problem / Separation of Variables Summary

3. Wheeler
   - Lecture Notes: TBA

Problems: (TBA)