Phys 3750 -- Wave Phenomena Fall 2019

November 15, 2019

1 Overview

Lecture 1 Course / Syllabus Overview

1.1 Physical models, wave and other

1.2 Linear vs. nonlinear

1.3 Superposition

2 Simple harmonic oscillation

1. Mass-spring (see notes)
2. Simple pendulum: Taylor series
3. General potentials
4. Complex representations

Readings:

1. Torre, Introduction
2. Torre: Harmonic Oscillations, Complex numbers ... Ch 1
3. Riffe: Lecture 2

Problems: (due Thursday, Sept. 5)

- Torre, problems 1.2, 1.4, 1.5, 1.6, 1.8, 1.9, 1.11, 1.18

3 Coupled oscillators and normal modes

Readings:

1. Torre:
   - 2. Two Coupled Oscillators
   - 3. How to Find Normal Modes
   - 4. Linear Chain of Coupled Oscillators
2. Riffe:

- Lecture 3 Two Coupled Oscillators / Normal Modes
- Lecture 4 Normal Mode Coordinates / Initial Value Problem. Homework 1
- Lecture 5 Linear Chain / Normal Modes

3. Lecture Notes

Problems: (due Thursday, Sept. 12)

1. Torre, problems 2.1, 2.3, 2.5
2. Riffe, problems 2.6, 2.7, 3.1
3. Completely solve the 3 mass case, by solving all parts of Torre 2.6. Once you've found the eigenvectors and eigenvalues, describe the motions of each of the three masses in each of the normal modes. Write the general solution for the motion as a vector superposition of eigenmodes.

4 Two approaches to the wave equation

Approach 1: The continuum limit of coupled oscillators

Read: Note that Torre and Riffe give slightly different approaches here.

1. Torre
   - 5. The Continuum Limit and the Wave Equation
   - 6. Elementary Solutions to the Wave Equation
2. Riffe
   - Lecture 7: Long Wavelength Limit / Normal Modes
3. Wheeler
   - Lecture Notes: The continuum limit

Approach 2: Waves on a string

Read my notes:

- Lecture Notes: Waves on a string

5 Solutions to the wave equation

Read:

1. Torre
   - 6. Elementary Solutions to the Wave Equation
   - 7. General Solution to the One Dimensional Wave Equation
2. Riffe
   - Lecture 9 Traveling Waves, Standing Waves, and the Dispersion Relation
   - Lecture 10 1D Wave Equation - General Solution / Gaussian Function

- Lecture 9 General Solution with Boundary Conditions.

3. Wheeler

    - Lecture Notes: The continuum limit
    - Lecture Notes: Solutions to the wave equation

Problems: (Due Thursday, September 19)

1. Torre: problems 3.1, 3.2, 3.3, 3.4
2. Riffe:
    (a) Lecture 4: problems 4.1, 4.3
    (b) Lecture 6: problem 6.3
    (c) Lecture 8: problems 8.1,
3. Find the eigenvalues and eigenvectors of the matrix

$$M = \begin{pmatrix} 3 & 4 \\ 4 & 3 \end{pmatrix}$$

First Midterm Exam, Tuesday, September 24 (see Lecture Notes)

The exam will consist of 6-8 short-to-medium length problems similar to the problem assignments. Or maybe 9. Therefore the problems are your best study guide. You may bring a 3×3 card with formulas; you may use both sides. Bring a pen/pencil. Paper will be provided.

6 Vector spaces; function spaces

Read:

1. Torre

    - Appendix B. Vector Spaces

2. Riffe

    - Lecture 13 (Vector Spaces / Real Space)
    - Lecture 14 (A Vector Space of Functions)

3. Wheeler

    - Lecture Notes: Vector spaces

Problems: (Due Thursday, October 3)

1. Riffe:
    (a) Lecture 13: problems 13.2, 13.7
    (b) Lecture 14: problem 14.3
2. Wheeler:

(a) Use Gram-Schmidt orthogonalization to find the third unit vector to complete the basis $(\hat{\mathbf{u}}, \hat{\mathbf{v}}_\perp)$ found in the lecture notes.

(b) Derive the cubic Legendre polynomial $P_3(x)$ by starting with

$$P_3(x) = ax^3 + bx^2 + cx + d$$

and imposing the orthogonality conditions

$$\begin{aligned} \langle P_3(x), P_0(x)\rangle &= 0 \\ \langle P_3(x), P_1(x)\rangle &= 0 \\ \langle P_3(x), P_2(x)\rangle &= 0 \end{aligned}$$

and finally requiring the normalization condition $P_3(1) = 1$.

(c) Show that x^3 and x^5 are not orthogonal.

(d) Show that the space of 2×2 matrices is a vector space by showing that they satisfiy all the properties listed in Section 1.1 of the lecture notes.

7 Fourier series

Read:

1. Riffe
   - Lecture 11 (Introduction to Fourier Series)
   - Lecture 12 (Complex Fourier Series)
2. Wheeler
   - Lecture Notes: Fourier series

Problems: (Due Thursday, October 10) Work all exercises from my lecture notes. See the notes for additional context.

1. Complete the proof that $\left\{ \sqrt{\frac{2}{L}} \sin \frac{n\pi y}{L}, \sqrt{\frac{2}{L}} \cos \frac{n\pi y}{L} \middle| n = 0, 1, 2, \ldots \right\}$ is an orthonormal set when integrated over a full period. By "full period", we mean a full oscillation of the fundamental $(n = 1)$ mode of the sine or cosine, given by

$$\begin{aligned} \sqrt{\frac{2}{L}} \sin \frac{\pi y}{L} &\Rightarrow \sqrt{\frac{2}{L}} \sin \left(\frac{\pi y}{L} + 2\pi \right) \\ &= \sqrt{\frac{2}{L}} \sin \left(\frac{\pi (y + 2L)}{L} \right) \\ y &\Rightarrow y + 2L \end{aligned}$$

Therefore, orthonormality means showing that for any a,

$$\begin{aligned} \frac{1}{L} \int\limits_a^{a+2L} \sin \frac{n\pi x}{L} \cos \frac{m\pi x}{L} dx &= 0 \\ \frac{1}{L} \int\limits_a^{a+2L} \cos \frac{n\pi x}{L} \cos \frac{m\pi x}{L} dx &= \delta_{nm} \end{aligned}$$

2. Show that the complex Fourier modes, $\left\{ \frac{1}{\sqrt{2\pi}} e^{ikx} \middle| k = 0, \pm 1, \pm 2, \ldots \right\}$, are orthonormal on the interval $[-\pi, \pi]$, using the complex inner product $\langle f, g \rangle = \int_{-\pi}^{\pi} f^* g \, dx$.

3. For the square wave solution in the lecture notes, let $a = \frac{1}{3}$ and $b = \frac{2}{3}$. Plot the partial sums,

$$f_N(x) \quad = \quad \frac{1}{2}(b-a) + \frac{1}{\pi} \sum_{k=1}^{N} \frac{1}{k} \left((\sin k\pi (x-a)) - \sin k\pi (x-b) \right)$$

for $N = 2, 5, 10, 100,$ *and* 1000 to see how the series approaches a unit step.

4. Prove that the only Fourier series that gives the zero function, $f(x) = 0$, has all zero coefficients.

5. Using the previous exercise, and without integrating to calculate any of the coefficients, prove that any symmetric function, $f(-x) = f(x)$, on a symmetric interval (you may take $[-\pi, \pi]$) may be written as a cosine series and that any odd function, $f(-x) = -f(x)$ on the same interval may be written as a sine series.

6. We argued by symmetry that the even terms in the Fourier series of

$$f(x) = \begin{cases} x & 0 < x < \frac{L}{2} \\ L - x & \frac{L}{2} < x < L \\ 0 & elsewhere \end{cases}$$

on the interval $[0, L]$ must vanish. Compute the coefficients of the even terms

$$a_{2m} \quad = \quad \sqrt{\frac{2}{L}} \int_0^L f(x) \sin \frac{2m\pi x}{L} dx$$

explicitly to show that they do indeed all vanish.

7. Compute the Fourier series of the function

$$f(x) = \begin{cases} A\left(x - \frac{L}{2}\right)^2 & 0 < x < L \\ 0 & elsewhere \end{cases}$$

on the interval $[0, L]$

8 Fourier transforms

Read:

1. Torre
   - 8. Fourier analysis
2. Riffe
   - Lecture 16 (Introduction to Fourier Transforms)
   - Lecture 17 (Fourier Transforms and the Wave Equation)
3. Wheeler
   - Lecture Notes: Fourier analysis

Problems: (Due Thursday, October 17)

1. Use contour integration to compute the integral

$$\int_{-\infty}^{\infty} \frac{e^{ikx}}{x^2 - ix} dx$$

2. Dirac delta

   a) Choose the constants A_n to normalize each of the functions

$$h_n(x) = \begin{cases} A_n\left(\frac{\pi}{n^2}^2 - x^2\right) & -\frac{\pi}{n} < x < \frac{\pi}{n} \\ 0 & otherwise \end{cases}$$

   to one, so that

$$\int_{-\infty}^{\infty} h_n(x)\, dx = 1$$

   b) Show that the limit as $n \to \infty$ is a Dirac delta function,

$$\delta(x) = \lim_{n\to\infty} h_n(x)$$

   by proving that for any test function $f(x)$,

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} h_n(x) f(x)\, dx = f(0)$$

   (Note: Because $h_n(x)$ vanishes outside of a shrinking interval, you should be able to make a very rigorous proof.)

3. By studying $\int \delta(g(x)) f(x)\, dx$ in a sufficiently small neighborhood of each zero, prove that for any smooth function $g(x)$ with isolated simple zeros at points $x_i, i = 1, 2, \ldots, n$,

$$\delta(g(x)) = \sum_{i=1}^{n} \frac{1}{|g'(x_i)|} \delta(x - x_i)$$

   where $g'(x_i)$ is the first derivative of $g(x)$ evaluated at the i^{th} pole.

4. Suppose we pluck a guitar string of length L and fixed ends by raising the center to form a triangle, then releasing it from rest. We have already seen that the initial triangle wave

$$f(x) = \begin{cases} x & 0 < x < \frac{L}{2} \\ L - x & \frac{L}{2} < x < L \\ 0 & elsewhere \end{cases}$$

   may be represented by a Fourier series,

$$f(x) = \frac{4L}{\pi^2} \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m+1)^2} \sin \frac{(2m+1)\pi x}{L}$$

   (a) Find the time evolution of the guitar string if we release the it at rest from its stretched triangular position.

(b) Write the solution as a sum of right moving and left moving waves.

(c) Describe the resulting waves.

5. Find the Fourier transform of the function

$$f(x) = \begin{cases} e^{ax} & x < 0 \\ e^{-ax} & x \geq 0 \end{cases}$$

where a is a positive real number. Verify that $\tilde{f}(k) = \tilde{f}^*(-k)$.

9 Waves in 3 dimensions: Cartesian

Read:

1. Torre
   - 9. The Wave Equation in 3 Dimensions
   - 10. Plane Waves
   - 11. Separation of Variables
2. Riffe
   - Lecture 18 3D Wave Equation and Plane Waves / 3D Differential Operators
   - Lecture 19 Separation of Variables in Cartesian Coordinates
3. Wheeler
   - Lecture Notes

Problems: (Due Thursday, October 31)

1. Find the gradient of

   (a) $f(x, y, z) = \frac{ax^2}{y+z^3}$

   (b) $f(x, y, z) = ax^2 + by^2 \sin z$

   (c) Let $f(x) = axy + bz$.

      i. Find the set of all vectors at the origin that are orthogonal to $\boldsymbol{\nabla} f$.

      ii. Do the same at the point $\mathcal{P} = \hat{\mathbf{i}} + 2\hat{\mathbf{j}}$, that is, find all vectors orthogonal to grad f at the point $(x, y, z) = (1, 2.0))$.

2. (Torre 5.4) Divergence. Using the Cartesian-coordinate form of the divergence, $\boldsymbol{\nabla} = \hat{\mathbf{i}}\frac{\partial}{\partial x} + \hat{\mathbf{j}}\frac{\partial}{\partial y} + \hat{\mathbf{k}}\frac{\partial}{\partial z}$, compute the following:

   (a) $\mathbf{E}(\mathbf{r}) = \frac{\mathbf{r}}{r^3}$,where $r = \sqrt{x^2 + y^2 + z^2} > 0$. (Coulomb electric field)

   (b) $\mathbf{B}(\mathbf{r}) = -\frac{y}{x^2+y^2}\hat{\mathbf{i}} + \frac{x}{x^2+y^2}\hat{\mathbf{j}}$. (magnetic field outside a long wire)

   (c) $\mathbf{D}(\mathbf{r}) = \vec{\mathbf{r}}$ (electric field inside a uniform ball of charge).

3. (Torre 5.8) Let f and g be two functions. We can take the gradients of f and g to get vector fields, $\boldsymbol{\nabla} f$ and $\boldsymbol{\nabla} g$. We can multiply these vector fields by the functions f and g to get more vector fields, e.g.,

$$f \boldsymbol{\nabla} g$$

As with any vector field, we can make a function by taking a divergence, e.g., $\boldsymbol{\nabla} \cdot (f \boldsymbol{\nabla} g)$. Using the definitions of gradient, divergence and Laplacian show that

$$\boldsymbol{\nabla} \cdot (f \boldsymbol{\nabla} g) = \boldsymbol{\nabla} f \cdot \boldsymbol{\nabla} g + f \nabla^2 g$$

and

$$f \nabla^2 g - g \nabla^2 f = \boldsymbol{\nabla} \cdot (f \boldsymbol{\nabla} g - g \boldsymbol{\nabla} f)$$

4. (Riffe 19.1) Heat Equation. A partial differential equation known as the heat equation, which is used to describe heat or temperature flow in an object, is given by

$$\nabla^2 q = \frac{1}{\lambda} \frac{\partial q}{\partial t}$$

where $\lambda > 0$.

(a) If there is no y or z dependence to the problem, write down a simplified version of this equation.

(b) Use separation of variables to find two ordinary differential equations, one in x and one in t . What are the orders of these two equations? Are they linear or nonlinear?

(c) Find the general solutions to the two ordinary differential equations. Thus write down the general separable solutions to the heat equation. [Note: there should be two linearly independent solutions.]

(d) In solving the heat equation, you should have found that $X''(x)X(x) = C$, where C is some constant. Assume that this constant is real. If $C > 0$, describe the behavior of the solutions in both x and t . In what ways are these solutions like solutions to the wave equation? In what ways are they different?

(e) If $C < 0$, describe the behavior of the solutions in both x and t. In what ways are these solutions like solutions to the wave equation? In what ways are they different?

Second Midterm Exam, Thursday, October 24 (see Lecture Notes)

The exam will consist of 6-8 short-to-medium length problems similar to the problem assignments. Or maybe 9. Therefore the problems are your best study guide. You may bring a 3×3 card with formulas; you may use both sides. Bring a pen/pencil. Paper will be provided.

10 Waves in 3 dimensions: Cylindrical

Read:

1. Torre
   - 12. Cylindrical Coordinates
2. Riffe
   - Lecture 21 Separation of Variables in Cylindrical Coordinates
3. Wheeler
   - Lecture Notes: Separation of variables in cylindrical coordinates. Bessel functions.

Problems: (Due Thursday, November 7)

1. (Torre 6.1). Use the method of separation of variables to find a nonzero solution of the 3-dimensional wave equation in the interior of a cube and which vanishes on the faces of a cube. (You can think of this as a mathematical model of sound waves in a room.) (Hint: are you going to use Cartesian or cylindrical coordinates?)

2. Suppose we seek time-independent and z-independent solutions to the wave equation in cylindrical coordinates. Then the wave equation reduces to the *Laplace equation*,

$$\nabla^2 q(\rho, \varphi) = 0$$

which in cylindrical coordinates is

$$\frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial q}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 q}{\partial \varphi^2} = 0$$

Apply separation of variables to this equation and solve both resulting equations. (Hint: the equation for ρ is much simpler here. Try simple powers, ρ^α and show that for each value of m there are two solutions for α. Handle the $m = 0$ case separately by directly integrating the corresponding equation.)

3. Suppose we have a cylindrically symmetric (that is, there is no φ-dependence) wave solution, $q(\rho, z, t)$. Let q vanish at $\rho = \rho_0$ and on the z-axis let

$$q(\rho = 0, z, t = 0) = \begin{cases} L & 0 < z < \pi \\ -L & -\pi < z < 0 \\ \textit{periodic} & \textit{period } 2\pi \end{cases}$$

which is periodic in z with and vanishes at $\rho = \rho_0$.

(a) We may write $q(0, z, 0)$ as a Fourier series

$$q(0, z, 0) = \sum_{n=0}^{\infty} (a_n \sin nz + b_n \cos nz)$$

Show that $b_n = 0$ for all n and find all coefficients a_n.

(b) We may extend this solution for $q(0, z, 0)$ to a solution for $0 \le \rho \le \rho_0$ by including appropriate Bessel functions,

$$q(\rho, z, 0) = \sum_{n=0}^{\infty}\sum_{l=0}^{\infty} a_{nl} \sin(nz) J_0\left(\frac{x_{0l}\rho}{\rho_0}\right)$$

(We can determine a_{nl} if at the initial time and for some constant $z = z_1$, we know the dependence on ρ, $q(\rho, z_1, 0) = f(\rho)$.)

i. Why do we only need $J_0(x)$?

ii. What condition relates a_{nl} to a_n?

(c) Suppose we have found all the coefficients a_{nl}. Then $q(\rho, z, 0)$ is a sum over normal mode solutions. We can put in the time dependence given the initial conditions. If $q(\rho, z, 0)$ is as described above and $\dot{q}(\rho, z, 0) = 0$ then

$$q(\rho, z, t) = \sum_{n=0}^{\infty}\sum_{l=0}^{\infty} a_{nl} \sin(nz) J_0\left(x_{0l}\frac{\rho}{\rho_0}\right) \cos \omega_{nl} t$$

What is the frequency, ω_{nl}, of the (n, l) normal mode?

4. Next, we will study the wave equation in spherical coordinates, r, θ, φ.
   (a) The Cartesian coordinates are

$$\begin{aligned} x &= r\sin\theta\cos\varphi \\ y &= r\sin\theta\sin\varphi \\ z &= r\cos\theta \end{aligned}$$

   Solve for r, θ, φ in terms of x, y, z.

   (b) Find the partial derivatives

$$\frac{\partial r}{\partial x}, \frac{\partial r}{\partial y}, \frac{\partial r}{\partial z}$$
$$\frac{\partial \theta}{\partial x}, \frac{\partial \theta}{\partial y}, \frac{\partial \theta}{\partial z}$$

   (We already know the derivatives of φ from our study of cylindrical coordinates.)

11 Waves in 3 dimensions: Spherical

Read:

1. Torre
   - 13. Spherical Coordinates
2. Riffe
   - Lecture 22 Separation of Variables in Spherical Coordinates
   - Lecture 23 Spherical Coordinates II / A B. V. Problem / Separation of Variables Summary
3. Wheeler
   - Lecture Notes: The Laplace and Helmholz equations in spherical coordinates

Problems: (Due Thursday, November 14)

1. - 8. There are eight exercises in the Lecture notes. Work all eight. You are better off reading them within the notes since they frequently amplify results given there.

Problems: (Due Thursday, November 21)

1. Show that
$$j_1(x) = \frac{\sin x}{x^2} - \frac{\cos x}{x}$$
   is finite at $x = 0$ and find its value there.
2. (Riffe 25.4: Normal Modes Inside a Sphere (modified)) Starting with the general solution near the origin for spherical waves,
$$\psi(r, \theta, \varphi, t) = \psi(r, t) = \sum_{n=0}^{\infty}\sum_{l=0}^{\infty}\sum_{m=-l}^{l} j_l(k_n r)\, Y_{lm}(\theta, \varphi)\left(A_{lm}\sin\omega_n t + B_{lm}\cos\omega_n t\right)$$
   find the most general solution to the wave equation inside a sphere satisfying the conditions below. You will need
$$j_0(x) = \frac{\sin x}{x}$$
   Find all solutions satisfying the following:

(a) $\psi(r, \theta, \varphi, t)$ is spherically symmetric. (This means no angular dependence!)
(b) $\psi(R, t) = 0$
(c) $\psi(r, t)$ is finite everywhere.
(d) $\psi(r, t)$ is real.

12 Third Midterm Exam, Thursday, November 21 (see Lecture Notes)

The exam will consist of 6-8 short-to-medium length problems similar to the problem assignments. Or maybe 9. Therefore the problems are your best study guide. You may bring a 3×3 card with formulas; you may use both sides. Bring a pen/pencil. Paper will be provided.

13 The continuity equation and the divergence theorem; Stokes' theorem. More to be announced.

You do ***not*** need these readings before the third Midterm Exam.

Read:

1. Torre
   - 14. Conservation of Energy
2. Riffe
   - Lecture 24 Energy Density/Energy Flux/Total Energy in 1D
   - Lecture 24 Energy Density/Energy Flux/Total Energy in 3D
3. Wheeler
   - Lecture Notes: The continuity equation; Stokes' theorem

Problems: (Due Thursday, December 5) TBA