Dynamics in special relativity

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Contents

1 Spacetime

The elements of this section are developed in greater depth in a power point presentation, available HERE.

The first thing to understand clearly is the difference between physical quantities such as the length of a ruler or the elapsed time on a clock, and the coordinates we use to label locations in the world. In 3-dim Euclidean geometry, for example, the length of a ruler is given in terms of coordinate intervals using the Pythagorean theorem. Thus, if the positions of the two ends of the ruler are (x_1, y_1, z_1) and (x_2, y_2, z_2), the length is

$$L = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

Observe that the actual values of (x_1, y_1, z_1) are irrelevant. Sometimes we choose our coordinates cleverly, say, by aligning the x-axis with the ruler and placing one end at the origin so that the endpoints are at $(0,0,0)$ and $(x_2, 0, 0)$. Then the *calculation* of L is trivial:

$$\begin{aligned} L &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2} \\ &= x_2 \end{aligned}$$

but it is still important to recognize the difference between the coordinates and the length.

With this concept clear, we next need a set of labels for spacetime. Starting with a blank page to represent spacetime, we start to construct a set of labels. First, since all observers agree on the motion of light, let's agree that (with time flowing roughly upward in the diagram and space extending left and right) light beams always move at 45 degrees in a straight line. An inertial observer (whose constant rate of motion has no absolute reality; we only consider the relative motions of two observers) will move in a straight line at a steeper angle than 45 degrees – a lesser angle would correspond to motion faster than the speed of light. For any such inertial observer, we let the time coordinate be the time as measured by a clock they carry. The ticks of this clock provide a time scale along the straight, angled world line of the observer. To set spatial coordinates, we use the constancy of the speed of light. Suppose our inertial observer send out a pulse of light at 3 minutes before noon, and suppose the nearby spacetime is dusty enough that bits of that pulse are reflected back continuously. Then some reflected light will arrive back at the observer at 3 minutes after noon. Since the trip out and the trip back must have taken the same length of time and occurred with the light moving at constant velocity, the reflection of the light by the dust particle must have occurred at noon in our observer's frame of reference. It must have occurred at a distance of 3 light minutes away. If we take the x direction to be the direction the light was initially sent, the location of the dust particle has coordinates $(noon, 3\, light\, minutes, 0, 0)$. In a similar way, we find the locus of all points with time coordinate $t = noon$ and both $y = 0$ and $z = 0$. These points form our x axis. We find the y and z axes in the same way. It is somewhat startling to realize when we draw a careful diagram of this construction, that the x axis seems to make an acute angle with the time axis, as if the time axis has been reflected about the 45 degree path of a light beam. We quickly notice that this must always be the case if all observers are to measure the same speed ($c = 1$ in our construction) for light.

This gives us our labels for spacetime *events*. Any other set of labels would work just as well. In particular, we are interested in those other sets of coordinates we get by choosing a different initial world line of an different inertial observer. Suppose we consider two inertial observers moving with *relative* velocity v. Using such devices as mirror clocks and other thought experiments, most elementary treatments of special relativity quickly arrive at the relationship between such a set of coordinates. If the relative motion is in the x direction, the transformation between the two frames of reference is the familiar Lorentz transformation:

$$\begin{aligned} t' &= \gamma\left(t - \frac{vx}{c^2}\right) \\ x' &= \gamma\,(x - vt) \\ y' &= y \\ z' &= z \end{aligned}$$

where

$$\gamma \equiv \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$$

The next step is the most important: we must find a way to write physically meaningful quantities. These quantities, like length in Euclidean geometry, must be independent of the labels, the coordinates, that we put on different points. If we get on the right track by forming a quadratic expression similar to the Pythagorean theorem, then it doesn't take long to arrive at the correct answer. In spacetime, we have a *pseudo*-Euclidean length interval, given by

$$c^2\tau^2 = c^2t^2 - x^2 - y^2 - z^2 \tag{1}$$

Computing the same quantity in the primed frame, we find

$$\begin{aligned} c^2\tau'^2 &= c^2t'^2 - x'^2 - y'^2 - z'^2 \\ &= c^2\gamma^2\left(t - \frac{vx}{c^2}\right)^2 - \gamma^2\,(x - vt)^2 - y^2 - z^2 \\ &= c^2\gamma^2\left(t^2 - \frac{2vxt}{c^2} + \frac{v^2x^2}{c^4}\right) \end{aligned}$$

$$
\begin{aligned}
& -\gamma^2\left(x^2-2xvt+v^2t^2\right)-y^2-z^2 \\
= & \ \gamma^2\left(c^2t^2-v^2t^2-x^2+\frac{v^2x^2}{c^2}\right)-y^2-z^2 \\
= & \ c^2t^2-x^2-y^2-z^2 \\
= & \ c^2\tau^2
\end{aligned}
$$

so that $\tau=\tau'$. Tau is called the proper time, and is invariant under Lorentz transformations. It plays the role of L in spacetime geometry, and becomes the defining property of spacetime symmetry: *we define Lorentz transformations to be those transformations that leave τ invariant.*

From the power point presentation, you know that spacetime is a four dimensional vector space vector length

$$
\begin{aligned}
s^2 &= -c^2(\Delta t)^2+(\Delta x)^2+(\Delta y)^2+(\Delta z)^2 \\
c^2\tau^2 &= c^2(\Delta t)^2-(\Delta x)^2-(\Delta y)^2-(\Delta z)^2
\end{aligned}
$$

where $\Delta t=t_2-t_1, \Delta x=x_2-x_1, \Delta y=y_2-y_1, \Delta z=z_2-z_1$ are the coordinate differences of the events at the ends of the vector. We may also write this length for infinitesimal proper time $d\tau$ and infinitesimal proper length ds along an infinitesimal interval,

$$
\begin{aligned}
c^2d\tau^2 &= c^2dt^2-dx^2-dy^2-dz^2 && (2)\\
ds^2 &= -c^2dt^2+dx^2+dy^2+dz^2 && (3)
\end{aligned}
$$

and these are agreed upon by all observers. The latter form allows us to find the proper time or length of an arbitrary curve in spacetime by integrating $d\tau$ or ds along the curve.

The elapsed physical time experienced traveling along any timelike curve is given by integrating $d\tau$ along that curve. Similarly, the proper distance along any spacelike curve is found by integrating ds.

The set of points at zero proper interval, $s^2=0$, from a given point, P, is the light cone of that point. The light cone divides spacetime into regions. Points lying inside the light cone and having later time than P lie in the *future* of P. Points inside the cone with earlier times lie in the *past* of P. Points outside the cone are called *elsewhere*.

Timelike vectors from P connect P to past or future points. Timelike curves, $x^\alpha(\lambda)$, are curves whose tangent vector $\frac{dx^\alpha}{d\lambda}(\lambda)$ at any point are timelike vectors, while spacelike curves have tangents lying outside the lightcones of their points. The elapsed physical time experienced traveling along any timelike curve is given by integrating $d\tau$ along that curve. Similarly, the proper distance along any spacelike curve is found by integrating ds.

We refer to the coordinates of an event in spacetime using the four coordinates

$$x^\alpha=(ct,x,y,z)$$

where $\alpha=0,1,2,3$. We may also write x^α in any of the following ways:

$$
\begin{aligned}
x^\alpha &= (ct,\mathbf{x}) \\
&= \left(ct,x^i\right) \\
&= \left(x^0,x^i\right)
\end{aligned}
$$

where $i=1,2,3$. This neatly separates the familiar three spatial components of the vector x^α from the time component, allowing us to recognize familiar relations from non-relativistic mechanics.

The invariant interval allows us to define a metric,

$$
\eta_{\alpha\beta}\equiv\left(\begin{array}{cccc} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{array}\right)
$$

so that we may write the invariant interval as

$$s^2 = \eta_{\alpha\beta} x^\alpha x^\beta$$

or infinitesimally

$$ds^2 = \eta_{\alpha\beta} dx^\alpha dx^\beta$$

where our use of the Einstein summation convention tells us that these expressions are summed over $\alpha, \beta = 0, 1, 2, 3$. A Lorentz transformation may be defined as any transformation of the coordinates that leaves the length-squared s^2 unchanged. It follows that a linear transformation,

$$y^\alpha = \Lambda^\alpha{}_\beta x^\beta$$

is a Lorentz transformation if and only if

$$\eta_{\alpha\beta} x^\alpha x^\beta = \eta_{\alpha\beta} y^\alpha y^\beta$$

Substituting for y^α and equating the coefficients of the arbitrary symmetric $x^\alpha x^\beta$ we have

$$\eta_{\mu\nu} = \eta_{\alpha\beta} \Lambda^\alpha{}_\mu \Lambda^\beta{}_\nu \tag{4}$$

as the necessary and sufficient condition for $\Lambda^\beta{}_\nu$ to be a Lorentz transformation.

2 Relativistic dynamics

2.1 Curves

We now turn to look at motion in spacetime. Consider a particle moving along a curve in spacetime. We can write that curve parametrically by giving each coordinate as a function of some parameter λ:

$$x^\alpha = x^\alpha (\lambda)$$

Such a path is called the *world line* of the particle. Here λ can be any parameter that increases monotonically along the curve. We note two particularly convenient choices for λ. First, we may use the time coordinate, t, in relative to our frame of reference. In this case, we have

$$x^\alpha (t) = (ct, x(t), y(t), z(t))$$

As we shall see, the proper time τ experienced by the particle is often a better choice. Then we have

$$x^\alpha (\tau) = (ct(\tau), x(\tau), y(\tau), z(\tau))$$

The proper time is an excellent choice because it provides the same parameterization in *any* frame of reference.

To calculate the proper time experienced along the world line of the particle between events A and B, just add up the infinitesimal displacements $d\tau$ along the path. Thus

$$\begin{aligned} \tau_{AB} &= \int_A^B d\tau \\ &= \int_A^B \sqrt{dt^2 - \frac{1}{c^2} (dx^i)^2} \\ &= \int_{t_A}^{t_B} dt \sqrt{1 - \frac{1}{c^2} \left(\frac{dx^i}{dt} \right)^2} \\ &= \int_{t_A}^{t_B} dt \sqrt{1 - \frac{\mathbf{v}^2}{c^2}} \end{aligned}$$

where $\mathbf{v}^2$ is the usual squared magnitude of the 3-velocity. Notice that if $\mathbf{v}^2$ is ever different from zero, then τ_{AB} is *smaller* than the time difference $t_B - t_A$:

$$\tau_{AB} = \int_{t_A}^{t_B} dt \sqrt{1 - \frac{\mathbf{v}^2}{c^2}} \leq \int_{t_A}^{t_B} dt = t_B - t_A$$

Equality holds only if the particle remains at rest in the given frame of reference. This difference has been measured to high accuracy. One excellent test is to study the number of muons reaching the surface of the earth after being formed by cosmic ray impacts on the top of the atmosphere. These particles have a halflife on the order of 10^{-11} seconds, so they would normally travel only a few centimeters before decaying. However, because they are produced in a high energy collision that leaves them travelling toward the ground at nearly the speed of light, many of them are detected at the surface of the earth.

In order to discuss dynamics in spacetime, we need a clearer definition of a spacetime vector. So far, we have taken a vector to be the directed segment between two spacetime events. This has a useful consequence. As we have seen, the components of the vectors to the endpoints of such segments transform via Lorentz transformations, $y^\alpha = \Lambda^\alpha{}_\beta x^\beta$. We now generalize from a vector as a directed spacetime interval to define a 4-vector as *any* set of four functions, $v^\alpha = (a, b, c, d)$, as long as in any other frame of reference the components $\tilde{v}^\alpha = \left(\tilde{a}, \tilde{b}, \tilde{c}, \tilde{d}\right)$ are given by

$$\tilde{v}^\alpha = \Lambda^\alpha{}_\beta v^\beta \tag{5}$$

This generalization is important because we are interested in velocities, forces and other physical quantities which are not simply spacetime intervals.

Identifying objects which transform in the same way as the coordinates allows us to generalize the invariance of the interval to the invariance of 4-vector length,

$$\eta_{\alpha\beta} v^\alpha v^\beta = \eta_{\alpha\beta} \tilde{v}^\alpha \tilde{v}^\beta$$

which follows by combining our new definition of a Lorentz transformation, eq.(4), with the transformation property of v^α, eq.(5).

2.2 The 4-velocity

We next define the 4-velocity of a particle, i.e., a 4-vector that characterizes the velocity. We can get the direction of the particle's motion in spacetime by looking at the tangent vector to the curve,

$$t^\alpha = \frac{dx^\alpha(\lambda)}{d\lambda}$$

We can see that this tangent vector is closely related to the ordinary 3-velocity of the particle by expanding with the chain rule,

$$\begin{aligned} t^\alpha &= \frac{dx^\alpha(\lambda)}{d\lambda} \\ &= \frac{dt}{d\lambda} \frac{dx^\alpha}{dt} \\ &= \frac{dt}{d\lambda} \frac{d}{dt}\left(ct, x^i\right) \\ &= \frac{dt}{d\lambda}\left(c, v^i\right) \end{aligned}$$

where v^i is the usual Newtonian 3-velocity. This is close to what we need, but since λ is arbitrary, so is $\frac{dt}{d\lambda}$. This means that t^α may or may not be a vector. For example, suppose, in some frame of reference, we have chosen coordinate time t as the parameter. Then in that frame of reference, we have

$$\begin{aligned}
t^\alpha &= \frac{dx^\alpha(t)}{dt} \\
&= \frac{d}{dt}\left(ct, x(t), y(t), z(t)\right) \\
&= \left(c, \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}\right) \\
&= (c, \mathbf{v})
\end{aligned}$$

This is not a vector by our definition, because when we transform to a new frame of reference, $\tilde{x}^\alpha = \Lambda^\alpha{}_\beta x^\beta$, we have

$$\begin{aligned}
\tilde{t}^\alpha &= \frac{d\tilde{x}^\alpha}{d\tilde{t}} \\
& \frac{d\left(\Lambda^\alpha{}_\beta x^\beta\right)}{\frac{1}{c} d\left(\Lambda^0{}_\mu x^\mu\right)} \\
&= \frac{\Lambda^\alpha{}_\beta dx^\beta}{\frac{1}{c}\Lambda^0{}_\mu dx^\mu} \\
&= \frac{\Lambda^\alpha{}_\beta dx^\beta}{\frac{1}{c}\left(\Lambda^0{}_0 dx^0 + \Lambda^0{}_i dx^i\right)} \\
&= \frac{\Lambda^\alpha{}_\beta dx^\beta}{\Lambda^0{}_0 dt + \frac{1}{c}\Lambda^0{}_i dx^i} \\
&= \frac{1}{\Lambda^0{}_0 + \frac{1}{c}\Lambda^0{}_i v^i}\Lambda^\alpha{}_\beta \frac{dx^\beta}{dt}
\end{aligned}$$

so we get an extraneous factor of $\left(\Lambda^0{}_0 + \frac{1}{c}\Lambda^0{}_i v^i\right)^{-1}$.

We can define a true 4-vector by using the proper time as the parameter. Let the world line be parameterized by the elapsed proper time, τ, of the particle. Then define the *4-velocity,*

$$u^\alpha = \frac{dx^\alpha(\tau)}{d\tau}$$

Since $\tau = \tilde{\tau}$ for all observers, we immediately have

$$\begin{aligned}
\tilde{u}^\alpha &= \frac{d\tilde{x}^\alpha(\tau)}{d\tilde{\tau}} \\
&= \frac{d\left(\Lambda^\alpha{}_\beta x^\beta\right)}{d\tau} \\
&= \Lambda^\alpha{}_\beta \frac{dx^\beta}{d\tau} \\
&= \Lambda^\alpha{}_\beta u^\beta
\end{aligned}$$

and u^α is a 4-vector.

A very convenient form for the 4-velocity is given by our expansion of the tangent vector. Just as for general λ, we have

$$u^\alpha = \frac{dt}{d\tau}\left(c, v^i\right)$$

but now we know what the function in front is. Compute

$$\begin{aligned} d\tau &= \sqrt{dt^2 - \frac{1}{c^2}\left(dx^i\right)^2} \\ &= dt\sqrt{1 - \frac{\mathbf{v}^2}{c^2}} \end{aligned}$$

Then we see that

$$\frac{dt}{d\tau} = \frac{1}{\sqrt{1 - \frac{\mathbf{v}^2}{c^2}}} = \gamma$$

Therefore,

$$u^\alpha = \gamma\left(c, v^i\right) \tag{6}$$

This is an extremely useful form for the 4-velocity. It is used frequently.

Since u^α is a 4-vector, its magnitude

$$\eta_{\alpha\beta}u^\alpha u^\beta$$

must be invariant! This means that the velocity of every particle in spacetime has the same particular value. Let's compute it:

$$\begin{aligned} \eta_{\alpha\beta}u^\alpha u^\beta &= -\left(u^0\right)^2 + \sum_i \left(u^i\right)^2 \\ &= -\gamma^2 c^2 + \gamma^2 \mathbf{v}^2 \\ &= \frac{-c^2 + \mathbf{v}^2}{1 - \frac{\mathbf{v}^2}{c^2}} \\ &= -c^2 \end{aligned}$$

This is indeed invariant! Our formalism is doing what it is supposed to do.

Now let's look at how the 4-velocity is related to the usual 3-velocity. If $\mathbf{v}^2 \ll c^2$, the components of the 4-velocity are just

$$u^\alpha = \gamma\left(c, v^i\right) \approx \left(c, v^i\right)$$

The speed of light, c, is just a constant, and the spatial components reduce to precisely the Newtonian velocity. This is just right. Moreover, it takes no new information to write the general form of u^α once we know v^i – there is no new information, just a different form.

2.3 Energy and momentum

From the 4-velocity it is natural to define the 4-momentum by multiplying by the mass,

$$p^\alpha = mu^\alpha$$

In order for the 4-momentum to be a vector, we require $\tilde{p}^\alpha = \Lambda^\alpha{}_\beta p^\beta$. Since u^α is itself a vector, we have

$$\begin{aligned} \tilde{p}^\alpha &= \tilde{m}\tilde{u}^\alpha \\ &= \tilde{m}\Lambda^\alpha{}_\beta u^\beta \end{aligned}$$

Since this must equal $\Lambda^\alpha{}_\beta p^\beta$, the mass of a particle is *invariant,*

$$\tilde{m} = m$$

As we might expect, the 3-momentum part of p^α is closely related to the Newtonian expression mv^i. In general it is

$$p^i = \frac{mv^i}{\sqrt{1-\frac{v^2}{c^2}}}$$

If $v \ll c$ we may expand the denominator to get

$$\begin{aligned} p^i &\approx mv^i\left(1+\frac{1}{2}\frac{v^2}{c^2}+\cdots\right) \\ &\approx mv^i \end{aligned}$$

Thus, while relativistic momentum differs from Newtonian momentum, they only differ at order $\frac{v^2}{c^2}$. Even for the 7 $mi/\sec$ velocity of a spacecraft which escapes Earth's gravity this ratio is only

$$\frac{v^2}{c^2} = 1.4\times 10^{-9}$$

so the Newtonian momentum is correct to parts per billion. In particle accelerators, however, where near-light speeds are commonplace, the difference is substantial (see exercises).

Now consider the remaining component of the 4-momentum. Multiplying by c and expanding γ we find

$$\begin{aligned} p^0 c &= mc^2\gamma \\ &= mc^2\left(1+\frac{1}{2}\frac{v^2}{c^2}+\frac{3}{8}\frac{v^4}{c^4}\cdots\right) \\ &\approx mc^2+\frac{1}{2}mv^2+\frac{3}{8}mv^2\frac{v^2}{c^2} \end{aligned}$$

The third term is negligible at ordinary velocities, while we recognize the second term as the usual Newtonian kinetic energy. We therefore identify $E = p^0c$. Since the first term is constant it plays no measurable role in classical mechanics but it suggests that there is intrinsic energy associated with the mass of an object. This conjecture is confirmed by observations of nuclear decay. In such decays, the mass of the initial particle is greater than the sum of the masses of the product particles, with the energy difference

$$\Delta E = m_{initial}c^2 - \sum m_{final}c^2$$

correctly showing up as kinetic energy.

Notice the difference between the mass and energy of a particle. The *mass* is independent of frame of reference – the mass of an electron is always 511 keV – but the energy, $E = mc^2\gamma$, depends on the velocity. In earlier texts the factor of γ is often associated with the mass, but this is not consistent with $p^\alpha = mu^\alpha$ for 4-vectors p^α, u^α.

3 Acceleration

Next we consider acceleration. We define the acceleration 4-vector to be the proper-time rate of change of 4-velocity,

$$\begin{aligned} a^\alpha &= \frac{du^\alpha}{d\tau} \\ &= \frac{dt}{d\tau}\frac{d\left(\gamma\left(c,v^i\right)\right)}{dt} \\ &= \gamma\left(-\frac{1}{2}\gamma^3\left(-2\frac{v^m a_m}{c^2}\right)\left(c,v^i\right)+\gamma\left(0,a^i\right)\right) \\ &= \frac{v^m a_m}{c^2}\gamma^3 u^\alpha+\gamma^2\left(0,a^i\right) \end{aligned}$$

Is this consistent with our expectations?

If we are in the instantaneous rest frame of the particle (or the particle momentarily at rest in our frame) then $v^i = 0$ and

$$a^\alpha = \left(0, a^i\right)$$

and the first term remains small except when $v^m a_m$ approaches c^2. This is consistent with classical experiments.

We also know that

$$u^\alpha u_\alpha = -c^2$$

which means that

$$\begin{aligned} 0 &= \frac{d}{d\tau}\left(-c^2\right) \\ &= 2\frac{du^\alpha}{d\tau}u_\alpha \end{aligned}$$

Therefore, the 4-velocity and 4-acceleration are orthogonal, which we easily verify directly,

$$\begin{aligned} u^\alpha a_\alpha &= u_\alpha\left(\frac{v^m a_m}{c^2}\gamma^3 u^\alpha + \gamma^2\left(0, a^i\right)\right) \\ &= \left(\frac{v^m a_m}{c^2}\gamma^3\left(-c^2\right) + \gamma\left(-c, v_i\right)\cdot\gamma^2\left(0, a^i\right)\right) \\ &= -v^m a_m \gamma^3 + \gamma^3 a^i v_i \\ &= 0 \end{aligned}$$

Now compute $a^\alpha a_\alpha$:

$$\begin{aligned} a^\alpha a_\alpha &= \left(\frac{v^m a_m}{c^2}\gamma^3 u^\alpha + \gamma^2\left(0, a^i\right)\right)a_\alpha \\ &= \gamma^2\left(0, a^i\right)a_\alpha \\ &= \gamma^2\left(0, a^i\right)\cdot\left(\frac{v^m a_m}{c^2}\gamma^4\left(c, v_i\right) + \gamma^2\left(0, a_i\right)\right) \\ &= \frac{v^m a_m}{c^2}\gamma^6 a^i v_i + \gamma^4 a^i a_i \\ &= \gamma^4\left(a^i a_i + \gamma^2\frac{\left(v^m a_m\right)^2}{c^2}\right) \end{aligned}$$

This expression gives the acceleration of a particle moving with relative velocity v^i when the acceleration in the instantaneous rest frame of the particle is given by the $v^i = 0$ expression

$$a^\alpha a_\alpha = a^i a_i$$

We consider two cases. First, suppose v^i is parallel to a^i. Then since $a^\alpha a_\alpha$ is invariant, the 3-acceleration is given by

$$\begin{aligned} a^\alpha a_\alpha &= \gamma^4\left(a^2 + \gamma^2\frac{v^2 a^2}{c^2}\right) \\ &= \gamma^6 a^i a_i \end{aligned}$$

or

$$a^i a_i = a^\alpha a_\alpha\left(1 - \frac{v^2}{c^2}\right)^3$$

where $a^\alpha a_\alpha$ is independent of v^i. Therefore, as the particle nears the speed of light, its apparent 3-acceleration decreases dramatically. When the acceleration is orthogonal to the velocity, the exponent is reduced,

$$a^i a_i = a^\alpha a_\alpha \left(1 - \frac{v^2}{c^2}\right)^2$$

4 Equations of motion from an action

4.1 Free particle

The relativistic action for a free particle is surprisingly simple. To derive a suitable equation of motion, we once again start with arc length. Suppose we have a timelike curve $x^\alpha(\lambda)$. Then distance along the curve is given by

$$\tau = -\frac{1}{c^2}\int \sqrt{(-v^\alpha v_\alpha)}d\lambda$$

where

$$v^\alpha = \frac{dx^\alpha}{d\lambda}$$

Since the integral is reparameterization invariant, there is no loss of generality if we use the 4-velocity in place of v^a and write

$$\tau_C = -\frac{1}{c^2}\int_C \sqrt{(-u^\alpha u_\alpha)}d\tau$$

Then the path of extremal proper time is given by the Euler-Lagrange equation

$$\frac{d}{d\tau}\frac{\partial}{\partial u^\beta}\left(-\frac{1}{c^2}u^\alpha u_\alpha\right) = 0$$

that is, vanishing 4-acceleration,

$$\frac{du^\alpha}{d\tau} = 0$$

4.2 Relativistic action with a potential

We can easily generalize this expression to include a potential. For relativistic problems it is possible to keep the action reparameterization invariant. To do so, we must multiply the line element by a function instead of adding the function. This gives

$$\tau_C = \frac{1}{c}\int_C \phi\sqrt{(-u^\alpha u_\alpha)}d\tau$$

The Euler-Lagrange equation is

$$\begin{aligned}
\frac{d}{d\tau}\left(-\phi\left(-u^\alpha u_\alpha\right)^{-1/2}u_\alpha\right) - \left(-u^\alpha u_\alpha\right)^{1/2}\frac{\partial\phi}{\partial x^\alpha} &= 0 \\
\frac{1}{c^2}\frac{d}{d\tau}\left(\phi u_\alpha\right) + \frac{\partial\phi}{\partial x^\alpha} &= 0
\end{aligned}$$

where we have simplified using the normalization condition $c = \left(-u^\alpha u_\alpha\right)^{1/2}$. Expanding the derivatives, and rearranging,

$$\begin{aligned}
0 &= \frac{1}{c^2}\frac{du_\alpha}{d\tau}\phi + \frac{1}{c^2}u_\alpha\frac{d\phi}{d\tau} + \frac{\partial\phi}{\partial x^\alpha} \\
&= \frac{1}{c^2}\frac{du_\alpha}{d\tau}\phi + \frac{1}{c^2}u_\alpha\frac{dx^\beta}{d\tau}\frac{\partial\phi}{\partial x^\beta} + \delta^\beta_\alpha\frac{\partial\phi}{\partial x^\beta} \\
&= \frac{1}{c^2}\frac{du_\alpha}{d\tau}\phi + \left(\frac{1}{c^2}u^\beta u_\alpha + \delta^\beta_\alpha\right)\frac{\partial\phi}{\partial x^\beta}
\end{aligned}$$

Notice that

$$P^{\beta}{}_{\alpha} = \delta^{\beta}_{\alpha} + \frac{1}{c^2} u^{\beta} u_{\alpha}$$

is a projection operator, because

$$\begin{aligned}
P^{\mu}{}_{\beta} P^{\beta}{}_{\alpha} &= \left(\delta^{\mu}_{\beta} + \frac{1}{c^2} u^{\mu} u_{\beta}\right)\left(\delta^{\beta}_{\alpha} + \frac{1}{c^2} u^{\beta} u_{\alpha}\right) \\
&= \delta^{\mu}_{\beta}\delta^{\beta}_{\alpha} + \frac{1}{c^2}\delta^{\mu}_{\beta} u^{\beta} u_{\alpha} + \frac{1}{c^2} u^{\mu} u_{\beta}\delta^{\beta}_{\alpha} + \frac{1}{c^4} u^{\mu} u_{\beta} u^{\beta} u_{\alpha} \\
&= \delta^{\mu}_{\alpha} + \frac{1}{c^2} u^{\mu} u_{\alpha} + \frac{1}{c^2} u^{\mu} u_{\alpha} - \frac{1}{c^2} u^{\mu} u_{\alpha} \\
&= P^{\mu}{}_{\alpha}
\end{aligned}$$

Indeed, it projects into directions orthogonal to the 4-velocity, giving zero when we act on u^{β}:

$$\begin{aligned}
P^{\alpha}{}_{\beta} u^{\beta} &= \left(\delta^{\alpha}_{\beta} + \frac{1}{c^2} u^{\alpha} u_{\beta}\right) u^{\beta} \\
&= \left(u^{\alpha} + \frac{1}{c^2} u^{\alpha}\left(u_{\beta} u^{\beta}\right)\right) \\
&= 0
\end{aligned}$$

Notice that the projection is symmetric, $P^{\alpha}{}_{\beta} = P_{\beta}{}^{\alpha}$

Now we may write the equation of motion as

$$\frac{1}{c^2}\frac{du_{\alpha}}{d\tau}\phi = -P_{\alpha}{}^{\beta}\frac{\partial\phi}{\partial x^{\beta}}$$

The projection operator is necessary because the acceleration term is orthogonal to u^{α}. Dividing by $\frac{\phi}{c^2}$, we see that

$$\frac{du_{\alpha}}{d\tau} = -c^2 P_{\alpha}{}^{\beta}\frac{\partial \ln\phi}{\partial x^{\beta}}$$

If we identify

$$\phi = \exp\left(\frac{V}{mc^2}\right)$$

then we arrive at the desired equation of motion

$$m\frac{du_{\alpha}}{d\tau} = -P_{\alpha}{}^{\beta}\frac{\partial V}{\partial x^{\beta}}$$

which now is seen to follow as the extremum of the functional

$$S\left[x^a\right] = \frac{1}{c}\int_C e^{\frac{V}{mc^2}}\left(-u^{\alpha}u_{\alpha}\right)^{1/2} d\tau \tag{7}$$

See the exercises for other ways of arriving at this result.

It is suggestive to notice that the integrand is simply the usual line element multiplied by a scale factor.

$$\begin{aligned}
d\sigma^2 &= \frac{1}{c^2} e^{\frac{2V}{mc^2}}\left(-u^{\alpha}u_{\alpha}\right) d\tau^2 \\
&= -e^{\frac{2V}{mc^2}} ds^2
\end{aligned}$$

This is called a *conformal line element* because it is formed from a metric which is related to the flat space metric by a conformal factor, $e^{\frac{V}{mc^2}}$,

$$g_{\alpha\beta} = e^{\frac{2V}{mc^2}}\eta_{\alpha\beta}$$

Conformal transformations also appear in the study of Hamiltonian mechanics.

We can generalize the action further by observing that the potential is the integral of the force along a curve,

$$V = -\int_C F_\alpha dx^\alpha$$

The potential is defined only when this integral is single valued. By Stoke's theorem, this occurs if and only if the force is curl-free. But even for general forces we can write the action as

$$S[x^a] = \frac{1}{c}\int_C e^{-\frac{1}{mc^2}\int_C F_\alpha dx^\alpha} (-u^\alpha u_\alpha)^{1/2} d\tau$$

In this case, variation leads to

$$\begin{aligned} 0 &= \delta S[x^a] \\ &= \frac{1}{c}\int_C \left(-\frac{1}{mc^2} e^{-\frac{1}{mc^2}\int_C F_\alpha dx^\alpha} \left(\frac{\partial}{\partial x^\alpha}\int_C F_\beta dx^\beta\right)\delta x^\alpha\right)(-u^\alpha u_\alpha)^{1/2} d\tau \\ &\quad + \frac{1}{c}\int_C e^{-\frac{1}{mc^2}\int_C F_\alpha dx^\alpha} \frac{1}{2(-u^\alpha u_\alpha)^{1/2}}(-2u_\alpha \delta u^\alpha) d\tau \end{aligned}$$

Now, replacing $-u^\alpha u_\alpha = c^2$, and integrating the second term by parts,

$$\begin{aligned} 0 &= \int_C \left(-\frac{1}{mc^2} e^{-\frac{1}{mc^2}\int_C F_\alpha dx^\alpha} \left(\frac{\partial}{\partial x^\alpha}\int_C F_\beta dx^\beta\right)\delta x^\alpha\right) d\tau \\ &\quad + \int_C \frac{d}{d\tau}\left(e^{-\frac{1}{mc^2}\int_C F_\alpha dx^\alpha}\frac{1}{c^2}u_\alpha\right)\delta x^\alpha d\tau \\ &= \int_C \left(-\frac{1}{mc^2} e^{-\frac{1}{mc^2}\int_C F_\alpha dx^\alpha} \left(\frac{\partial}{\partial x^\alpha}\int_C F_\beta dx^\beta\right)\delta x^\alpha\right) d\tau \\ &\quad + \int_C \left(\frac{1}{c^2}u_\alpha \frac{d}{d\tau} e^{-\frac{1}{mc^2}\int_C F_\alpha dx^\alpha} + e^{-\frac{1}{mc^2}\int_C F_\alpha dx^\alpha}\frac{1}{c^2}\frac{du_\alpha}{d\tau}\right)\delta x^\alpha d\tau \end{aligned}$$

We need to evaluate the derivatives of the integrals, remembering that the integral is along the curve C with tangent u^α. For the first,

$$\frac{\partial}{\partial x^\alpha}\int_C F_\beta dx^\beta = F_\alpha$$

while for the second,

$$\begin{aligned} \frac{d}{d\tau} e^{-\frac{1}{mc^2}\int_C F_\alpha dx^\alpha} &= -\frac{1}{mc^2} e^{-\frac{1}{mc^2}\int_C F_\alpha dx^\alpha}\frac{d}{d\tau}\int_C F_\alpha dx^\alpha \\ &= -\frac{1}{mc^2} e^{-\frac{1}{mc^2}\int_C F_\alpha dx^\alpha}\frac{d}{d\tau}\int_C F_\alpha \frac{dx^\alpha}{d\tau} d\tau \\ &= -\frac{1}{mc^2} e^{-\frac{1}{mc^2}\int_C F_\alpha dx^\alpha} F_\alpha u^\alpha \end{aligned}$$

Combining these with the variation,

$$\begin{aligned} 0 &= \int_C \left(-\frac{1}{mc^2} e^{-\frac{1}{mc^2}\int_C F_\alpha dx^\alpha} F_\alpha \delta x^\alpha\right) d\tau \\ &\quad + \int_C \left(\frac{1}{c^2}u_\alpha\left(-\frac{1}{mc^2} e^{-\frac{1}{mc^2}\int_C F_\mu dx^\mu} F_\beta u^\beta\right) + e^{-\frac{1}{mc^2}\int_C F_\alpha dx^\alpha}\frac{1}{c^2}\frac{du_\alpha}{d\tau}\right)\delta x^\alpha d\tau \\ &= -\frac{1}{mc^2}\int_C e^{-\frac{1}{mc^2}\int_C F_\alpha dx^\alpha}\left(F_\alpha + \frac{1}{c^2}u_\alpha u^\beta F_\beta - m\frac{du_\alpha}{d\tau}\right)\delta x^\alpha d\tau \end{aligned}$$

Cancelling the overall factor of $-\frac{1}{mc^2}e^{-\frac{1}{mc^2}\int_C F_\alpha dx^\alpha}$, the equation of motion is

$$m\frac{du_\alpha}{d\tau} = F_\alpha + \frac{1}{c^2}u_\alpha u^\beta F_\beta$$

and therefore,

$$m\frac{du_\alpha}{d\tau} = P_\alpha{}^\beta F_\beta$$

This time the equation holds for an arbitrary relativistic force.

Finally, consider the non-relativistic limit of the action. If $v \ll c$ and $V \ll mc^2$ then to lowest order,

$$\begin{aligned}
S[x^a] &= \int_C e^{\frac{V}{mc^2}} d\tau \\
&= \int_C \left(1 + \frac{V}{mc^2}\right)\frac{1}{\gamma}dt \\
&= \frac{1}{mc^2}\int_C \left(mc^2 + V\right)\sqrt{1 - \frac{v^2}{c^2}}dt \\
&= -\frac{1}{mc^2}\int_C \left(mc^2\left(-1 + \frac{v^2}{2c^2}\right) - V\right)dt \\
&= -\frac{1}{mc^2}\int_C \left(-mc^2 + \frac{1}{2}mv^2 - V\right)dt \\
&= t_f - t_i - \frac{1}{mc^2}\int_C \left(\frac{1}{2}mv^2 - V\right)dt
\end{aligned}$$

Discarding the multiplier and irrelevant constant $t_f - t_i$, we recover

$$S_{Cl} = \int_C \left(\frac{1}{2}mv^2 - V\right)dt = \int_C (T - V)\,dt$$

Since the conformal line element is a more fundamental object than the classical action, this may be regarded as another derivation of the classical form of the Lagrangian, $L = T - V$.

Exercises

1. Suppose a muon is produced in the upper atmosphere moving downward at $v = .99c$ relative to the surface of Earth. If it decays after a proper time $\tau = 2.2 \times 10^{-6}$ seconds, how far would it travel if there were no time dilation? Would it reach Earth's surface? How far does it actually travel relative to Earth? Note that many muons are seen reaching Earth's surface.

2. A free neutron typically decays into a proton, an electron, and an antineutrino. How much kinetic energy is shared by the final particles?

3. Suppose a proton at Fermilab travels at $.99c$. Compare Newtonian energy, $\frac{1}{2}mv^2$ to the relativistic energy p^0c.

4. A proton at the LHC at CERN may currently be given an energy of 7 TeV. What is the speed of the proton?

5. A projection operator is an operator which is idempotent, that is, it is its own square.

   (a) Write $P^\alpha{}_\beta = \delta^\alpha_\beta + \frac{1}{c^2}u^\alpha u_\beta$ as a matrix in the rest frame (i.e., the inertial frame where u^α is simply $u^\alpha = (c, \mathbf{0})$.

(b) We showed that $P^{\beta}{}_{\alpha}u^{\alpha} = 0$. Show that if $u_{\alpha}w^{\alpha} = 0$, that $P^{\alpha}{}_{\beta}w^{\beta} = w^{\alpha}$.

6. Show that the 4-velocity takes the form $u^{\alpha} = (c, \mathbf{0})$ if and only if the 3-velocity vanishes.

7. Consider the action

$$S[x^a] = \int (mu^{\alpha}u_{\alpha} + \phi)\, d\tau$$

This is no longer reparameterization invariant, so we need an additional Lagrange multiplier term to enforce the constraint,

$$+\lambda\left(u^{\alpha}u_{\alpha} + c^2\right)$$

so the action becomes

$$S_1[x^a] = \int \left(mu^{\alpha}u_{\alpha} + c^2\phi + \lambda\left(u^{\alpha}u_{\alpha} + c^2\right)\right) d\tau$$

(a) Write the Euler-Lagrange equations (including the one arising from the variation of λ).

(b) The constraint implies $u^{\alpha}\frac{du_{\alpha}}{d\tau} = 0$. Solve for λ by contracting the equation of motion with u^{α}, using $u^{\alpha}\frac{du_{\alpha}}{d\tau} = 0$, and integrating. You should find that

$$\lambda = -\frac{1}{2}(\phi + a)$$

(c) Substitute λ back into the equation of motion and show that the choice

$$\ln\left(\frac{\phi - 2m + a}{\phi_0 - 2m + a}\right) = \frac{1}{mc^2}V$$

gives the correct equation of motion.

(d) Show, using the constraint freely, that S_1 is a multiple of the action of eq.(7).

8. Consider the action

$$S_2[x^a] = \int \left(mc^2\sqrt{1 - \frac{v^2}{c^2}} + V\right) dt$$

Show that S_2 has the correct low-velocity limit, $L = T - V$. Show that the Euler-Lagrange equation following from the variation of S_2 is *not* covariant. S_2 is therefore unsatisfactory.

9. Consider the action

$$S_3[x^a] = \int (mu^{\alpha}u_{\alpha} - 2V)\, d\tau$$

(a) Show that the Euler-Lagrange equation for S_3 is

$$\frac{d}{d\tau}mu_{\alpha} = -\frac{\partial V}{\partial x^{\alpha}}$$

(b) Show that the constraint, $u^{\alpha}u_{\alpha} = -c^2$ is not satisfied for general potentials V.

(c) Show that S_3 has the wrong low-velocity limit.

5 Lorentz tensors and invariant tensors

5.1 Lorentz tensors

Recall that we have define Lorentz transformations as those linear transformations $\Lambda^{\alpha}{}_{\mu}$ that leave the Minkowski metric invariant, eq.(4). A Lorentz vector is then any set of four quantities that transform just like the coordinates,

$$(w')^{\alpha} = \Lambda^{\alpha}{}_{\beta} w^{\beta}$$

It follows immediately that $w_{\alpha} w^{\alpha}$ is invariant under Lorentz transformations. As long as we are careful to use only quantities that have such simple transformations (i.e., linear and homogeneous) it is easy to construct Lorentz invariant quantities by "contracting" indices. Any time we sum one contravariant vector index with one covariant vector index, we produce an invariant.

We will shortly see other objects with linear, homogeneous transformations under the Lorentz group. Some have multiple indices,

$$T^{\alpha\beta\ldots\mu}$$

and transform linearly on each index,

$$(T')^{\alpha\beta\ldots\mu} = \Lambda^{\alpha}{}_{\rho} \Lambda^{\beta}{}_{\sigma} \Lambda^{\mu}{}_{\nu} T^{\rho\sigma\ldots\nu}$$

The collection of all such objects is called the set of Lorentz tensors. More specifically, we are discussing the group of transformations (Exercise: prove that the Lorentz transformations form a group!) that preserves the matrix $diag(-1,1,1,1)$. This group is name $O(3,1)$, meaning the pseudo-orthogonal group that preserves the 4-dimensional metric with 3 plus and 1 minus sign. In general the group of transformations preserving $diag(1,\ldots 1,-1,\ldots -1)$ with p plus signs and q plus signs is named $O(p,q)$. From the definition of $\Lambda^{\alpha}{}_{\mu}$ via $\eta_{\mu\nu} = \eta_{\alpha\beta} \Lambda^{\alpha}{}_{\mu} \Lambda^{\beta}{}_{\nu}$ or, more concisely

$$\eta = \Lambda^{t} \eta \Lambda \tag{8}$$

we see that $(\det \Lambda)^2 = 1$ so that $\det \Lambda = \pm 1$. If we restrict to $\det \Lambda = +1$, the corresponding group is called $SO(3,1)$, where the S stands for "special".

5.2 Lorentz invariant tensors

Notice that the defining property of Lorentz transformations, eq.(4) or eq.(8), states the invariance of the metric $\eta_{\alpha\beta}$ under Lorentz transformations. This is a very special property – in general, the components of tensors are shuffled linearly by Lorentz transformations.

The Levi-Civita tensor, defined to be the unique, totally antisymmetric rank four tensor $\varepsilon_{\alpha\beta\mu\nu}$ with

$$\varepsilon_{0123} = 1$$

is the only other independent tensor which is Lorentz invariant. To see that $\varepsilon_{\alpha\beta\mu\nu}$ is invariant, we first note that it may be used to define determinants. For any matrix $M^{\alpha\beta}$, we may write

$$\begin{aligned} \det M &= \varepsilon_{\alpha\beta\mu\nu} M^{\alpha 0} M^{\beta 1} M^{\mu 2} M^{\nu 3} \\ &= \frac{1}{4!} \varepsilon_{\gamma\delta\rho\sigma} \varepsilon_{\alpha\beta\mu\nu} M^{\alpha\gamma} M^{\beta\delta} M^{\mu\rho} M^{\nu\sigma} \\ &= \frac{1}{4!} \varepsilon^{\gamma\delta\rho\sigma} \varepsilon_{\alpha\beta\mu\nu} M^{\alpha}{}_{\gamma} M^{\beta}{}_{\delta} M^{\mu}{}_{\rho} M^{\nu}{}_{\sigma} \end{aligned}$$

because the required antisymmetrizations are accomplished by the Levi-Civita tensor. An alternative way to write this is

$$(\det M)\, \varepsilon_{\gamma\delta\rho\sigma} = \varepsilon_{\alpha\beta\mu\nu} M^{\alpha}{}_{\gamma} M^{\beta}{}_{\delta} M^{\mu}{}_{\rho} M^{\nu}{}_{\sigma}$$

because the right side is totally antisymmetric on $\gamma\delta\rho\sigma$ and if we set $\gamma\delta\rho\sigma = 0123$ we get our original expression for $\det M$. Since this last expression holds for any matrix $M^{\alpha}{}_{\gamma}$, it holds for the Lorentz transformation matrix, $\Lambda^{\alpha}{}_{\gamma}$:

$$(\det \Lambda)\, \varepsilon_{\gamma\delta\rho\sigma} = \varepsilon_{\alpha\beta\mu\nu} \Lambda^{\alpha}{}_{\gamma} \Lambda^{\beta}{}_{\delta} \Lambda^{\mu}{}_{\rho} \Lambda^{\nu}{}_{\sigma}$$

However, since the determinant of a proper Lorentz transformation is $+1$, we have $(\det \Lambda) = 1$ and the invariance of the Levi-Civita tensor,

$$\varepsilon_{\gamma\delta\rho\sigma} = \varepsilon_{\alpha\beta\mu\nu} \Lambda^{\alpha}{}_{\gamma} \Lambda^{\beta}{}_{\delta} \Lambda^{\mu}{}_{\rho} \Lambda^{\nu}{}_{\sigma} \tag{9}$$

This also shows that under spatial inversion, which has $\det \Lambda = -1$, the Levi-Civita tensor changes sign. The presence of an odd number of Levi-Civita tensors in any relativistic expression therefore shows that that expression is odd under parity.

In fact, we need only know this parity argument for a single Levi-Civita tensor, because any pair of them may always be replaced by four antisymmetrized Kronecker deltas using

$$\varepsilon^{\alpha\beta\mu\nu} \varepsilon_{\gamma\delta\rho\sigma} = \delta^{\alpha}_{[\gamma} \delta^{\beta}_{\delta} \delta^{\mu}_{\rho} \delta^{\nu}_{\sigma]} \tag{10}$$

where the square brackets around the indices indicate antisymmetrization over all 24 permutations of $\gamma\delta\rho\sigma$, with the normalization $\frac{1}{4!}$. By taking one, two, three or four contractions we obtain the following identities:

$$\begin{aligned}
\varepsilon^{\alpha\beta\mu\nu} \varepsilon_{\alpha\delta\rho\sigma} &= 6\delta^{\beta}_{[\delta} \delta^{\mu}_{\rho} \delta^{\nu}_{\sigma]} && (11)\\
\varepsilon^{\alpha\beta\mu\nu} \varepsilon_{\alpha\beta\rho\sigma} &= 2\left(\delta^{\mu}_{\rho} \delta^{\nu}_{\sigma} - \delta^{\mu}_{\sigma} \delta^{\nu}_{\rho}\right) && (12)\\
\varepsilon^{\alpha\beta\mu\nu} \varepsilon_{\alpha\beta\mu\sigma} &= 6\delta^{\nu}_{\sigma} && (13)\\
\varepsilon^{\alpha\beta\mu\nu} \varepsilon_{\alpha\beta\mu\nu} &= 24 && (14)
\end{aligned}$$

Similar identities hold in every dimension. In n dimensions, the Levi-Civita tensor is of rank n. For example, the Levi-Civita tensor of Euclidean 3-space is ε_{ijk}, where $\varepsilon_{123} = 1$ and all other components follow using the antisymmetry. Along with the metric, $g_{ij} = \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix}$, ε_{ijk} is invariant under $SO(3)$. It is again odd under parity, and satisfies the following identities

$$\begin{aligned}
\varepsilon^{ijk} \varepsilon_{lmn} &= \delta^{i}_{[l} \delta^{j}_{m} \delta^{k}_{n]}\\
\varepsilon^{ijk} \varepsilon_{imn} &= \delta^{j}_{m} \delta^{k}_{n} - \delta^{j}_{n} \delta^{k}_{m}\\
\varepsilon^{ijk} \varepsilon_{ijn} &= 2\delta^{k}_{n}\\
\varepsilon^{ijk} \varepsilon_{ijk} &= 6
\end{aligned}$$

These identities will be useful in our discussion of the rotation group.

5.3 Discrete Lorentz transformations

5.3.1 Parity

In addition to rotations and boosts, there are two additional discrete transformations which preserve τ. Normally these are taken to be parity ($\mathcal{P}$) and time reversal ($\mathcal{T}$). Parity is defined as spatial inversion,

$$\mathcal{P} : (t, \mathbf{x}) \to (t, -\mathbf{x}) \tag{15}$$

We do not achieve new symmetries by reflecting only two of the spatial coordinates, e.g., $(t, x, y, z) \to (t, -x, -y, z)$ because this effect is achieved by a rotation by π about the z axis. For the same reason, reflection of a single coordinate is equivalent to reflecting all three. The effect of the parity on energy and momentum follows easily. Since the 4-momentum is defined by

$$p^\beta = m\frac{dx^\beta}{d\tau}$$

and because m and τ are Lorentz invariant, we have

$$\begin{aligned}
\mathcal{P}(E/c, \mathbf{p}) &= \mathcal{P}\left(m\frac{d(t, \mathbf{x})}{d\tau}\right) \\
&= m\frac{d}{d\tau}\mathcal{P}(t, \mathbf{x}) \\
&= m\frac{d}{d\tau}(t, -\mathbf{x}) \\
&= (E/c, -\mathbf{p})
\end{aligned}$$

We may represent parity by the matrix

$$\pi^\alpha{}_\beta = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$$

and then we easily see that

$$\pi\eta\pi = \eta$$

However, acting on the Levi-Civita tensor, we have

$$\begin{aligned}
\mathcal{P}(\varepsilon_{\alpha\beta\mu\nu}) &= \varepsilon_{\alpha\beta\mu\nu}\pi^\alpha{}_\rho\pi^\beta{}_\sigma\pi^\mu{}_\lambda\pi^\nu{}_\theta \\
&= \varepsilon_{\rho\sigma\lambda\theta}\det\pi \\
&= -\varepsilon_{\rho\sigma\lambda\theta}
\end{aligned}$$

Generally, the parity of an expression depends on whether it contains an even or odd number of Levi-Civita tensors.

5.3.2 Reflection and chronicity

There is one more independent discrete Lorentz symmetry. We begin with the definition,

Define: Reflection, $\mathcal{R}$ is the discrete Lorentz transformation takes every 4-vector to its negative,

$$\mathcal{R} : (t, \mathbf{x}) \to (-t, -\mathbf{x}) \tag{16}$$

This clearly preserves the invariant interval. Notice that it take future pointing timelike vectors to past pointing ones.

A more convenient form of this transformation is called *chronicity.* We may define it as the combined effect of reflection and parity, but the net effect is simply to reverse the sign of the time component of all 4-vectors.

Define: *Chronicity,* $\Theta \equiv \mathcal{RP}$, is the reversal of the time component of 4-vectors,

$$\Theta : (t, \mathbf{x}) \to (-t, \mathbf{x}) \tag{17}$$

This is clearly a Lorentz transformation, $\Theta\tau = \tau$.

Notice that the Lorentz invariance makes chronicity consistent with other operations. For example the effect of chronicity on energy and momentum follows in two ways. Acting directly on p^α according to the definition,

$$\Theta (E/c, \mathbf{p}) \quad = \quad (-E/c, \mathbf{p}) \tag{18}$$

but we also have

$$\Theta (E/c, \mathbf{p}) = \Theta \left(m \frac{d (t, \mathbf{x})}{d\tau} \right) = m \frac{d}{d\tau} \Theta (t, \mathbf{x}) = m \frac{d}{d\tau} (-t, \mathbf{x}) = (-E/c, \mathbf{p})$$

Notice the importance of the Lorentz invariance of the operators,

$$\Theta \frac{d}{d\tau} = \frac{d}{d\tau} \Theta$$

5.3.3 Time reversal

There is additional discrete symmetry which, though important, is not a Lorentz transformation: time reversal. Time reversal is a generalization of Newtonian time reversal, in which we simply replace t by $-t$,

$$\mathcal{T}_N : (t, \mathbf{x}) \rightarrow (-t, \mathbf{x})$$

This makes sense in the Newtonian context because time is simply a parameter. Acting on non-relativistic energy and momentum, Newtonian time reversal gives

$$\begin{aligned} \mathcal{T}_N E &= \mathcal{T}_N \left(\frac{1}{2} m \left(\frac{d\mathbf{x}}{dt} \right)^2 \right) = \frac{1}{2} m \left(\frac{d\mathbf{x}}{d(-t)} \right)^2 = E \\ \mathcal{T}_N \mathbf{p} &= \mathcal{T}_N m \left(\frac{d\mathbf{x}}{dt} \right) = m \frac{d\mathbf{x}}{d(-t)} = -\mathbf{p} \end{aligned}$$

so that

$$\mathcal{T}_N : (E, \mathbf{p}) \rightarrow (E, -\mathbf{p})$$

This means that while Newtonian time reversal preserves the norm of some 4-vectors, it does not preserve all. If we take a linear combination of the position and momentum 4-vectors, then compute the norm,

$$(\alpha x^\alpha + \beta p^\alpha) (\alpha x_\alpha + \beta p_\alpha) = \alpha^2 s^2 + 2\alpha\beta x^\alpha p_\alpha - \beta^2 m^2 c^2$$

However, the time-reversal is

$$\begin{aligned} \mathcal{T} (\alpha x^\alpha + \beta p^\alpha) &= \alpha (-t, \mathbf{x}) + \beta (E, -\mathbf{p}) \\ &= (-\alpha t + \beta E, \alpha \mathbf{x} - \beta \mathbf{p}) \end{aligned}$$

with norm

$$\begin{aligned} -(-\alpha t + \beta E)^2 + (\alpha \mathbf{x} - \beta \mathbf{p})^2 &= -\alpha^2 t^2 + 2\alpha\beta E t - \beta^2 E^2 + \alpha^2 \mathbf{x}^2 - 2\alpha\beta \mathbf{x} \cdot \mathbf{p} + \beta^2 \mathbf{p}^2 \\ &= \alpha^2 s^2 - 2\alpha\beta x^\alpha p_\alpha - \beta^2 m^2 c^2 \end{aligned}$$

Time reversal in this sense therefore violates the vector space character of spacetime.

The resolution of this difficulty lies in recognizing that time reversal is a transformation of *curves*, not vectors. If we have a spacetime curve, $\mathcal{C}(\lambda) : \mathbb{R} \rightarrow x^\alpha (\lambda)$, then as we increase the parameter λ we trace the progression of the curve. Typically, the parameter is the proper time, τ, along a timelike curve followed by a particle. Time reversal in the Newtonian sense runs this curve in the opposite direction,

$$\mathcal{T} : C(\tau) \rightarrow C(-\tau)$$

Even though this is not a Lorentz transformation, it has an invariant meaning for relativistic or non-relativistic systems. Notice that it replicates the Newtonian effect on position and momentum:

$$\mathcal{T}(\mathbf{x}(t)) = \mathbf{x}(-t)$$

while

$$\begin{aligned}\mathcal{T}(\dot{\mathbf{x}}(t)) &= \dot{\mathbf{x}}(-t) \\ &= -\dot{\mathbf{x}}(t)\end{aligned}$$

Define: Relativistic time reversal, $\mathcal{T}$, is a mapping from curves to curves, given by $\mathcal{T}C(\tau) = C(-\tau)$.

For a relativistic system,

$$\begin{aligned}\mathcal{T}x^{\alpha}(\tau) &= x^{\alpha}(-\tau) \\ \frac{d}{d\tau}(\mathcal{T}x^{\alpha}(\tau)) &= m\frac{d}{d\tau}(x^{\alpha}(-\tau)) \\ &= -m\frac{dx^{\alpha}}{d\tau}(-\tau) \\ &= -p^{\alpha}(-\tau)\end{aligned}$$

Notice the unexpected role played by the invariance of the proper time. By contrast with Newtonian time reversal, with the invariance of τ and the linearity of both E and $\mathbf{p}$ in τ, only the energy reverses sense. The difference is easy to see in a spacetime diagram, where the old "run the movie backward" prescription is seen to require some fine tuning. In spacetime, the "motion" of the particle is replaced by a world line. Under chronicity, this world line flips into the past light cone. An observer (still moving forward in time in either the Newtonian or the relativistic version) experiences this flipped world line in reverse order, so negative energy appears to depart the endpoint and later arrive at the initial point of the motion. A collision at the endpoint, however, imparts momentum in the same direction regardless of the time orientation (see fig.(1)).

In discussing the inevitable negative energy states that arise in field and their relation to antiparticles, chronicity plays a central role.

The subgroup of Lorentz transformations for which the coordinate system remains right handed is called the *proper Lorentz group*, and the subgroup of Lorentz transformations which maintains the orientation of time is called the *orthochronous Lorentz group.* The simply connected subgroup which maintains both the direction of time and the handedness of the spatial coordinates is the *proper orthochronous Lorentz group.*