1 Quantization of the Dirac field

1.1 The Dirac action

We have written the Dirac action as

$$S = \int d^4x \; \bar{\psi} \left(i\gamma^\mu \partial_\mu - m \right) \psi$$

where $\bar{\psi} \equiv \psi^\dagger \gamma^0$. It is clear that γ^0 is required here to make the action real, since, with $\gamma^{0\dagger} = \gamma^0$ and $\gamma^{i\dagger} = -\gamma^i$, we have

$$\begin{aligned}
\left[\bar{\psi} \left(i\gamma^\mu \partial_\mu \psi - m\psi \right) \right]^* &= \left(-i\partial_\mu \psi^\dagger \gamma^{\mu\dagger} - \psi^\dagger m \right) \gamma^0 \psi \\
&= \left(-i\partial_0 \psi^\dagger \gamma^0 - i\partial_i \psi^\dagger \gamma^{i\dagger} - \psi^\dagger m \right) \gamma^0 \psi \\
&= -i\partial_0 \psi^\dagger \gamma^0 \gamma^0 \psi + i\partial_i \psi^\dagger \gamma^i \gamma^0 \psi - m\psi^\dagger \gamma^0 \psi \\
&= -i\partial_0 \psi^\dagger \gamma^0 \gamma^0 \psi - i\partial_i \psi^\dagger \gamma^0 \gamma^i \psi - m\bar{\psi}\psi \\
&= -i\partial_0 \bar{\psi} \gamma^0 \psi - i\partial_i \bar{\psi} \gamma^i \psi - m\bar{\psi}\psi \\
&= -i\partial_\mu \bar{\psi} \gamma^\mu \psi - m\bar{\psi}\psi
\end{aligned}$$

so that, integrating by parts,

$$\begin{aligned}
S^* &= \int d^4x \; \left(-i\partial_\mu \bar{\psi} \gamma^\mu - m\bar{\psi} \right) \psi \\
&= \int d^4x \; \bar{\psi} \left(i\gamma^\mu \partial_\mu - m \right) \psi \\
&= \int d^4x \; \bar{\psi} \left(i\gamma^\mu \partial_\mu - m \right) \psi \\
\gamma^\mu \partial_\mu &= \gamma^\mu \partial
\end{aligned}$$

1.2 Hamiltonian formulation

Now we turn to the quantization of the Dirac field. The action is

$$S = \int d^4x \; \bar{\psi} \left(i\gamma^\mu \partial_\mu - m \right) \psi$$

The conjugate momentum to ψ is the spinor field

$$\begin{aligned}
\pi_A &= \frac{\delta L}{\delta \left(\partial_0 \psi^A \right)} = i\bar{\psi}\gamma^0 \\
&= i \left[\psi^\dagger \right]^B h_{BC} \left[\gamma^0 \right]^C{}_A
\end{aligned}$$

We can also write this as

$$\pi \gamma^0 = i\bar{\psi} \tag{1}$$

Undaunted by the peculiar lack of a time derivative in the momentum, we press on with the Hamiltonian:

$$\begin{aligned}
H &= \int d^3x \; i\bar{\psi}\gamma^0 \partial_0 \psi - \int d^3x \bar{\psi} \left(i\gamma^\mu \partial_\mu - m \right) \psi \\
&= \int d^3x \; \left(i\bar{\psi}\gamma^0 \partial_0 \psi - i\bar{\psi}\gamma^0 \partial_0 \psi - i\bar{\psi}\gamma^i \partial_i \psi + m\bar{\psi}\psi \right) \\
&= \int d^3x \; \left(-i\bar{\psi}\gamma^i \partial_i \psi + m\bar{\psi}\psi \right)
\end{aligned}$$

$$
\begin{aligned}
&= \int d^3x \; i\bar{\psi}\left(-\gamma^i\partial_i\psi - im\psi\right) \\
&= \int d^3x \; \pi\gamma^0\left(-\gamma^i\partial_i - im\right)\psi \\
&= i\int d^3x \; \pi\gamma^0\left(i\gamma^i\partial_i - m\right)\psi \qquad (2)
\end{aligned}
$$

Once again, we are struck by the absence of time derivatives in the energy. This is somewhat illusory, since we may rewrite H using the field equation as

$$
\begin{aligned}
H &= i\int d^3x \; \pi\gamma^0\left(i\gamma^i\partial_i - m\right)\psi \\
&= \int d^3x \; \pi\gamma^0\left(\gamma^0\partial_0\right)\psi \\
&= \int d^3x \; \pi\partial_0\psi \qquad (3)
\end{aligned}
$$

As we show below, only the first form of the Hamiltionian is suitable for deriving the field equations, since we used the field equations to write this simplified form. However, eq.(3) is useful for computing the operator form of the Hamiltonian from solutions.

We can check the field equation using either form of H. Thus, we have

$$
\begin{aligned}
\partial_0\psi &= \{H,\psi\} \\
&= \int d^3x' \left(\frac{\delta H(x)}{\delta\pi(x')}\frac{\delta\psi(x)}{\delta\psi(x')} - \frac{\delta H(x)}{\delta\psi(x')}\frac{\delta\psi(x)}{\delta\pi(x')}\right) \\
&= i\int d^3x' \; \gamma^0\left(i\gamma^i\partial_i - m\right)\psi(x') \; \delta^3(x-x') \\
&= i\gamma^0\left(i\gamma^i\partial_i - m\right)\psi(x) \qquad (4)
\end{aligned}
$$

Multiplying by $i\gamma^0$ this becomes

$$
\begin{aligned}
i\gamma^0\partial_0\psi &= -\left(i\gamma^i\partial_i - m\right)\psi(x) \qquad (5) \\
\left(i\gamma^\alpha\partial_\alpha - m\right)\psi(x) &= 0 \qquad (6)
\end{aligned}
$$

Notice that, had we used eq.(3) for the Hamiltonian, we find only an identity:

$$
\begin{aligned}
\partial_0\psi &= \int d^3x' \left(\frac{\delta H(x)}{\delta\pi(x')}\frac{\delta\psi(x)}{\delta\psi(x')} - \frac{\delta H(x)}{\delta\psi(x')}\frac{\delta\psi(x)}{\delta\pi(x')}\right) \qquad (7) \\
&= \int d^3x' \left(\partial_0\psi(x')\right)\delta^3(x-x') \qquad (8) \\
&= \partial_0\psi(x) \qquad (9)
\end{aligned}
$$

As already noted, the identity occurs because we have already used the field equation to write the Hamiltonian in the simplified form.

For the momentum we find the conjugate field equation:

$$
\begin{aligned}
\partial_0\pi &= \{H,\pi\} \qquad (10) \\
&= \int d^3x' \left(\frac{\delta H(x)}{\delta\pi(x')}\frac{\delta\pi(x)}{\delta\psi(x')} - \frac{\delta H(x)}{\delta\psi(x')}\frac{\delta\pi(x)}{\delta\pi(x')}\right) \qquad (11) \\
&= \int d^3x' \left(-\frac{\delta H(x)}{\delta\psi(x')}\frac{\delta\pi(x)}{\delta\pi(x')}\right) \qquad (12)
\end{aligned}
$$

$$= \int d^3x' \left(-\left(i\left(-i\partial_i \pi\gamma^0\gamma^i - \pi\gamma^0 m\right)\right)\delta^3(x-x')\right) \tag{13}$$
$$= i\left(i\partial_i\pi\gamma^0\gamma^i + \pi\gamma^0 m\right) \tag{14}$$
$$\equiv i\pi(x)\gamma^0\left(i\gamma^i\overleftarrow{\partial}_i + m\right) \tag{15}$$

where the arrow to the left over the derivative is standard notation indicating that the derivative acts to the left on π. This lets us write the final result more compactly. Replacing $\pi\gamma^0 = i\bar{\psi}$ and inserting $\gamma^0\gamma^0 = 1$ on the left, we find:

$$\partial_0\pi\gamma^0\gamma^0 = -\bar{\psi}\left(i\gamma^i\overleftarrow{\partial}_i + m\right) \tag{16}$$
$$i\partial_0\bar{\psi}\gamma^0 = -\bar{\psi}\left(i\gamma^i\overleftarrow{\partial}_i + m\right) \tag{17}$$

and therefore gathering terms

$$i\partial_0\bar{\psi}\gamma^0 + \bar{\psi}\left(i\gamma^i\overleftarrow{\partial}_i + m\right) = 0 \tag{18}$$
$$\bar{\psi}\left(i\gamma^\alpha\overleftarrow{\partial}_\alpha + m\right) = 0 \tag{19}$$

thereby arriving at the conjugate Dirac equation. Once again, if we use H as given in eq.(3), we find only an identity.

Finally, we write the fundamental Poisson brackets,

$$\left\{\pi_A(\mathbf{x},t), \psi^B(\mathbf{x}',t)\right\}_{PB} = \delta^B_A\delta^3\left(\mathbf{x}-\mathbf{x}'\right) \tag{20}$$

Before we can proceed, we need to solve the classical Dirac equation.

1.3 Solution to the free classical Dirac equation

As with the scalar field, we can solve using a Fourier integral. First consider a single value of the momentum. Then we can write two plane wave solutions with fixed, positive energy 4-momentum p^α in the form

$$\psi(\mathbf{x},t) = u(p^\alpha)e^{-ip_\alpha x^\alpha} + v(p^\alpha)e^{ip_\alpha x^\alpha} \tag{21}$$

where $u(p^\alpha)$ and $v(p^\alpha)$ are spinors, $p^\alpha = \left(E, p^i\right)$and $p_\alpha = (E, p_i) = (E, -p^i)$. Substituting,

$$0 = \left(i\gamma^\alpha\partial_\alpha - m\right)\psi(\mathbf{x},t) \tag{22}$$
$$= \left(i\gamma^\alpha\partial_\alpha - m\right)\left(u(p^\alpha)e^{-ip_\alpha x^\alpha} + v(p^\alpha)e^{ip_\alpha x^\alpha}\right) \tag{23}$$
$$= \left(\gamma^\alpha p_\alpha - m\right)u(p^\alpha)e^{-ip_\alpha x^\alpha} - \left(\gamma^\alpha p_\alpha + m\right)v(p^\alpha)e^{ip_\alpha x^\alpha} \tag{24}$$

we find the pair of equations

$$\left(\gamma^\alpha p_\alpha - m\right)u(p^\alpha) = 0 \tag{25}$$
$$\left(\gamma^\alpha p_\alpha + m\right)v(p^\alpha) = 0 \tag{26}$$

for the $u(p^\alpha)$ and $v(p^\alpha)$ modes, respectively.

We'll begin by writing out the equation using the Dirac matrices as given in eqs.(**??**),

$$\gamma^0 = \begin{pmatrix} \mathbf{1} & \\ & -\mathbf{1} \end{pmatrix}, \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix} \tag{27}$$

and solve first for $u(p^\alpha)$. If we set

$$\left[u(p_\alpha)\right]^A = \begin{pmatrix} \alpha(p_\alpha) \\ \beta(p_\alpha) \end{pmatrix} \tag{28}$$

where $A = 1, 2, 3, 4$, then we get the matrix equation

$$\begin{aligned} 0 &= (\gamma^\alpha p_\alpha - m)\, w(p^\alpha) && (29) \\ &= \begin{pmatrix} E-m & \sigma^i p_i \\ -\sigma^i p_i & -E-m \end{pmatrix} \begin{pmatrix} \alpha(p_\alpha) \\ \beta(p_\alpha) \end{pmatrix} && (30) \end{aligned}$$

which gives the set of 2×2 equations

$$\begin{aligned} (E-m)\,\alpha(p_\alpha) + \sigma^i p_i \beta(p_\alpha) &= 0 && (31) \\ -\sigma^i p_i \alpha(p_\alpha) - (E+m)\,\beta(p_\alpha) &= 0 && (32) \end{aligned}$$

Since $E > 0$, the quantity $E + m$ is nonzero so the second equation may be solved for $\beta(p_\alpha)$ and substituted into the first:

$$\begin{aligned} \beta(p_\alpha) &= -\left(\frac{\sigma^i p_i}{E+m}\right)\alpha(p_\alpha) && (33) \\ (E-m)\,\alpha(p_\alpha) &= \sigma^i p_i \left(\frac{\sigma^i p_i}{E+m}\right)\alpha(p_\alpha) && (34) \\ \left(E^2 - \mathbf{p}^2 - m^2\right)\alpha(p_\alpha) &= 0 && (35) \end{aligned}$$

where we use $\left(\sigma^i p_i\right)^2 = (-p^i)(-p^i) = \mathbf{p}^2$ in the last line. This just gives the usual relativistic expression relating mass, energy and momentum, with positive energy solution

$$E = \sqrt{\mathbf{p}^2 + m^2} \qquad (36)$$

This determines the energy; now we need the eigenstates. These must satisfy

$$\begin{aligned} \beta(p_\alpha) &= -\left(\frac{\sigma^i p_i}{E+m}\right)\alpha(p_\alpha) && (37) \\ 0 &= E^2 - \mathbf{p}^2 - m^2 && (38) \end{aligned}$$

with no further constraint on $\alpha(p_\alpha)$. We are free to choose any convenient independent 2-spinors for $\alpha(p_\alpha)$. Therefore, let

$$\alpha_1(p_\alpha) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}; \alpha_2(p_\alpha) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad (39)$$

For $\alpha_1(p_\alpha)$, (remembering that $p_i = -p^i$) we must have

$$\begin{aligned} \beta_1(p_\alpha) &= -\left(\frac{\sigma^i p_i}{E+m}\right)\alpha_1(p_\alpha) && (40) \\ &= \frac{1}{E+m}\begin{pmatrix} p^z & p^x - ip^y \\ p^x + ip^y & -p^z \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} && (41) \\ &= \frac{1}{E+m}\begin{pmatrix} p^z \\ p^x + ip^y \end{pmatrix} && (42) \end{aligned}$$

while for $\alpha_2(p_\alpha)$ we find

$$\begin{aligned} \beta_2(p_\alpha) &= -\left(\frac{\sigma^i p_i}{E+m}\right)\alpha_2(p_\alpha) && (43) \\ &= \frac{1}{E+m}\begin{pmatrix} p^z & p^x - ip^y \\ p^x + ip^y & -p^z \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} && (44) \\ &= \frac{1}{E+m}\begin{pmatrix} p^x - ip^y \\ -p^z \end{pmatrix} && (45) \end{aligned}$$

These relations define two independent, normalized, positive energy solutions, which we denote by $u_a(p^\alpha)$:

$$[u_1(p^\alpha)]^A = \sqrt{\frac{E+m}{2m}} \begin{pmatrix} 1 \\ 0 \\ \frac{p^z}{E+m} \\ \frac{p^x+ip^y}{E+m} \end{pmatrix} \tag{46}$$

$$[u_2(p^\alpha)]^A = \sqrt{\frac{E+m}{2m}} \begin{pmatrix} 0 \\ 1 \\ \frac{p^x-ip^y}{E+m} \\ \frac{-p^z}{E+m} \end{pmatrix} \tag{47}$$

Exercise: Show that $u_1(p^\alpha)$ and $u_2(p^\alpha)$ are orthonormal, where the inner product of two spinors is given by

$$\langle \chi, \psi \rangle \equiv \chi^\dagger h \psi \tag{48}$$

with h given by eq.(**??**). Notice that this inner product is Lorentz invariant, so our spinor basis remains orthonormal in every frame of reference.

For the second set of mode amplitudes, we solve

$$0 = (\gamma^\alpha p_\alpha + m)\, v(p^\alpha) \tag{49}$$

$$= \begin{pmatrix} E+m & \sigma^i p_i \\ -\sigma^i p_i & -E+m \end{pmatrix} \begin{pmatrix} \alpha(p_\alpha) \\ \beta(p_\alpha) \end{pmatrix} \tag{50}$$

for $\alpha(p_\alpha)$ first instead:

$$\alpha(p_\alpha) = -\frac{\sigma^i p_i}{E+m} \beta(p_\alpha) \tag{51}$$

Once again this leads to $E^2 - \mathbf{p}^2 - m^2 = 0$, so that $E = \sqrt{\mathbf{p}^2 + m^2}$. There are again two solutions. Since $\beta(p_\alpha)$ is arbitrary and $\alpha(p_\alpha)$ is given by eq.(51), we choose

$$\beta_1(p_\alpha) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}; \beta_2(p_\alpha) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{52}$$

leading to two more independent, normalized solutions, $v_a(p^\alpha)$,

$$[v_1(p^\alpha)]^A = \sqrt{\frac{m+E}{2m}} \begin{pmatrix} \frac{p^z}{E+m} \\ \frac{p^x+ip^y}{E+m} \\ 1 \\ 0 \end{pmatrix} \tag{53}$$

$$[v_2(p^\alpha)]^A = \sqrt{\frac{m+E}{2m}} \begin{pmatrix} \frac{p^x-ip^y}{E+m} \\ \frac{-p^z}{E+m} \\ 0 \\ 1 \end{pmatrix} \tag{54}$$

The entire set of four spinors, $u_a(p^\alpha), v_a(p^\alpha)$, is a complete, pseudo-orthonormal basis.

Exercise: Check that $v_1(p^\alpha)$ and $v_2(p^\alpha)$ satisfy

$$\langle v_a(p^\alpha), v_b(p^\alpha) \rangle = -\delta_{ab} \tag{55}$$

$$\langle u_a(p^\alpha), v_b(p^\alpha) \rangle = 0 \tag{56}$$

Exercise: Prove the completeness relation,

$$\sum_{a=1}^{2}\left(\left[u_a(p^\alpha)\right]^A\left[\bar{u}_a(p^\alpha)\right]_B-\left[v_a(p^\alpha)\right]^A\left[\bar{v}_a(p^\alpha)\right]_B\right)=\delta^A_B \tag{57}$$

where $A, B = 1, \ldots, 4$ index the components of the basis spinors.

Using this basis, we now have a complete solution to the free Dirac equation. Using $\Theta(E)$ to enforce positive energy condition, we have

$$\begin{aligned}
\psi(\mathbf{x},t) &= \frac{1}{(2\pi)^{3/2}}\sum_{\Sigma=1}^{4}\int d^4k\delta\left(E^2-\mathbf{p}^2-m^2\right)\Theta(E)\left(a_\Sigma(p^\alpha)w_\Sigma(p^\alpha)e^{-\frac{i}{\hbar}p_\alpha x^\alpha}\right. && (58)\\
&\quad\left.+\,c^\dagger_A(p^\alpha)w^\dagger_\Sigma(p^\alpha)e^{\frac{i}{\hbar}p_\alpha x^\alpha}\right) && (59)\\
&= \sum_{i=1}^{2}\int d^3k\sqrt{\frac{\omega}{m}}\left(b_i(\mathbf{k})u_i(\mathbf{k})e^{-i(\omega t-\mathbf{k}\cdot\mathbf{x})}+d^\dagger_i(\mathbf{k})v_i(\mathbf{k})e^{i(\omega t-\mathbf{k}\cdot\mathbf{x})}\right) && (60)
\end{aligned}$$

where we introduce the conventional normalization for the Fourier amplitudes, $a_i(\mathbf{k})=\sqrt{\frac{2}{m}\omega}\;b_i(\mathbf{k})$ and $c^\dagger_i(\mathbf{k})=\sqrt{\frac{2}{m}\omega}\;d^\dagger_i(\mathbf{k})$ and set $\omega=+\sqrt{\mathbf{k}^2+m^2}$ as before. Before turning to quantization, let's consider the spin.

1.4 The spin of spinors

The basis spinors $(u_a(p^\alpha), v_a(p^\alpha))$ may be thought of as eigenvectors of the operator $p_\alpha\gamma^\alpha$. For $u_a(p^\alpha)$ we have:

$$\begin{aligned}
0 &= (\gamma^\alpha p_\alpha-m)\,u_a(p^\alpha)\\
0 &= (\gamma^\alpha p_\alpha+m)\,v_a(p^\alpha)
\end{aligned}$$

and therefore

$$\begin{aligned}
\gamma^\alpha p_\alpha u_a(p^\alpha) &= m u_a(p^\alpha) && (61)\\
\gamma^\alpha p_\alpha v_a(p^\alpha) &= -m v_a(p^\alpha) && (62)
\end{aligned}$$

This means we can construct projection operators that single out the $u_a(p^\alpha)$- and $v_a(p^\alpha)$-type spinors. If we write

$$P_+=\frac{1}{2}\left(\mathbf{1}+\frac{1}{m}\gamma^\alpha p_\alpha\right) \tag{63}$$

then

$$\begin{aligned}
P_+^2 &= \frac{1}{4}\left(\mathbf{1}+\frac{1}{m}\gamma^\alpha p_\alpha\right)\left(\mathbf{1}+\frac{1}{m}\gamma^\beta p_\beta\right)\\
&= \frac{1}{4}\left(\mathbf{1}+\frac{2}{m}\gamma^\alpha p_\alpha+\frac{1}{m^2}\gamma^\alpha p_\alpha\gamma^\beta p_\beta\right)\\
&= \frac{1}{4}\left(\mathbf{1}+\frac{2}{m}\gamma^\alpha p_\alpha+\frac{1}{m^2}p^2\right)\\
&= P_+ && (64)
\end{aligned}$$

Clearly, we have

$$\begin{aligned} P_+ u_a(p^\alpha) &= u_a(p^\alpha) \\ P_+ v_a(p^\alpha) &= 0 \\ P_+ &= \sum_{a=1}^{2} u_a(p^\alpha)\bar{u}_a(p^\alpha) \end{aligned} \tag{65}$$

Similarly, we define

$$P_- = \frac{1}{2}\left(\mathbf{1} - \frac{1}{m}\gamma^\alpha p_\alpha\right) \tag{66}$$

satisfying

$$\begin{aligned} P_- u_a(p^\alpha) &= 0 \\ P_- v_a(p^\alpha) &= v_a(p^\alpha) \\ P_- &= \sum_{a=1}^{2} v_a(p^\alpha)\bar{v}_a(p^\alpha) \end{aligned} \tag{67}$$

These projections span the spinor space since $P_+ + P_- = \mathbf{1}$.

Next, we seek a pair of operators which distinguishes between u_1 and u_2 and between v_1 and v_2. Since u_a and v_a are pseudo-orthonormal, we can simply write

$$[\Pi_+]^A{}_B = u_1 \otimes \bar{u}_1 - v_2 \otimes \bar{v}_2 = [u_1]^A \left[\gamma^0\right]_{BC} \left[u_1^\dagger\right]^C - [v_2]^A \left[\gamma^0\right]_{BC} \left[v_2^\dagger\right]^C \tag{68}$$

In the rest frame of the particle, where the 4-momentum is given by

$$p^\alpha = (mc, 0) \tag{69}$$

we have

$$\begin{aligned} u_1\left(p^\alpha\right) &= (1,0,0,0) \\ u_2\left(p^\alpha\right) &= (0,1,0,0) \\ v_1\left(p^\alpha\right) &= (0,0,1,0) \\ v_2\left(p^\alpha\right) &= (0,0,0,1) \end{aligned} \tag{70}$$

so that

$$[\Pi_+]^A{}_B = \begin{pmatrix} 1 & & & \\ & 0 & & \\ & & 0 & \\ & & & 1 \end{pmatrix} \tag{71}$$

This combination is easy to construct from the gamma matrices. With

$$\gamma^0 = \begin{pmatrix} \mathbf{1} & \\ & -\mathbf{1} \end{pmatrix}, \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}, \gamma_5 = \begin{pmatrix} & 1 \\ 1 & \end{pmatrix} \tag{72}$$

we note that

$$\gamma^3\gamma_5 = \begin{pmatrix} \sigma^3 & 0 \\ 0 & -\sigma^3 \end{pmatrix} = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & 1 \end{pmatrix} \tag{73}$$

We see that *all* of the basis vectors, eq.(70), are eigenvectors of $\gamma^3\gamma_5$:

$$\gamma^3\gamma_5 u_1 = u_1 \tag{74}$$
$$\gamma^3\gamma_5 u_2 = -u_2 \tag{75}$$
$$\gamma^3\gamma_5 v_1 = -v_1 \tag{76}$$
$$\gamma^3\gamma_5 v_2 = v_2 \tag{77}$$

Therefore, we have two projection operators,

$$\Pi_+ = \frac{1}{2}\left(1+\gamma^3\gamma_5\right) = \frac{1}{2}\left(1+n_\alpha\gamma^\alpha\gamma_5\right) \tag{78}$$
$$\Pi_- = \frac{1}{2}\left(1-\gamma^3\gamma_5\right) = \frac{1}{2}\left(1-n_\alpha\gamma^\alpha\gamma_5\right) \tag{79}$$

where $n_\alpha = (0,0,0,1)$. Notice that n_α is spacelike, with $n^2 = -1$, and that $p^\alpha n_\alpha = 0$.

Now, we generalize these new projections by writing

$$\Pi_+ = \frac{1}{2}\left(1+s_\mu\gamma^\mu\gamma_5\right) \tag{80}$$
$$\Pi_- = \frac{1}{2}\left(1-s_\mu\gamma^\mu\gamma_5\right) \tag{81}$$

where s_μ is any 4-vector. These are still projection operators provided $s_\mu s_\nu \eta^{\mu\nu} = s^2 = -1$, since then we have

$$\Pi_+^2 = \frac{1}{4}\left(1+s_\mu\gamma^\mu\gamma_5\right)\left(1+s_\nu\gamma^\nu\gamma_5\right) \tag{82}$$
$$= \frac{1}{4}\left(1+2s_\mu\gamma^\mu\gamma_5 + s_\mu\gamma^\mu\gamma_5 s_\nu\gamma^\nu\gamma_5\right) \tag{83}$$
$$= \frac{1}{4}\left(1+2s_\mu\gamma^\mu\gamma_5 - s_\mu s_\nu\gamma^\mu\gamma^\nu\gamma_5\gamma_5\right) \tag{84}$$
$$= \frac{1}{4}\left(1+2s_\mu\gamma^\mu\gamma_5 - s_\mu s_\nu\eta^{\mu\nu}\right) \tag{85}$$
$$= \Pi_+ \tag{86}$$

and similarly for Π_-. In addition, we can make these projections commute with P_+ and P_- . Consider

$$[\Pi_+, P_+] = \left[\frac{1}{2}\left(1+s_\mu\gamma^\mu\gamma_5\right), \frac{1}{2}\left(1+\frac{1}{m}\gamma^\alpha p_\alpha\right)\right] \tag{87}$$
$$= \frac{1}{4}\left(1+s_\mu\gamma^\mu\gamma_5 + \frac{1}{m}\gamma^\alpha p_\alpha + \frac{1}{m}s_\mu p_\alpha\gamma^\mu\gamma_5\gamma^\alpha\right) \tag{88}$$
$$-\frac{1}{4}\left(1+\frac{1}{m}\gamma^\alpha p_\alpha + s_\mu\gamma^\mu\gamma_5 + \frac{1}{m}p_\alpha s_\mu\gamma^\alpha\gamma^\mu\gamma_5\right) \tag{89}$$
$$= -\frac{1}{4m}\left(s_\mu p_\alpha\gamma^\mu\gamma^\alpha\gamma_5 + p_\alpha s_\mu\gamma^\alpha\gamma^\mu\gamma_5\right) \tag{90}$$
$$= -\frac{1}{4m}s_\mu p_\alpha\left(\gamma^\mu\gamma^\alpha + \gamma^\alpha\gamma^\mu\right)\gamma_5 \tag{91}$$
$$= -\frac{1}{2m}s_\mu p_\alpha\eta^{\mu\alpha}\gamma_5 \tag{92}$$

This will vanish if s^α and p_α are orthogonal, $s^\alpha p_\alpha = 0$. Since, $P_+P_- = 0$ and $\Pi_+\Pi_- = 0$, the set of projection operators,

$$\{P_+, P_-, \Pi_+, \Pi_-\} \tag{93}$$

is fully commuting and therefore simultaneously diagonalizable. Moreover, they are independent. To see this, consider the products

$$\{P_+\Pi_+, P_+\Pi_-, P_-\Pi_+, P_-\Pi_-\} \tag{94}$$

These are mutually orthogonal, i.e., $(P_+\Pi_+)(P_+\Pi_-) = P_+P_+\Pi_+\Pi_- = 0$ and so on. Each combination projects into a 1-dimensional subspace of the spinor space since

$$\begin{aligned} tr\,(P_+\Pi_+) &= \frac{1}{4}tr\left(1 + s_\mu\gamma^\mu\gamma_5 + \frac{1}{m}\gamma^\alpha p_\alpha + \frac{1}{m}s_\mu p_\alpha\gamma^\mu\gamma_5\gamma^\alpha\right) && (95)\\ &= \frac{1}{4}(4+0+0+0) && (96)\\ &= 1 && (97)\end{aligned}$$

and similarly

$$tr\,(P_+\Pi_-) = tr\,(P_-\Pi_+) = tr\,(P_-\Pi_-) = 1 \tag{98}$$

Moreover, they span the space as we see from the completeness relation:

$$\begin{aligned} P_+\Pi_+ + P_+\Pi_- + P_-\Pi_+ + P_-\Pi_- &= P_+(\Pi_+ + \Pi_-) + P_-(\Pi_+ + \Pi_-) && (99)\\ &= P_+ + P_- && (100)\\ &= \mathbf{1} && (101)\end{aligned}$$

We interpret all of this as follows. The vector s_α is the 4-dimensional generalization of the spin vector, s^i, and in the rest frame, u and v are eigenvectors of the z-component of spin. We are free to choose u and v to be eigenvectors of any 3-vector s^i, and therefore eigenspinors of the corresponding $\Pi_+(s^\alpha), \Pi_-(s^\alpha)$. As a result, we can label the spinors by their 4-momentum and their spin vectors,

$$\begin{aligned} &u_a\left(p^\alpha, s^\beta\right) && (102)\\ &v_a\left(p^\alpha, s^\beta\right) && (103)\end{aligned}$$

In the rest frame, with $s_\alpha = (0,0,0,1) = (0,n^i) \equiv n_\alpha$, we have:

$$\begin{aligned} \Pi_+ &= u_1\left(p^\alpha, n^\beta\right)\bar{u}_1\left(p^\alpha, n^\beta\right) - v_2\left(p^\alpha, n^\beta\right)\bar{v}_2\left(p^\alpha, n^\beta\right) && (104)\\ \Pi_- &= u_2\left(p^\alpha, n^\beta\right)\bar{u}_2\left(p^\alpha, n^\beta\right) - v_1\left(p^\alpha, n^\beta\right)\bar{v}_1\left(p^\alpha, n^\beta\right) && (105)\end{aligned}$$

and since both sides transform in the same way under Lorentz transformations, we have

$$\begin{aligned} \Pi_+ &= u_1\left(p^\alpha, s^\beta\right)\bar{u}_1\left(p^\alpha, n^\beta\right) - v_2\left(p^\alpha, s^\beta\right)\bar{v}_2\left(p^\alpha, s^\beta\right) && (106)\\ \Pi_- &= u_2\left(p^\alpha, s^\beta\right)\bar{u}_2\left(p^\alpha, s^\beta\right) - v_1\left(p^\alpha, s^\beta\right)\bar{v}_1\left(p^\alpha, s^\beta\right) && (107)\end{aligned}$$

in any frame of reference and for any choice of spin direction.

Using these expressions for the spin projection operators together with the corresponding expressions, eqs.(65) and (67), for the energy, we can rewrite the outer products of the completeness relation, eq.(57), as

$$\begin{aligned} P_+\Pi_+ &= u_1\left(p^\alpha, s^\beta\right)\bar{u}_1\left(p^\alpha, s^\beta\right) && (108)\\ &= \frac{1}{2}\left(\mathbf{1} + \frac{1}{m}\gamma^\alpha p_\alpha\right)\frac{1}{2}\left(1 + s_\mu\gamma^\mu\gamma_5\right) && (109)\\ P_+\Pi_- &= u_2\left(p^\alpha, s^\beta\right)\bar{u}_2\left(p^\alpha, s^\beta\right) && (110)\\ &= \frac{1}{2}\left(\mathbf{1} + \frac{1}{m}\gamma^\alpha p_\alpha\right)\frac{1}{2}\left(1 - s_\mu\gamma^\mu\gamma_5\right) && (111)\\ &= u_1\left(p^\alpha, -s^\beta\right)\bar{u}_1\left(p^\alpha, -s^\beta\right) && (112)\end{aligned}$$

$$
\begin{aligned}
P_-\Pi_- &= -v_1\left(p^\alpha, s^\beta\right)\bar{v}_1\left(p^\alpha, s^\beta\right) &(113)\\
&= \frac{1}{2}\left(\mathbf{1} - \frac{1}{m}\gamma^\alpha p_\alpha\right)\frac{1}{2}\left(1 - s_\mu\gamma^\mu\gamma_5\right) &(114)\\
P_-\Pi_+ &= -v_1\left(p^\alpha, s^\beta\right)\bar{v}_1\left(p^\alpha, s^\beta\right) &(115)\\
&= \frac{1}{2}\left(\mathbf{1} - \frac{1}{m}\gamma^\alpha p_\alpha\right)\frac{1}{2}\left(1 + s_\mu\gamma^\mu\gamma_5\right) &(116)\\
&= -v_1\left(p^\alpha, -s^\beta\right)\bar{v}_1\left(p^\alpha, -s^\beta\right) &(117)
\end{aligned}
$$

These identities will be useful for calculating scattering amplitudes.

1.5 Quantization of the Dirac field

The fundamental commutator of the spinor field follows from the fundamental Poisson brackets, eq.(20) as

$$
\left[\hat{\pi}_A(\mathbf{x}, t), \hat{\psi}^B(\mathbf{x}', t)\right] = i\delta_A^B\delta^3\left(\mathbf{x} - \mathbf{x}'\right) \tag{118}
$$

and we can immediately turn to our examination of the commutation relations of the mode amplitudes. The classical solution is

$$
\begin{aligned}
\psi(\mathbf{x}, t) &= \frac{1}{(2\pi)^{3/2}}\sum_{a=1}^{2}\int d^3k\sqrt{\frac{m}{\omega}}\Big(b_a(\mathbf{k})u_a(\mathbf{k})e^{-i(\omega t - \mathbf{k}\cdot\mathbf{x})} &(119)\\
&\quad + d_a^\dagger(\mathbf{k})v_a(\mathbf{k})e^{i(\omega t - \mathbf{k}\cdot\mathbf{x})}\Big) &(120)\\
\psi^\dagger(\mathbf{x}, t) &= \frac{1}{(2\pi)^{3/2}}\sum_{a=1}^{2}\int d^3k\sqrt{\frac{m}{\omega}}\Big(b_a^\dagger(\mathbf{k})u_a^\dagger(\mathbf{k})e^{i(\omega t - \mathbf{k}\cdot\mathbf{x})} &(121)\\
&\quad + d_a(\mathbf{k})v_a^\dagger(\mathbf{k})e^{-i(\omega t - \mathbf{k}\cdot\mathbf{x})}\Big) &(122)\\
\pi(\mathbf{x}, t) &= \frac{i}{(2\pi)^{3/2}}\sum_{a=1}^{2}\int d^3k\sqrt{\frac{m}{\omega}}\Big(b_a^\dagger(\mathbf{k})\bar{u}_a(\mathbf{k})e^{-i(\omega t - \mathbf{k}\cdot\mathbf{x})} &(123)\\
&\quad + d_a^\dagger(\mathbf{k})\bar{v}_a(\mathbf{k})e^{i(\omega t - \mathbf{k}\cdot\mathbf{x})}\Big)\gamma^0 &(124)
\end{aligned}
$$

and we may solve for the amplitudes as usual. Notice that our writing $d_i^\dagger(\mathbf{k})$ instead of $d_i(\mathbf{k})$ in the expansion of ψ, while perfectly allowable, has no justification at this point. It is purely a matter of definition. However, when we look at the commutation relations of the corresponding operators, this part of the field operator $\hat{\psi}$ should create an antiparticle, and therefore is most appropriately called $d_i^\dagger(\mathbf{k})$. This is consistent with $CP\Theta$ symmetry of the field.

Setting $t = t' = 0$, we first invert the Fourier transform:

$$
\begin{aligned}
\tilde{\psi}(\mathbf{k}) &\equiv \frac{1}{(2\pi)^{3/2}}\int\psi(\mathbf{x}, 0)e^{-i\mathbf{k}\cdot\mathbf{x}}d^3x &(125)\\
&= \frac{1}{(2\pi)^3}\sum_{j=1}^{2}\int\int d^3xd^3k'\sqrt{\frac{m}{\omega'}}\Big(b_j(\mathbf{k}')u_j(\mathbf{k}')e^{i\left(\mathbf{k}' - \mathbf{k}\right)\cdot\mathbf{x}} &(126)\\
&\quad + d_j^\dagger(\mathbf{k}')v_j(\mathbf{k}')e^{-i\left(\mathbf{k}' + \mathbf{k}\right)\cdot\mathbf{x}}\Big) &(127)\\
&= \sum_{j=1}^{2}\int d^3k'\sqrt{\frac{m}{\omega'}}\left(b_i(\mathbf{k}')u_i(\mathbf{k}')\delta^3\left(\mathbf{k}' - \mathbf{k}\right)\right. &(128)
\end{aligned}
$$

$$+ d_j^\dagger(\mathbf{k}')v_j(\mathbf{k}')\delta^3\left(\mathbf{k}'+\mathbf{k}\right)\Big) \tag{129}$$

$$= \sum_{j=1}^{2}\sqrt{\frac{m}{\omega}}\left(b_j(\mathbf{k})u_j(\mathbf{k}) + d_j^\dagger(-\mathbf{k})v_j(-\mathbf{k})\right) \tag{130}$$

We immediately find

$$\tilde{\psi}^\dagger(\mathbf{k}) \equiv \frac{1}{(2\pi)^{3/2}}\int \psi^\dagger(\mathbf{x},0)e^{i\mathbf{k}\cdot\mathbf{x}}d^3x \tag{131}$$

$$= \sum_{j=1}^{2}\sqrt{\frac{m}{\omega}}\left(b_j^\dagger(\mathbf{k})u_j^\dagger(\mathbf{k}) + d_j(-\mathbf{k})v_j^\dagger(-\mathbf{k})\right) \tag{132}$$

so that

$$\tilde{\pi}(\mathbf{k}) = i\tilde{\psi}^\dagger(\mathbf{k})h\gamma^0 \tag{133}$$

$$= \frac{i}{(2\pi)^{3/2}}\int \psi^\dagger(\mathbf{x},0)h\gamma^0 e^{i\mathbf{k}\cdot\mathbf{x}}d^3x \tag{134}$$

$$= i\sum_{j=1}^{2}\sqrt{\frac{\omega}{m}}\left(b_j^\dagger(\mathbf{k})u_j^\dagger(\mathbf{k})h\gamma^0 + d_j(-\mathbf{k})v_j^\dagger(-\mathbf{k})h\gamma^0\right) \tag{135}$$

Now, we would like to use the spinor inner product to isolate b_i and d_i. However, since $\tilde{\psi}(\mathbf{k})$ involves $v_j(-\mathbf{k})$ instead of $v_j(\mathbf{k})$, we need a modified form of the orthonormality relation. From the form of our solution for $v_j(\mathbf{k})$, we immediately see that

$$v_1(-\mathbf{k}) = \sqrt{\frac{m+\omega}{2m}}\begin{pmatrix} \frac{-k^z}{\omega+m} \\ -\frac{k^x+ik^y}{\omega+m} \\ 1 \\ 0 \end{pmatrix} = -\gamma^0 v_1(\mathbf{k}) \tag{136}$$

$$v_2(-\mathbf{k}) = \sqrt{\frac{m+\omega}{2m}}\begin{pmatrix} -\frac{k^x-ik^y}{\omega+m} \\ \frac{k^z}{\omega+m} \\ 0 \\ 1 \end{pmatrix} = -\gamma^0 v_2(\mathbf{k}) \tag{137}$$

We also need

$$\bar{v}_i(-\mathbf{k}) = v_i^\dagger(-\mathbf{k})h = \left(-\gamma^0 v_i(\mathbf{k})\right)^\dagger h = -v_i^\dagger(\mathbf{k})\gamma^0 h \tag{138}$$

as well as two more identities to reach our goal.

Exercise: Show that

$$u_{jB}^\dagger(\mathbf{k})\left[\gamma^0\right]^B{}_A u_i^A(\mathbf{k}) = \frac{\omega}{m}\delta_{ij} \tag{139}$$

and

$$v_{jB}^\dagger(\mathbf{k})\left[\gamma^0\right]^B{}_A v_i^A(\mathbf{k}) = \frac{\omega}{m}\delta_{ij} \tag{140}$$

where $u_{jB}^\dagger = \left[u_j^\dagger\right]^A h_{AB}$ and $v_{jB}^\dagger(\mathbf{k}) = \left[v_j^\dagger\right]^A h_{AB}(\mathbf{k})$.

Continuing, we may write the Fourier transforms as

$$\tilde{\psi}(\mathbf{k}) = \sum_{j=1}^{2}\sqrt{\frac{m}{\omega}}\left(b_j(\mathbf{k})u_j(\mathbf{k}) - d_j^\dagger(-\mathbf{k})\gamma^0 v_j(\mathbf{k})\right) \tag{141}$$

$$\tilde{\pi}(\mathbf{k}) = i\sum_{j=1}^{2}\sqrt{\frac{m}{\omega}}\left(b_j^\dagger(\mathbf{k})u_j^\dagger(\mathbf{k})h\gamma^0 - d_j(-\mathbf{k})v_j^\dagger(\mathbf{k})h\right) \tag{142}$$

where we used $\gamma^0 h \gamma^0 = h$. As a result,

$$\begin{aligned}
\bar{u}_i(\mathbf{k})\,\gamma^0\tilde{\psi}(\mathbf{k}) &= \sum_{j=1}^{2}\sqrt{\frac{m}{\omega}}\,\Big(b_j(\mathbf{k})\bar{u}_i(\mathbf{k})\gamma^0 u_j(\mathbf{k}) && (143)\\
&\quad - d_j^\dagger(-\mathbf{k})\bar{u}_i(\mathbf{k})v_j(\mathbf{k})\Big) && (144)\\
&= \sum_{j=1}^{2}\sqrt{\frac{m}{\omega}}b_j(\mathbf{k})\bar{u}_i(\mathbf{k})\,\gamma^0 u_j(\mathbf{k}) && (145)\\
&= \sqrt{\frac{\omega}{m}}b_i(\mathbf{k}) && (146)
\end{aligned}$$

and similarly

$$\begin{aligned}
\bar{v}_i(\mathbf{k})\,\tilde{\psi}(\mathbf{k}) &= \sum_{j=1}^{2}\sqrt{\frac{m}{\omega}}\,(b_j(\mathbf{k})\bar{v}_i(\mathbf{k})\,u_j(\mathbf{k}) && (147)\\
&\quad - d_j^\dagger(-\mathbf{k})\bar{v}_i(\mathbf{k})\,\gamma^0 v_j(\mathbf{k})\Big) && (148)\\
&= -\sum_{j=1}^{2}\sqrt{\frac{m}{\omega}}d_j^\dagger(-\mathbf{k})\bar{v}_i(\mathbf{k})\,\gamma^0 v_j(\mathbf{k}) && (149)\\
&= -\sqrt{\frac{\omega}{m}}d_j^\dagger(-\mathbf{k}) && (150)
\end{aligned}$$

while for the momentum,

$$\begin{aligned}
\tilde{\pi}(\mathbf{k})u_i(\mathbf{k}) &= i\sum_{j=1}^{2}\sqrt{\frac{m}{\omega}}\,\Big(b_j^\dagger(\mathbf{k})u_j^\dagger(\mathbf{k})h\gamma^0 u_i(\mathbf{k}) && (151)\\
&\quad - d_j(-\mathbf{k})v_j^\dagger(\mathbf{k})hu_i(\mathbf{k})\Big) && (152)\\
&= i\sum_{j=1}^{2}\sqrt{\frac{m}{\omega}}b_j^\dagger(\mathbf{k})u_j^\dagger(\mathbf{k})h\gamma^0 u_i(\mathbf{k}) && (153)\\
&= i\sqrt{\frac{\omega}{m}}b_i^\dagger(\mathbf{k}) && (154)
\end{aligned}$$

and

$$\begin{aligned}
\tilde{\pi}(\mathbf{k})\gamma^0 v_i(\mathbf{k}) &= i\sum_{j=1}^{2}\sqrt{\frac{m}{\omega}}\,\Big(b_j^\dagger(\mathbf{k})u_j^\dagger(\mathbf{k})h\gamma^0\gamma^0 v_i(\mathbf{k}) && (155)\\
&\quad - d_j(-\mathbf{k})v_j^\dagger(\mathbf{k})h\gamma^0 v_i(\mathbf{k})\Big) && (156)\\
&= -i\sum_{j=1}^{2}\sqrt{\frac{m}{\omega}}\,\Big(d_j(-\mathbf{k})v_j^\dagger(\mathbf{k})h\gamma^0 v_i(\mathbf{k})\Big) && (157)\\
&= -i\sqrt{\frac{\omega}{m}}d_j(-\mathbf{k}) && (158)
\end{aligned}$$

Noting that

$$\tilde{\psi}(\mathbf{k}) \equiv \frac{1}{(2\pi)^{3/2}}\int \psi(\mathbf{x},0)e^{-i\mathbf{k}\cdot\mathbf{x}}d^3x$$

$$\tilde{\psi}^{\dagger}(\mathbf{k}) = \frac{1}{(2\pi)^{3/2}} \int \psi^{\dagger}(\mathbf{x},0) e^{i\mathbf{k}\cdot\mathbf{x}} d^3x$$

we collect terms and replace the mode amplitudes by operators:

$$\hat{b}_i(\mathbf{k}) = \sqrt{\frac{m}{\omega}} \frac{1}{(2\pi)^{3/2}} \int \bar{u}_i(\mathbf{k}) \gamma^0 \psi(\mathbf{x},0) e^{-i\mathbf{k}\cdot\mathbf{x}} d^3x \tag{159}$$

$$\hat{d}_j^{\dagger}(\mathbf{k}) = -\sqrt{\frac{m}{\omega}} \frac{1}{(2\pi)^{3/2}} \int \bar{v}_i(-\mathbf{k}) \psi(\mathbf{x},0) e^{i\mathbf{k}\cdot\mathbf{x}} d^3x \tag{160}$$

$$= \sqrt{\frac{m}{\omega}} \frac{1}{(2\pi)^{3/2}} \int \bar{v}_i(\mathbf{k}) \gamma^0 \psi(\mathbf{x},0) e^{i\mathbf{k}\cdot\mathbf{x}} d^3x \tag{161}$$

$$\hat{b}_i^{\dagger}(\mathbf{k}) = \sqrt{\frac{m}{\omega}} \frac{1}{(2\pi)^{3/2}} \int \psi^{\dagger}(\mathbf{x},0) h\gamma^0 u_i(\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{x}} d^3x \tag{162}$$

$$\hat{d}_j(\mathbf{k}) = -\sqrt{\frac{m}{\omega}} \frac{1}{(2\pi)^{3/2}} \int \psi^{\dagger}(\mathbf{x},0) h v_i(-\mathbf{k}) e^{-i\mathbf{k}\cdot\mathbf{x}} d^3x \tag{163}$$

$$= \sqrt{\frac{m}{\omega}} \frac{1}{(2\pi)^{3/2}} \int \psi^{\dagger}(\mathbf{x},0) h\gamma^0 v_i(\mathbf{k}) e^{-i\mathbf{k}\cdot\mathbf{x}} d^3x \tag{164}$$

Next we want to find the commutation relations satisfied by these mode amplitudes. For this it is convenient to rewrite the fundamental commutator,

$$\left[\hat{\pi}_A(\mathbf{x},t), \hat{\psi}^B(\mathbf{x}',t)\right] = i\delta_A^B \delta^3(\mathbf{x}-\mathbf{x}') \tag{165}$$

in terms of $\hat{\psi}$ and $\hat{\psi}^{\dagger}$. Replacing $\hat{\pi}_A(\mathbf{x},t)$ by $i\hat{\psi}^{\dagger}(\mathbf{x},t)h\gamma^0$ we have

$$\begin{aligned}
i\left[\left[\hat{\psi}^{\dagger}(\mathbf{x},t)\right]^C h_{CD} \left[\gamma^0\right]^D{}_A, \left[\hat{\psi}(\mathbf{x}',t)\right]^B\right] &= i\delta_A^B \delta^3(\mathbf{x}-\mathbf{x}') \\
i\left[\left[\hat{\psi}^{\dagger}(\mathbf{x},t)\right]_D, \left[\hat{\psi}(\mathbf{x}',t)\right]^B\right] \left[\gamma^0\right]^D{}_A &= i\delta_A^B \delta^3(\mathbf{x}-\mathbf{x}') \\
\left[\left[\hat{\psi}^{\dagger}(\mathbf{x},t)\right]_C, \left[\hat{\psi}(\mathbf{x}',t)\right]^B\right] &= \left[\gamma^0\right]^B{}_C \delta^3(\mathbf{x}-\mathbf{x}')
\end{aligned}$$

or simply

$$\left[\hat{\psi}^{\dagger}(\mathbf{x},t)h, \hat{\psi}(\mathbf{x}',t)\right] = \gamma^0 \delta^3(\mathbf{x}-\mathbf{x}') \tag{166}$$

We are now in a position to compute the commutators of the mode operators

1.5.1 Anticommutation

Now consider the $\hat{b}_a(\mathbf{k}'), \hat{b}_b^{\dagger}(\mathbf{k})$ and $\hat{d}_i^{\dagger}(\mathbf{k}), \hat{d}_j(\mathbf{k}')$ commutators:

$$\begin{aligned}
\left[\hat{b}_a(\mathbf{k}'), \hat{b}_b^{\dagger}(\mathbf{k})\right] &= \frac{m}{(2\pi)^3 \omega} \int\int d^3x d^3x' e^{i\mathbf{k}\cdot\mathbf{x} - i\mathbf{k}'\cdot\mathbf{x}'} \bar{u}_{aC}(\mathbf{k}') \left[\gamma^0\right]^C{}_D \\
&\quad \times \left[\psi^D(\mathbf{x}',0), \left(\psi^{\dagger}(\mathbf{x},0)h\right)_A\right] \left[\gamma^0\right]^A{}_B \left[u_b(\mathbf{k})\right]^B \\
&= -\frac{m}{(2\pi)^3 \omega} \int\int d^3x d^3x' e^{i\mathbf{k}\cdot\mathbf{x} - i\mathbf{k}'\cdot\mathbf{x}'} \\
&\quad \times \bar{u}_{aC}(\mathbf{k}') \left[\gamma^0\right]^C{}_D \left[\gamma^0\right]^D{}_A \left[\gamma^0\right]^A{}_B \left[u_b(\mathbf{k})\right]^B \delta^3(\mathbf{x}-\mathbf{x}')
\end{aligned}$$

$$
\begin{aligned}
&= -\frac{m}{(2\pi)^3\omega}\int d^3x e^{i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{x}}\bar{u}_{aC}(\mathbf{k}')\left[\gamma^0\right]^C{}_B\left[u_b(\mathbf{k})\right]^B \\
&= -\frac{m}{\omega}\frac{1}{(2\pi)^3}\int d^3x\frac{\omega}{m}\delta_{ab}e^{i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{x}} \\
&= -\delta^3(\mathbf{k}-\mathbf{k}')\delta_{ab}
\end{aligned}
$$

and

$$
\begin{aligned}
\left[\hat{d}_a(\mathbf{k}),\hat{d}_b^\dagger(\mathbf{k}')\right] &= \frac{m}{\omega}\frac{1}{(2\pi)^3}\int\int d^3xd^3x'e^{-i\mathbf{k}\cdot\mathbf{x}+i\mathbf{k}'\cdot\mathbf{x}'}\left[\gamma^0\right]^D{}_E v_a^E(\mathbf{k}') \\
&\quad\times\left[\left[\psi^\dagger\right]^C(\mathbf{x}',0)h_{CD},\psi^B(\mathbf{x},0)\right]\bar{v}_{bA}(\mathbf{k})\left[\gamma^0\right]^A{}_B \\
&= \frac{m}{\omega}\frac{1}{(2\pi)^3}\int\int d^3xd^3x'e^{-i\mathbf{k}\cdot\mathbf{x}+i\mathbf{k}'\cdot\mathbf{x}'}\left[\gamma^0\right]^D{}_E v_a^E(\mathbf{k}') \\
&\quad\times\left[\gamma^0\right]^B{}_D\delta^3(\mathbf{x}-\mathbf{x}')\bar{v}_{bA}(\mathbf{k})\left[\gamma^0\right]^A{}_B \\
&= \frac{m}{\omega}\frac{1}{(2\pi)^3}\int\int d^3xd^3x'e^{-i\mathbf{k}\cdot\mathbf{x}+i\mathbf{k}'\cdot\mathbf{x}'}\bar{v}_{bA}(\mathbf{k})\gamma^0v_a(\mathbf{k}')\delta^3(\mathbf{x}-\mathbf{x}') \\
&= \frac{m}{\omega}\delta^3(\mathbf{k}-\mathbf{k}')\bar{v}_{bA}(\mathbf{k})\left[\gamma^0\right]^A{}_E v_b^E(\mathbf{k}') \\
&= \delta_{ab}\delta^3(\mathbf{k}-\mathbf{k}')
\end{aligned}
$$

This is just the relationship we expect for $\hat{d}_a(\mathbf{k})$ – the mode amplitudes $\hat{d}_a(\mathbf{k})$ and $\hat{d}_a^\dagger(\mathbf{k})$ act as annihilation and creation operators, respectively. However, commutator

$$
\left[\hat{b}_a(\mathbf{k}),\hat{b}_b^\dagger(\mathbf{k}')\right] = -\delta^3(\mathbf{k}-\mathbf{k}')\delta_{ab}
$$

has the wrong sign, with $\hat{b}_a(\mathbf{k})$ rather than $\hat{b}_b^\dagger(\mathbf{k})$ acting like the creation operator. However, $\hat{b}_a(\mathbf{k})$ multiplies $e^{-i(\omega t-\mathbf{k}\cdot\mathbf{x})}$ while $\hat{d}_a^\dagger(\mathbf{k})$ multiplies the $CP\Theta$ conjugate of $e^{-i(\omega t-\mathbf{k}\cdot\mathbf{x})}$. This is consistent with our identification of these modes as particles and antiparticles, respectively. As we shall see, this pairing of particle creation with antiparticle annihilation, and vice versa, is necessary for other reasons as well. The identification we have chosen is necessary for conservation of charge (How could the action ψ have the potential to either create an electron or create a positron, since these have opposite electrical charges?). In addition, particle-antiparticle annihilation would not work correctly – every interaction that created a particle would have to annihilate a particle. We do not observe this. What went wrong?

We have very little freedom for introducing a sign here. In particular, the bilinear form $v_i^\dagger\gamma^0v_j$ is governed by the Lorentz invariance properties of the spinor products. An overall sign on the field or the momentum would change the sign of the$\hat{d}_a(\mathbf{k})$ commutator as well as the $\hat{b}_a(\mathbf{k})$ commutator, thereby merely displacing the problem. Moreover, since $\hat{b}_a(\mathbf{k})$ and $\hat{b}_b^\dagger(\mathbf{k})$ enter the commutator together, a relative sign in the definition of $\hat{b}_a(\mathbf{k})$ is cancelled by a corresponding sign from $\hat{b}_b^\dagger(\mathbf{k})$. The only place a sign enters in a way that we could change the outcome is in our use of the antisymmetry of the commutator. If this "bracket" of conjugate variables were symmetric instead of antisymmetric, the proper relationship would be restored. But recall that this bracket was imposed by fiat – it is simply a rule that says we should take Poisson brackets to field commutators to arrive at the quantum field theory from the classical field theory.

Of course, we know that using anticommutators for fermionic fields *is* the right answer – essentially all of the rigid strucuture of the world, from the discretely stacked energy levels of nucleons in the nucleus and electrons in atoms to the endstates of stars as white dwarfs and neutron stars, relies on the Pauli exclusion principle. This principle states that no two fermions can occupy the same state and it is enforced mathematically by requiring fermion fields to anticommute. Here, we see the principle emerge from field theory as a condition of chronicity invariance. Below, we will see that the same conclusion follows from a consideration of energy.

Returning to the previous calculations, we see that nothing goes awry if we replace the canonical quantization rule with a sign change to an anticommutator in the case of fermions. The fundamental anticommutation relations for the Dirac field are then:

$$\left\{\hat{\pi}_A(\mathbf{x},t),\hat{\psi}^B(\mathbf{x}',t)\right\} \equiv \hat{\pi}_A(\mathbf{x},t)\hat{\psi}^B(\mathbf{x}',t)+\hat{\psi}^B(\mathbf{x}',t)\hat{\pi}_A(\mathbf{x},t) \quad (167)$$

$$= i\delta_A^B\delta^3\left(\mathbf{x}-\mathbf{x}'\right) \quad (168)$$

with the consequence

$$\left\{\hat{b}_i^\dagger(\mathbf{k}),\hat{b}_j(\mathbf{k}')\right\} = \delta_{ij}\delta^3\left(\mathbf{k}-\mathbf{k}'\right)$$

$$\left\{\hat{d}_i^\dagger(\mathbf{k}),\hat{d}_i(\mathbf{k}')\right\} = \delta_{ij}\delta^3\left(\mathbf{k}-\mathbf{k}'\right)$$

All other *anti*commutators vanish.

1.5.2 The Dirac Hamiltonian

Next, consider the Hamiltonian. We wish to express it as a quantum operator in terms of the creation and annihilation operators. It is now convenient to use the simplified form of the Dirac Hamiltonian, eq.(3):

$$H = i\int d^3x\ \pi\gamma^0\left(i\gamma^i\partial_i - m\right)\psi = \int d^3x\ \pi\partial_0\psi \quad (169)$$

so that

$$\hat{H} = i\int d^3x\ \pi\gamma^0\left(i\gamma^i\partial_i - m\right)\psi = \int d^3x\ \pi\partial_0\psi \quad (170)$$

We begin by substituting the field operator expansions,

$$\hat{\psi}(\mathbf{x},t) = \frac{1}{(2\pi)^{3/2}}\sum_{i=1}^{2}\int d^3k\sqrt{\frac{m}{\omega}}\left(\hat{b}_i(\mathbf{k})u_i(\mathbf{k})e^{-i(\omega t-\mathbf{k}\cdot\mathbf{x})}\right. \quad (171)$$

$$\left.+\ \hat{d}_i^\dagger(\mathbf{k})v_i(\mathbf{k})e^{i(\omega t-\mathbf{k}\cdot\mathbf{x})}\right) \quad (172)$$

$$\hat{\psi}^\dagger(\mathbf{x},t) = \frac{1}{(2\pi)^{3/2}}\sum_{i=1}^{2}\int d^3k\sqrt{\frac{m}{\omega}}\left(\hat{b}_i^\dagger(\mathbf{k})u_i^\dagger(\mathbf{k})e^{i(\omega t-\mathbf{k}\cdot\mathbf{x})}\right. \quad (173)$$

$$\left.+\ \hat{d}_i(\mathbf{k})v_i^\dagger(\mathbf{k})e^{-i(\omega t-\mathbf{k}\cdot\mathbf{x})}\right) \quad (174)$$

$$\hat{\pi}(\mathbf{x},t) = i\psi^\dagger(\mathbf{x},t)h\gamma^0$$

$$= \frac{i}{(2\pi)^{3/2}}\sum_{i=1}^{2}\int d^3k\sqrt{\frac{m}{\omega}}\left(\hat{d}_i(\mathbf{k})v_i^\dagger(\mathbf{k})h\gamma^0e^{-i(\omega t-\mathbf{k}\cdot\mathbf{x})}\right. \quad (175)$$

$$\left.+\ \hat{b}_i^\dagger(\mathbf{k})u_i^\dagger(\mathbf{k})h\gamma^0e^{i(\omega t-\mathbf{k}\cdot\mathbf{x})}\right) \quad (176)$$

into the integral for the Hamiltonian,

$$\hat{H} = \int d^3x\ :\hat{\pi}\partial_0\hat{\psi}: \quad (177)$$

$$= \frac{i}{(2\pi)^3}\sum_{a=1}^{2}\sum_{b=1}^{2}\int d^3x\int d^3k\int d^3k'\ \frac{m}{\sqrt{\omega\omega'}}:\left(\hat{d}_a(\mathbf{k})v_a^\dagger(\mathbf{k})h\gamma^0e^{-i(\omega t-\mathbf{k}\cdot\mathbf{x})}\right.$$

$$\left.+\ \hat{b}_a^\dagger(\mathbf{k})u_a^\dagger(\mathbf{k})h\gamma^0e^{i(\omega t-\mathbf{k}\cdot\mathbf{x})}\right)$$

$$\times\Big(-i\omega_b'\hat{b}(\mathbf{k}')u_b(\mathbf{k}')e^{-i\left(\omega' t-\mathbf{k}'\cdot\mathbf{x}\right)}$$
$$+i\omega'\ \hat{d}_b^\dagger(\mathbf{k}')v_b(\mathbf{k}')e^{i\left(\omega' t-\mathbf{k}'\cdot\mathbf{x}\right)}\Big): \tag{178}$$

Collecting terms we have

$$\begin{aligned}\hat{H} &= \frac{i}{(2\pi)^3}\sum_{a=1}^{2}\sum_{b=1}^{2}\int d^3x\int d^3k\int d^3k'\ \frac{i\omega' m}{\sqrt{\omega\omega'}}\\ &\times:\Big(-\hat{d}_a(\mathbf{k})\hat{b}_b(\mathbf{k}')v_a^\dagger(\mathbf{k})h\gamma^0u_b(\mathbf{k}')e^{-i\left(\omega+\omega'\right)t+i\left(\mathbf{k}+\mathbf{k}'\right)\cdot\mathbf{x}}\\ &-\hat{b}_a^\dagger(\mathbf{k})\hat{b}_b(\mathbf{k}')u_a^\dagger(\mathbf{k})h\gamma^0u_b(\mathbf{k}')e^{i\left(\omega-\omega'\right)t-i\left(\mathbf{k}-\mathbf{k}'\right)\cdot\mathbf{x}}\\ &+\hat{d}_a(\mathbf{k})\hat{d}_b^\dagger(\mathbf{k}')v_a^\dagger(\mathbf{k})h\gamma^0v_b(\mathbf{k}')e^{-i\left(\omega-\omega'\right)t+i\left(\mathbf{k}-\mathbf{k}'\right)\cdot\mathbf{x}}\\ &+\hat{b}_a^\dagger(\mathbf{k})\hat{d}_b^\dagger(\mathbf{k}')u_a^\dagger(\mathbf{k})h\gamma^0v_b(\mathbf{k}')e^{i\left(\omega+\omega'\right)t-i\left(\mathbf{k}+\mathbf{k}'\right)\cdot\mathbf{x}}\Big):\end{aligned} \tag{179}$$

Now, integrating over d^3x, we produce Dirac delta functions:

$$\begin{aligned}\hat{H} &= \sum_{a=1}^{2}\sum_{b=1}^{2}\int d^3k\int d^3k'\ \frac{\omega' m}{\sqrt{\omega\omega'}}\\ &\times:\Big(\hat{d}_a(\mathbf{k})b_b(\mathbf{k}')v_a^\dagger(\mathbf{k})h\gamma^0u_b(\mathbf{k}')\delta^3\left(\mathbf{k}+\mathbf{k}'\right)e^{-2i\omega t}\\ &+b_a^\dagger(\mathbf{k})b_b(\mathbf{k}')u_a^\dagger(\mathbf{k})h\gamma^0u_b(\mathbf{k}')\delta^3\left(\mathbf{k}-\mathbf{k}'\right)\\ &-\hat{d}_a(\mathbf{k})\hat{d}_b^\dagger(\mathbf{k}')v_a^\dagger(\mathbf{k})h\gamma^0v_b(\mathbf{k}')\delta^3\left(\mathbf{k}-\mathbf{k}'\right)\\ &-b_a^\dagger(\mathbf{k})\hat{d}_b^\dagger(\mathbf{k}')u_a^\dagger(\mathbf{k})h\gamma^0v_b(\mathbf{k}')\delta^3\left(\mathbf{k}+\mathbf{k}'\right)e^{2i\omega t}\Big):\end{aligned} \tag{180}$$

which immediately integrate to give

$$\begin{aligned}\hat{H} &= m\sum_{a=1}^{2}\sum_{b=1}^{2}\int d^3k\\ &\times:\Big(\hat{d}_a(\mathbf{k})\hat{b}_b(-\mathbf{k})v_a^\dagger(\mathbf{k})h\gamma^0u_b(-\mathbf{k})e^{-2i\omega t}\\ &+\hat{b}_a^\dagger(\mathbf{k})\hat{b}_b(\mathbf{k})u_a^\dagger(\mathbf{k})h\gamma^0u_b(\mathbf{k})\\ &-\hat{d}_a(\mathbf{k})\hat{d}_b^\dagger(\mathbf{k})v_a^\dagger(\mathbf{k})h\gamma^0v_b(\mathbf{k})\\ &-\hat{b}_a^\dagger(\mathbf{k})\hat{d}_b^\dagger(-\mathbf{k})u_a^\dagger(\mathbf{k})h\gamma^0v_b(-\mathbf{k})e^{2i\omega t}\Big):\end{aligned} \tag{181}$$

Finally, we replace the inner products using

$$v_a(-\mathbf{k}) = -\gamma^0v_a(\mathbf{k}) \tag{182}$$
$$u_a(-\mathbf{k}) = \gamma^0u_a(\mathbf{k}) \tag{183}$$

thereby arriving at

$$\begin{aligned}\hat{H} &= m\sum_{a=1}^{2}\sum_{b=1}^{2}\int d^3k\\ &\times:\Big(\hat{d}_a(\mathbf{k})\hat{b}_b(-\mathbf{k})v_a^\dagger(\mathbf{k})hu_b(\mathbf{k})e^{-2i\omega t}\\ &+\hat{b}_a^\dagger(\mathbf{k})\hat{b}_b(\mathbf{k})u_a^\dagger(\mathbf{k})h\gamma^0u_b(\mathbf{k})\\ &-\hat{d}_a(\mathbf{k})\hat{d}_b^\dagger(\mathbf{k})v_a^\dagger(\mathbf{k})h\gamma^0v_b(\mathbf{k})\end{aligned}$$

$$
\begin{aligned}
& + \hat{b}_a^\dagger(\mathbf{k})\hat{d}_b^\dagger(-\mathbf{k})u_a^\dagger(\mathbf{k})hv_b(\mathbf{k})e^{2i\omega t}\Big) : \\
= \;& m\sum_{a=1}^{2}\sum_{b=1}^{2}\int d^3k : \left(\hat{b}_a^\dagger(\mathbf{k})\hat{b}_b(\mathbf{k})\frac{\omega}{m}\delta_{ab} - \hat{d}_a(\mathbf{k})\hat{d}_b^\dagger(\mathbf{k})\frac{\omega}{m}\delta_{ab}\right) : \\
= \;& \sum_{a=1}^{2}\int d^3k\ \omega : \left(\hat{b}_a^\dagger(\mathbf{k})\hat{b}_a(\mathbf{k}) - \hat{d}_a(\mathbf{k})\hat{d}_a^\dagger(\mathbf{k})\right) : \qquad (184)
\end{aligned}
$$

This would be a troubling result if it weren't for the anticommutation relations. If we simply used the normal ordering procedure, the second term would be negative and the energy indefinite. However,

$$
\left\{\hat{d}_a^\dagger(\mathbf{k}), \hat{d}_b(\mathbf{k}')\right\} = \hat{d}_a^\dagger(\mathbf{k})\hat{d}_b(\mathbf{k}') + \hat{d}_b(\mathbf{k}')\hat{d}_a^\dagger(\mathbf{k}) = \delta_{ab}\delta^3\left(\mathbf{k} - \mathbf{k}'\right) \qquad (185)
$$

so the normal ordering prescription is taken to mean

$$
: \hat{d}_b(\mathbf{k}')\hat{d}_a^\dagger(\mathbf{k}) := -\hat{d}_a^\dagger(\mathbf{k})\hat{d}_b(\mathbf{k}') \qquad (186)
$$

We then can write the normal ordered Hamiltonian operator as

$$
\hat{H} = \sum_{a=1}^{2}\int d^3k\ \omega\left(\hat{b}_a^\dagger(\mathbf{k})\hat{b}_a(\mathbf{k}) + \hat{d}_a^\dagger(\mathbf{k})\hat{d}_a(\mathbf{k})\right) \qquad (187)
$$

This convention preserves the anticommutativity, while still eliminating the infinite delta function contribution to the vacuum energy.

1.6 Symmetries of the Dirac field

We'd now like to find the conserved currents of the Dirac field. There are two kinds – the spacetime symmetries, including Lorentz transformations and translations, and a $U(1)$ phase symmetry. We'll discuss the spacetime symmetries first. We put off our study of the phase symmetry to the next chapter, where it leads us systematically to Quantum Electrodynamics: QED.

1.6.1 Translations

Under a translation, $x^\alpha \to x^\alpha + a^\alpha$, the Dirac field changes by

$$
\psi\left(x^\alpha\right) \to \psi\left(x^\alpha + a^\alpha\right) = \psi\left(x^\alpha\right) + \frac{\partial\psi\left(x^\alpha\right)}{\partial x^\beta}a^\beta \qquad (188)
$$

so we identify Δ of eq.() as

$$
\Delta = \left(\partial_\beta\psi\right)a^\beta \qquad (189)
$$

The four conserved currents form the stress-energy tensor, given by eq.():

$$
\begin{aligned}
T^{\alpha\beta} &= \frac{\delta\mathcal{L}}{\delta\left(\partial_\alpha\psi\right)}\partial^\beta\psi - \mathcal{L}\eta^{\mu\beta} && (190) \\
&= i\bar{\psi}\gamma^\alpha\partial^\beta\psi - \eta^{\alpha\beta}\bar{\psi}\left(i\gamma^\mu\partial_\mu - m\right)\psi && (191) \\
&= i\bar{\psi}\gamma^\alpha\partial^\beta\psi && (192)
\end{aligned}
$$

since the Lagrangian density vanishes when the field equation is satisfied. For the conserved charges, we therefore find that the conserved energy is the Hamiltonian,

$$
\begin{aligned}
P^0 &= i\int d^3x\bar{\psi}\gamma^0\partial^0\psi && (193) \\
&= i\int d^3x\bar{\psi}\gamma^0\partial_0\psi && (194) \\
&= H && (195)
\end{aligned}
$$

while the conserved momentum is

$$
\begin{aligned}
P^i &= -i : \int d^3x \bar{\psi}\gamma^0 \partial_i \psi : && (196)\\
&= \sum_{a=1}^{2}\sum_{b=1}^{2} \int d^3k \frac{mk'^i}{\omega} && (197)\\
&\quad \times \Big(\hat{b}_a^\dagger(\mathbf{k})\hat{b}_b(\mathbf{k}) u_a^\dagger(\mathbf{k}) h\gamma^0 u_b(\mathbf{k}) && (198)\\
&\quad - \hat{b}_a^\dagger(\mathbf{k})\hat{d}_b^\dagger(-\mathbf{k}) u_a^\dagger(\mathbf{k}) h\gamma^0 e^{2i\omega t} v_b(-\mathbf{k}) && (199)\\
&\quad + \hat{d}_a(\mathbf{k})\hat{b}_b(-\mathbf{k}) v_a^\dagger(\mathbf{k}) h\gamma^0 e^{-2i\omega t} u_b(-\mathbf{k}) && (200)\\
&\quad - : \hat{d}_a(\mathbf{k})\hat{d}_b^\dagger(\mathbf{k}) : v_a^\dagger(\mathbf{k}) h\gamma^0 v_b(\mathbf{k}) \Big) && (201)\\
&= \sum_{a=1}^{2} \int d^3k \; k'^i \left(\hat{b}_a^\dagger(\mathbf{k})\hat{b}_a(\mathbf{k}) - : \hat{d}_a(\mathbf{k})\hat{d}_a^\dagger(\mathbf{k}) : \right) && (202)\\
&= \sum_{a=1}^{2} \int d^3k \; k'^i \left(\hat{b}_a^\dagger(\mathbf{k})\hat{b}_a(\mathbf{k}) + \hat{d}_a^\dagger(\mathbf{k})\hat{d}_a(\mathbf{k}) \right) && (203)
\end{aligned}
$$

This is just what we expect.