Spin States

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We have seen that the three spin operators may be written in terms of the Pauli matrices,

$$\hat{S}_i = \frac{\hbar}{2}\hat{\sigma}_i$$

or, equivalently, in bra-ket notation as

$$\begin{aligned}\hat{S}_x &= \frac{\hbar}{2}\left(|+\rangle\langle-| + |-\rangle\langle+|\right)\\ \hat{S}_y &= \frac{i\hbar}{2}\left(|-\rangle\langle+| - |+\rangle\langle-|\right)\\ \hat{S}_z &= \frac{\hbar}{2}\left(|+\rangle\langle+| - |-\rangle\langle-|\right)\end{aligned}$$

Consider the evolution of a beam of electrons, prepared in the normalized state,

$$|A\rangle = \alpha|+\rangle + \beta|-\rangle$$

where $|\alpha|^2 + |\beta|^2 = 1$, as we measure successive components of spin. If we make a measurement of the z-component of spin,

$$\begin{aligned}\hat{S}_z|A\rangle &= \left(\frac{\hbar}{2}\left(|+\rangle\langle+| - |-\rangle\langle-|\right)\right)\left(\alpha|+\rangle + \beta|-\rangle\right)\\ &= \frac{\alpha\hbar}{2}|+\rangle - \frac{\beta\hbar}{2}|-\rangle\end{aligned}$$

then the probablility that we measure the value $+\frac{\hbar}{2}$, and therefore find the state to be $|+\rangle$ is

$$\begin{aligned}|\langle+|A\rangle|^2 &= |\langle+|\left(\alpha|+\rangle - \beta|-\rangle\right)|^2\\ &= |\alpha|^2\end{aligned}$$

while the probability of measuring $-\frac{\hbar}{2}$ is

$$|\langle-|A\rangle|^2 = |\beta|^2$$

The expectation value of the spin, essentially the average of many measurements, is

$$\begin{aligned}\langle A|\hat{S}_z|A\rangle &= \left(\langle+|\alpha^* - \langle-|\beta^*\right)\left(\frac{\alpha\hbar}{2}|+\rangle - \frac{\beta\hbar}{2}|-\rangle\right)\\ &= \frac{\hbar}{2}\alpha^*\alpha - \frac{\hbar}{2}\beta^*\beta\end{aligned}$$

i.e., $+\frac{\hbar}{2}$ times the probability of measuring spin up, plus $-\frac{\hbar}{2}$ times the probability of measuring spin down.

Suppose we measure a given electron to have z-component of spin $+\frac{\hbar}{2}$. Then the subsequent state must reflect this, and is therefore

$$|A'\rangle = \alpha|+\rangle$$

for the resulting spin-up beam. This state no longer has the same normalization, because we have eliminated the spin-down portion of the beam of electrons. If we make another measurement of the z-component of this state, the result is

$$\begin{aligned} \hat{S}_z |A'\rangle &= \left(\frac{\hbar}{2} (|+\rangle \langle+| \ - \ |-\rangle \langle-|) \right) \alpha |+\rangle \\ &= \frac{\hbar}{2} \alpha |+\rangle \end{aligned}$$

This corresponds to measuring the value $+\frac{\hbar}{2}$ every time, and we see that $|A'\rangle$ is already an eigenstate of $\hat{S}_z$. The state is unchanged by the subsequent measurement of the same observable.

On the other hand, suppose we measure the x-component of spin for $|A'\rangle$,

$$\begin{aligned} \hat{S}_x |A'\rangle &= \frac{\hbar}{2} (|+\rangle \langle-| \ + \ |-\rangle \langle+|) \alpha |+\rangle \\ &= \frac{\alpha\hbar}{2} |-\rangle \end{aligned}$$

The state is altered by the measurement, so it is not an eigenstate of $\hat{S}_x$. To find the probabilities for measuring the x-component up or down, we need to write $|A'\rangle$ in terms of the eigenstates of $\hat{S}_x$. These are not hard to find. They satisfy

$$\hat{S}_x \left| \hat{S}_x, \lambda \right\rangle = \lambda \left| \hat{S}_x, \lambda \right\rangle$$

Expanding in terms of the z-basis,

$$\left| \hat{S}_x, \lambda \right\rangle = a |+\rangle - b |-\rangle$$

the eigenvector equation becomes

$$\begin{aligned} \frac{\hbar}{2} (|+\rangle \langle-| \ + \ |-\rangle \langle+|) (a |+\rangle \ - \ b |-\rangle) &= \lambda (a |+\rangle \ - \ b |-\rangle) \\ -\frac{\hbar}{2} b |+\rangle \ + \ \frac{\hbar}{2} a |-\rangle &= \lambda (a |+\rangle \ - \ b |-\rangle) \end{aligned}$$

so that, equating like components,

$$\begin{aligned} -\frac{\hbar}{2} b |+\rangle &= a\lambda |+\rangle \\ \frac{\hbar}{2} a |-\rangle &= -\lambda b |-\rangle \end{aligned}$$

Solving the second, we have

$$b = -\frac{\hbar}{2\lambda} a$$

and substituting this into the first gives

$$\begin{aligned} \frac{\hbar}{2} \frac{\hbar}{2\lambda} a &= a\lambda \\ \left(\frac{\hbar}{2} \right)^2 &= \lambda^2 \\ \lambda &= \pm \frac{\hbar}{2} \end{aligned}$$

With these values for λ, we find two states, having $b = \mp a$. Normalizing by requiring $a^2 + b^2 = 1$, the eigenstates are

$$\begin{aligned} \left|\hat{S}_x, \frac{\hbar}{2}\right\rangle &= \frac{1}{\sqrt{2}}\left(|+\rangle - |-\rangle\right) \\ \left|\hat{S}_x, -\frac{\hbar}{2}\right\rangle &= \frac{1}{\sqrt{2}}\left(|+\rangle + |-\rangle\right) \end{aligned}$$

Now return to the state $|A'\rangle = \alpha\,|+\rangle$. We may write this in terms of the eigenstates of $\hat{S}_x$, as

$$\begin{aligned} |A'\rangle &= \alpha\,|+\rangle \\ &= \frac{\alpha}{\sqrt{2}}\left(\left|\hat{S}_x, +\frac{\hbar}{2}\right\rangle + \left|\hat{S}_x, -\frac{\hbar}{2}\right\rangle\right) \end{aligned}$$

and now we see that the probability of measuring the x-component of spin to be $+\frac{\hbar}{2}$ is

$$\begin{aligned} \left|\left\langle \hat{S}_x, +\frac{\hbar}{2} \mid A'\right\rangle\right|^2 &= \left|\left\langle \hat{S}_x, +\frac{\hbar}{2}\right| \alpha\,|+\rangle\right|^2 \\ &= \left|\frac{\alpha}{\sqrt{2}}\right|^2 \\ &= \frac{1}{2}\,|\alpha|^2 \end{aligned}$$

The probability of measuring spin down is also $\frac{1}{2}\,|\alpha|^2$, where the factor $|\alpha|^2$ reflects the dimunition of the beam by the original z measurement. Therefore, a state which has its z-component of spin in the spin up state has equal probability of finding the x-component of spin in either the spin up or spin down state.