Solutions to the wave equation

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1 Constant potential

The stationary state Schrödinger equation is

$$-\frac{\hbar^2}{2m}\nabla^2\psi + V\psi = E\psi$$

which, for a constant potential V_0, we may rearrange as

$$\nabla^2\psi = -\frac{2m}{\hbar^2}\left(E - V_0\right)\psi$$

Let $\psi = Ae^{\frac{i}{\hbar}\mathbf{p}\cdot\mathbf{x}}$. Then

$$\begin{aligned}\left(\frac{i}{\hbar}\mathbf{p}\right)^2\psi &= -\frac{2m}{\hbar^2}\left(E - V_0\right)\psi \\ \mathbf{p}^2\psi &= 2m\left(E - V_0\right)\psi\end{aligned}$$

so we must have

$$E = \frac{\mathbf{p}^2}{2m} + V_0$$

This gives a stationary state wave function,

$$\psi\left(\mathbf{x}\right) = Ae^{\frac{i}{\hbar}\mathbf{p}\cdot\mathbf{x}}$$

Multiplying by the time dependence gives

$$\begin{aligned}\psi\left(\mathbf{x}, t\right) &= Ae^{\frac{i}{\hbar}\left(\mathbf{p}\cdot\mathbf{x} - Et\right)} \\ &= Ae^{\frac{i}{\hbar}\left(\mathbf{p}\cdot\mathbf{x} - \frac{\mathbf{p}^2}{2m}t + V_0 t\right)}\end{aligned}$$

A general solution is an arbitrary linear combination of these stationary states.

1.1 Free space solution

In empty space, $V_0 = 0$, and there are only boundary conditions at infinity. The general solution is therefore a superposition of the plane waves above over all values of $\mathbf{p}$. The contribution to the superposition may be different for each $\mathbf{p}$, so we allow the amplitude, A, to depend on momentum

$$\psi\left(\mathbf{x}, t\right) = \int d^3p A\left(\mathbf{p}\right) e^{\frac{i}{\hbar}\left(\mathbf{p}\cdot\mathbf{x} - \frac{\mathbf{p}^2}{2m}t\right)}$$

Suppose $A\left(\mathbf{p}\right)$ is a Gaussian centered on some momentum $\mathbf{p}_0$,

$$A\left(\mathbf{p}\right) = Ae^{-\frac{\left(\mathbf{p} - \mathbf{p}_0\right)^2}{2\lambda^2}}$$

where A is a single fixed normalization constant. This localizes the particle somewhat in momentum space, and will be shown in the one dimensional case below to do the same in configuration space. Knowing $A(\mathbf{p})$, we can now do the integral.

For simplicity, we restrict our attention to one dimension, so that

$$\psi(x,t) = A\int dp e^{-\frac{(p-p_0)^2}{2\lambda^2}} e^{\frac{i}{\hbar}\left(px - \frac{p^2}{2m}t\right)}$$

This avoids the angular integrals required by the dot products. Define

$$\alpha^2(t) \equiv \left(1 + \frac{i\lambda^2 t}{m\hbar}\right)$$

Then

$$\begin{aligned}
\psi(x,t) &= A\int dp \exp\left(-\frac{(p-p_0)^2}{2\lambda^2} + \frac{i}{\hbar}\left(px - \frac{p^2}{2m}t\right)\right) \\
&= A\int dp \exp\left(-\frac{1}{2\lambda^2}\left(p^2 - 2pp_0 + p_0^2 - \frac{2i\lambda^2}{\hbar}\left(px - \frac{p^2}{2m}t\right)\right)\right) \\
&= A\int dp \exp\left(-\frac{1}{2\lambda^2}\left(p^2\left(1 + \frac{i\lambda^2 t}{m\hbar}\right) - 2p\left(p_0 + \frac{i\lambda^2 x}{\hbar}\right) + p_0^2\right)\right) \\
&= A\int dp \exp\left(-\frac{1}{2\lambda^2}\left(\alpha^2 p^2 - 2p\left(p_0 + \frac{i\lambda^2 x}{\hbar}\right) + p_0^2\right)\right) \\
&= A\int dp \exp\left(-\frac{1}{2\lambda^2}\left(\left(\alpha p - \frac{1}{\alpha}\left(p_0 + \frac{i\lambda^2 x}{\hbar}\right)\right)^2 - \frac{1}{\alpha^2}\left(p_0 + \frac{i\lambda^2 x}{\hbar}\right)^2 + p_0^2\right)\right) \\
&= A\exp\left[\frac{1}{2\lambda^2\alpha^2}\left(p_0 + \frac{i\lambda^2 x}{\hbar}\right)^2 - \frac{p_0^2}{2\lambda^2}\right]\int dp \exp\left(-\frac{1}{2\lambda^2}\left(\alpha p - \frac{1}{\alpha}\left(p_0 + \frac{i\lambda^2 x}{\hbar}\right)\right)^2\right)
\end{aligned}$$

Now define

$$q \equiv \alpha p - \frac{1}{\alpha}\left(p_0 + \frac{i\lambda^2}{\hbar}x\right)$$

so that

$$dq = \alpha dp$$

so the integral becomes

$$\frac{1}{\alpha}\int dq \exp\left(-\frac{q^2}{2\lambda^2}\right)$$

For a Gaussian integral,

$$\begin{aligned}
I &= \int dq \exp\left(-\mu q^2\right) \\
I^2 &= \int dq \int dq' \exp\left(-\mu q^2 - \mu q'^2\right) \\
I^2 &= \int r dr d\varphi \exp\left(-\mu r^2\right) \\
&= 2\pi\frac{1}{2}\int_0^\infty d\left(r^2\right) \exp\left(-\mu r^2\right) \\
&= 2\pi\frac{1}{2}\int_0^\infty dy \exp\left(-\mu y\right)
\end{aligned}$$

$$
\begin{aligned}
&= 2\pi\frac{1}{2}\left(-\frac{1}{\mu}\exp\left(-\mu\infty\right)+\frac{1}{\mu}\exp\left(-\mu 0\right)\right) \\
&= \frac{\pi}{\mu} \\
I &= \sqrt{\frac{\pi}{\mu}}
\end{aligned}
$$

Therefore

$$
\frac{1}{\alpha}\int dq\,\exp\left(-\frac{q^2}{2\lambda^2}\right) = \frac{\lambda}{\alpha}\sqrt{2\pi}
$$

Returning to the wave function,

$$
\psi\left(x,t\right) = A\frac{\lambda}{\alpha}\sqrt{2\pi}\exp\left[\frac{1}{2\lambda^2\alpha^2}\left(p_0+\frac{i\lambda^2 x}{\hbar}\right)^2-\frac{p_0^2}{2\lambda^2}\right]
$$

Now separate the real and imaginary parts. Restore $\alpha^2 = 1+\frac{i\lambda^2 t}{m\hbar} = 1+i\beta t$ where $\beta\equiv\frac{\lambda^2}{m\hbar}$

$$
\begin{aligned}
\psi\left(x,t\right) &= A\frac{\lambda}{\alpha}\sqrt{2\pi}\exp\left[\frac{1}{2\lambda^2\alpha^2}\left(p_0+\frac{i\lambda^2 x}{\hbar}\right)^2-\frac{p_0^2}{2\lambda^2}\right] \\
&= A\frac{\sqrt{2\pi}\lambda}{\sqrt{1+i\beta t}}\exp\left[\frac{1}{2\lambda^2\left(1+i\beta t\right)}\left(p_0+\frac{i\lambda^2 x}{\hbar}\right)^2-\frac{p_0^2}{2\lambda^2}\right] \\
&= A\frac{\sqrt{2\pi}\lambda}{\sqrt{1+i\beta t}}\exp\left[\frac{\left(1-i\beta t\right)}{2\lambda^2\left(1+\beta^2 t^2\right)}\left(p_0^2+\frac{2i}{\hbar}\lambda^2 p_0 x-\frac{\lambda^4}{\hbar^2}x^2\right)-\frac{p_0^2}{2\lambda^2}\right] \\
&= A\frac{\sqrt{2\pi}\lambda}{\sqrt{1+i\beta t}}\exp\left[-\frac{1}{2\lambda^2\left(1+\beta^2 t^2\right)}\left(-p_0^2\left(1-i\beta t\right)-\frac{2i}{\hbar}\lambda^2 p_0 x\left(1-i\beta t\right)+\frac{\lambda^4}{\hbar^2}x^2\left(1-i\beta t\right)+p_0^2\left(1+\beta^2 t^2\right)\right)\right]
\end{aligned}
$$

Now reduce the factor in the exponent. The real part is:

$$
\begin{aligned}
-p_0^2-\frac{2}{\hbar}\lambda^2\beta p_0 t x+\frac{\lambda^4}{\hbar^2}x^2+p_0^2\left(1+\beta^2 t^2\right) &= \frac{\lambda^4}{\hbar^2}x^2-\frac{2}{\hbar}\lambda^2\beta p_0 t x+p_0^2\beta^2 t^2 \\
&= \frac{\lambda^4}{\hbar^2}\left(x^2-2x\frac{p_0}{m}t+\frac{p_0^2}{m^2}t^2\right) \\
&= \frac{\lambda^4}{\hbar^2}\left(x-\frac{p_0 t}{m}\right)^2
\end{aligned}
$$

while the imaginary part is

$$
\begin{aligned}
-p_0^2\left(-i\beta t\right)-\frac{2i}{\hbar}\lambda^2 p_0 x+\frac{\lambda^4}{\hbar^2}x^2\left(-i\beta t\right) &= -p_0^2\left(-i\frac{\lambda^2}{m\hbar}t\right)-\frac{2i}{\hbar}\lambda^2 p_0 x+\frac{\lambda^4}{\hbar^2}x^2\left(-i\frac{\lambda^2}{m\hbar}t\right) \\
&= i\frac{m^2\lambda^2}{\hbar}\left(\frac{p_0^2}{m^2}t-2\frac{p_0}{m}x-\frac{\lambda^4}{m^2\hbar^2}x^2 t\right)
\end{aligned}
$$

Then the wave function is

$$
\psi\left(x,t\right) = A\frac{\sqrt{2\pi}\lambda}{\left(1+\beta^2 t^2\right)^{1/4}}e^{i\varphi(x,t)}\exp\left[-\frac{\lambda^2}{2\hbar^2\left(1+\beta^2 t^2\right)}\left(x-\frac{p_0 t}{m}\right)^2\right]
$$

where the phase is

$$
\varphi\left(x,t\right) = \frac{1}{2}\tan^{-1}\beta t-\frac{1}{2\lambda^2\left(1+\beta^2 t^2\right)}\frac{m^2\lambda^2}{\hbar}\left(\frac{p_0^2}{m^2}t-2\frac{p_0}{m}x-\frac{\lambda^4}{m^2\hbar^2}x^2 t\right)
$$

At late times, $\beta t \gg 1$, this reduces to

$$\psi(\mathbf{x}, t) \approx \frac{A' e^{-\frac{i\pi}{4}}}{\sqrt{\beta t}} \exp\left[-\frac{\lambda^2}{2\hbar^2\beta^2 t^2}\left(x - \frac{p_0}{m}t\right)^2\right]$$

This describes a Gaussian wave packed centered at

$$x = \frac{p_0}{m}t$$

of width

$$\sigma = \frac{\hbar\beta t}{\lambda}$$

and amplitude decreasing as

$$\sim \frac{1}{\sqrt{\beta t}}$$

Notice that the standard deviation of the momentum distribution is $\sigma_p = \lambda$, while that of the spatial distribution is $\sigma_x = \frac{\hbar\sqrt{1+\left(\frac{\lambda^2 t}{m\hbar}\right)^2}}{\lambda}$, with product

$$\sigma_x \sigma_p = \hbar\sqrt{1 + \beta^2 t^2}$$

which grows from a minimum of $\hbar$ at $t = 0$, in agreement with the uncertainty relation.