But the right-hand side of (3.10.44) is the same as $\langle\alpha', jm|\mathbf{J}\cdot\mathbf{V}|\alpha, jm\rangle / \langle\alpha, jm|\mathbf{J}^2|\alpha, jm\rangle$ by (3.10.42) and (3.10.43). Moreover, the left-hand side of (3.10.43) is just $j(j+1)\hbar^2$. So

$$\langle\alpha', jm'|V_q|\alpha, jm\rangle = \frac{\langle\alpha', jm|\mathbf{J}\cdot\mathbf{V}|\alpha, jm\rangle}{\hbar^2 j(j+1)}\langle jm'|J_q|jm\rangle, \quad (3.10.45)$$

which proves the projection theorem. □

We will give applications of the theorem in subsequent sections.

PROBLEMS

1. Find the eigenvalues and eigenvectors of $\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$. Suppose an electron is in the spin state $\begin{pmatrix} \alpha \\ \beta \end{pmatrix}$. If s_y is measured, what is the probability of the result $\hbar/2$?
2. Consider the 2×2 matrix defined by
$$U = \frac{a_0 + i\boldsymbol{\sigma}\cdot\mathbf{a}}{a_0 - i\boldsymbol{\sigma}\cdot\mathbf{a}},$$
where a_0 is a real number and $\mathbf{a}$ is a three-dimensional vector with real components.
   a. Prove that U is unitary and unimodular.
   b. In general, a 2×2 unitary unimodular matrix represents a rotation in three dimensions. Find the axis and angle of rotation appropriate for U in terms of a_0, a_1, a_2, and a_3.
3. The spin-dependent Hamiltonian of an electron-positron system in the presence of a uniform magnetic field in the z-direction can be written as
$$H = A\mathbf{S}^{(e^-)}\cdot\mathbf{S}^{(e^+)} + \left(\frac{eB}{mc}\right)\left(S_z^{(e^-)} - S_z^{(e^+)}\right).$$
Suppose the spin function of the system is given by $\chi_+^{(e^-)}\chi_-^{(e^+)}$.
   a. Is this an eigenfunction of H in the limit $A\to0$, $eB/mc\neq0$? If it is, what is the energy eigenvalue? If it is not, what is the expectation value of H?
   b. Same problem when $eB/mc\to0$, $A\neq0$.
4. Consider a *spin* 1 particle. Evaluate the matrix elements of
$$S_z(S_z+\hbar)(S_z-\hbar) \quad\text{and}\quad S_x(S_x+\hbar)(S_x-\hbar).$$
5. Let the Hamiltonian of a rigid body be
$$H = \frac{1}{2}\left(\frac{K_1^2}{I_1} + \frac{K_2^2}{I_2} + \frac{K_3^2}{I_3}\right),$$
where $\mathbf{K}$ is the angular momentum in the body frame. From this expression obtain the Heisenberg equation of motion for $\mathbf{K}$ and then find Euler's equation of motion in the correspondence limit.

6. Let $U = e^{iG_3\alpha}e^{iG_2\beta}e^{iG_3\gamma}$, where (α,β,γ) are the Eulerian angles. In order that U represent a rotation (α,β,γ), what are the commutation rules satisfied by the G_k? Relate $\mathbf{G}$ to the angular momentum operators.
7. What is the meaning of the following equation:
$$U^{-1}A_kU = \sum R_{kl}A_l,$$
where the three components of $\mathbf{A}$ are matrices? From this equation show that matrix elements $\langle m|A_k|n\rangle$ transform like vectors.
8. Consider a sequence of Euler rotations represented by
$$\mathscr{D}^{(1/2)}(\alpha,\beta,\gamma) = \exp\left(\frac{-i\sigma_3\alpha}{2}\right)\exp\left(\frac{-i\sigma_2\beta}{2}\right)\exp\left(\frac{-i\sigma_3\gamma}{2}\right)$$
$$= \begin{pmatrix} e^{-i(\alpha+\gamma)/2}\cos\frac{\beta}{2} & -e^{-i(\alpha-\gamma)/2}\sin\frac{\beta}{2} \\ e^{i(\alpha-\gamma)/2}\sin\frac{\beta}{2} & e^{i(\alpha+\gamma)/2}\cos\frac{\beta}{2} \end{pmatrix}.$$
Because of the group properties of rotations, we expect that this sequence of operations is equivalent to a *single* rotation about some axis by an angle θ. Find θ.
9. a. Consider a pure ensemble of identically prepared spin $\frac{1}{2}$ systems. Suppose the expectation values $\langle S_x\rangle$ and $\langle S_z\rangle$ and the sign of $\langle S_y\rangle$ are known. Show how we may determine the state vector. Why is it unnecessary to know the magnitude of $\langle S_y\rangle$?
   b. Consider a mixed ensemble of spin $\frac{1}{2}$ systems. Suppose the ensemble averages $[S_x]$, $[S_y]$, and $[S_z]$ are all known. Show how we may construct the 2×2 density matrix that characterizes the ensemble.
10. a. Prove that the time evolution of the density operator ρ (in the Schrödinger picture) is given by
$$\rho(t) = \mathscr{U}(t,t_0)\rho(t_0)\mathscr{U}^\dagger(t,t_0).$$
   b. Suppose we have a pure ensemble at $t=0$. Prove that it cannot evolve into a mixed ensemble as long as the time evolution is governed by the Schrödinger equation.
11. Consider an ensemble of spin 1 systems. The density matrix is now a 3×3 matrix. How many independent (real) parameters are needed to characterize the density matrix? What must we know in addition to $[S_x]$, $[S_y]$, and $[S_z]$ to characterize the ensemble completely?
12. An angular-momentum eigenstate $|j, m=m_{\max}=j\rangle$ is rotated by an infinitesimal angle ε about the y-axis. Without using the explicit form of the $d_{m'm}^{(j)}$ function, obtain an expression for the probability for the new rotated state to be found in the original state up to terms of order ε^2.

13. Show that the 3×3 matrices G_i $(i=1,2,3)$ whose elements are given by

$$(G_i)_{jk} = -i\hbar\varepsilon_{ijk},$$

where j and k are the row and column indices, satisfy the angular momentum commutation relations. What is the physical (or geometric) significance of the transformation matrix that connects G_i to the more usual 3×3 representations of the angular-momentum operator J_i with J_3 taken to be diagonal? Relate your result to

$$\mathbf{V} \to \mathbf{V} + \hat{\mathbf{n}}\delta\phi \times \mathbf{V}$$

under infinitesimal rotations. (*Note*: This problem may be helpful in understanding the photon spin.)

14. a. Let $\mathbf{J}$ be angular momentum. It may stand for orbital $\mathbf{L}$, spin $\mathbf{S}$, or $\mathbf{J}_{\text{total}}$.) Using the fact that $J_x, J_y, J_z (J_\pm \equiv J_x \pm iJ_y)$ satisfy the usual angular-momentum commutation relations, prove

$$\mathbf{J}^2 = J_z^2 + J_+J_- - \hbar J_z.$$

b. Using (a) (or otherwise), derive the "famous" expression for the coefficient c_- that appears in

$$J_-\psi_{jm} = c_-\psi_{j,m-1}.$$

15. The wave function of a particle subjected to a spherically symmetrical potential $V(r)$ is given by

$$\psi(\mathbf{x}) = (x+y+3z)f(r).$$

a. Is ψ an eigenfunction of $\mathbf{L}^2$? If so, what is the l-value? If not, what are the possible values of l we may obtain when $\mathbf{L}^2$ is measured?

b. What are the probabilities for the particle to be found in various m_l states?

c. Suppose it is known somehow that $\psi(\mathbf{x})$ is an energy eigenfunction with eigenvalue E. Indicate how we may find $V(r)$.

16. A particle in a spherically symmetrical potential is known to be in an eigenstate of $\mathbf{L}^2$ and L_z with eigenvalues $\hbar^2 l(l+1)$ and $m\hbar$, respectively. Prove that the expectation values between $|lm\rangle$ states satisfy

$$\langle L_x\rangle = \langle L_y\rangle = 0, \qquad \langle L_x^2\rangle = \langle L_y^2\rangle = \frac{[l(l+1)\hbar^2 - m^2\hbar^2]}{2}.$$

Interpret this result semiclassically.

17. Suppose a half-integer l-value, say $\frac{1}{2}$, were allowed for orbital angular momentum. From

$$L_+Y_{1/2,1/2}(\theta,\phi) = 0,$$

we may deduce, as usual,

$$Y_{1/2,1/2}(\theta,\phi) \propto e^{i\phi/2}\sqrt{\sin\theta}.$$

Now try to construct $Y_{1/2,-1/2}(\theta,\phi)$; by (a) applying L_- to $Y_{1/2,1/2}(\theta,\phi)$; and (b) using $L_-Y_{1/2,-1/2}(\theta,\phi)=0$. Show that the two procedures lead to contradictory results. (This gives an argument against half-integer l-values for orbital angular momentum.)

18. Consider an orbital angular-momentum eigenstate $|l=2, m=0\rangle$. Suppose this state is rotated by an angle β about the y-axis. Find the probability for the new state to be found in $m=0, \pm 1$, and ± 2. (The spherical harmonics for $l=0$, 1, and 2 given in Appendix A may be useful.)

19. What is the physical significance of the operators

$$K_+ \equiv a_+^\dagger a_-^\dagger \quad \text{and} \quad K_- \equiv a_+ a_-$$

in Schwinger's scheme for angular momentum? Give the nonvanishing matrix elements of $K_\pm$.

20. We are to add angular momenta $j_1=1$ and $j_2=1$ to form $j=2$, 1, and 0 states. Using either the ladder operator method or the recursion relation, express all (nine) $\{j,m\}$ eigenkets in terms of $|j_1j_2; m_1m_2\rangle$. Write your answer as

$$|j=1,m=1\rangle = \frac{1}{\sqrt{2}}|+,0\rangle - \frac{1}{\sqrt{2}}|0,+\rangle, \ldots,$$

where $+$ and 0 stand for $m_{1,2}=1,0$, respectively.

21. a. Evaluate

$$\sum_{m=-j}^{j} |d_{mm'}^{(j)}(\beta)|^2 m$$

for *any* j (integer or half-integer); then check your answer for $j=\frac{1}{2}$.

b. Prove, for any j,

$$\sum_{m=-j}^{j} m^2|d_{m'm}^{(j)}(\beta)|^2 = \frac{1}{2}j(j+1)\sin^2\beta + m'^2\frac{1}{2}(3\cos^2\beta - 1).$$

[*Hint*: This can be proved in many ways. You may, for instance, examine the rotational properties of J_z^2 using the spherical (irreducible) tensor language.]

22. a. Consider a system with $j=1$. Explicitly write

$$\langle j=1, m'|J_y|j=1, m\rangle$$

in 3×3 matrix form.

b. Show that for $j=1$ only, it is legitimate to replace $e^{-iJ_y\beta/\hbar}$ by

$$1 - i\left(\frac{J_y}{\hbar}\right)\sin\beta - \left(\frac{J_y}{\hbar}\right)^2(1-\cos\beta).$$

c. Using (b), prove

$$d^{(j=1)}(\beta)=\begin{pmatrix}\left(\frac{1}{2}\right)(1+\cos\beta) & -\left(\frac{1}{\sqrt{2}}\right)\sin\beta & \left(\frac{1}{2}\right)(1-\cos\beta)\\ \left(\frac{1}{\sqrt{2}}\right)\sin\beta & \cos\beta & -\left(\frac{1}{\sqrt{2}}\right)\sin\beta\\ \left(\frac{1}{2}\right)(1-\cos\beta) & \left(\frac{1}{\sqrt{2}}\right)\sin\beta & \left(\frac{1}{2}\right)(1+\cos\beta)\end{pmatrix}.$$

23. Express the matrix element $\langle\alpha_2\beta_2\gamma_2|J_3^2|\alpha_1\beta_1\gamma_1\rangle$ in terms of a series in

$$\mathscr{D}^j_{mn}(\alpha\beta\gamma)=\langle\alpha\beta\gamma|jmn\rangle.$$

24. Consider a system made up of two spin $\frac{1}{2}$ particles. Observer A specializes in measuring the spin components of one of the particles (s_{1z}, s_{1x} and so on), while observer B measures the spin components of the other particle. Suppose the system is known to be in a spin-singlet state, that is, $S_{\text{total}}=0$.
   a. What is the probability for observer A to obtain $s_{1z}=\hbar/2$ when observer B makes no measurement? Same problem for $s_{1x}=\hbar/2$.
   b. Observer B determines the spin of particle 2 to be in the $s_{2z}=\hbar/2$ state with certainty. What can we then conclude about the outcome of observer A's measurement if (i) A measures s_{1z} and (ii) A measures s_{1x}? Justify your answer.

25. Consider a spherical tensor of rank 1 (that is, a vector)

$$V^{(1)}_{\pm1}=\mp\frac{V_x\pm iV_y}{\sqrt{2}},\qquad V^{(1)}_0=V_z.$$

Using the expression for $d^{(j=1)}$ given in Problem 22, evaluate

$$\sum_{q'}d^{(1)}_{qq'}(\beta)V^{(1)}_{q'}$$

and show that your results are just what you expect from the transformation properties of $V_{x,y,z}$ under rotations about the y-axis.

26. a. Construct a spherical tensor of rank 1 out of two different vectors $\mathbf{U}=(U_x,U_y,U_z)$ and $\mathbf{V}=(V_x,V_y,V_z)$. Explicitly write $T^{(1)}_{\pm1,0}$ in terms of $U_{x,y,z}$ and $V_{x,y,z}$.
   b. Construct a spherical tensor of rank 2 out of two different vectors $\mathbf{U}$ and $\mathbf{V}$. Write down explicitly $T^{(2)}_{\pm2,\pm1,0}$ in terms of $U_{x,y,z}$ and $V_{x,y,z}$.

27. Consider a spinless particle bound to a fixed center by a central force potential.

   a. Relate, as much as possible, the matrix elements

$$\langle n',l',m'|\mp\frac{1}{\sqrt{2}}(x\pm iy)|n,l,m\rangle\quad\text{and}\quad\langle n',l',m'|z|n,l,m\rangle$$

using *only* the Wigner-Eckart theorem. Make sure to state under what conditions the matrix elements are nonvanishing.
   b. Do the same problem using wave functions $\psi(\mathbf{x})=R_{nl}(r)Y_l^m(\theta,\phi)$.

28. a. Write xy, xz, and (x^2-y^2) as components of a spherical (irreducible) tensor of rank 2.
   b. The expectation value

$$Q\equiv e\langle\alpha,j,m=j|(3z^2-r^2)|\alpha,j,m=j\rangle$$

is known as the *quadrupole moment*. Evaluate

$$e\langle\alpha,j,m'|(x^2-y^2)|\alpha,j,m=j\rangle,$$

(where $m'=j,j-1,j-2,\ldots$) in terms of Q and appropriate Clebsch-Gordan coefficients.

29. A spin $\frac{3}{2}$ nucleus situated at the origin is subjected to an external inhomogeneous electric field. The basic electric quadrupole interaction may by taken to be

$$H_{\text{int}}=\frac{eQ}{2s(s-1)\hbar^2}\left[\left(\frac{\partial^2\phi}{\partial x^2}\right)_0S_x^2+\left(\frac{\partial^2\phi}{\partial y^2}\right)_0S_y^2+\left(\frac{\partial^2\phi}{\partial z^2}\right)_0S_z^2\right],$$

where ϕ is the electrostatic potential satisfying Laplace's equation and the coordinate axes are so chosen that

$$\left(\frac{\partial^2\phi}{\partial x\partial y}\right)_0=\left(\frac{\partial^2\phi}{\partial y\partial z}\right)_0=\left(\frac{\partial^2\phi}{\partial x\partial z}\right)_0=0.$$

Show that the interaction energy can be written as

$$A(3S_z^2-\mathbf{S}^2)+B(S_+^2+S_-^2),$$

and express A and B in terms of $(\partial^2\phi/\partial x^2)_0$ and so on. Determine the energy eigenkets (in terms of $|m\rangle$, where $m=\pm\frac{3}{2},\pm\frac{1}{2}$) and the corresponding energy eigenvalues. Is there any degeneracy?