Consider now what happens in the overlap region $\varepsilon < \theta < \pi - \varepsilon$, where we may use either $\mathbf{A}^{(\mathrm{I})}$ or $\mathbf{A}^{(\mathrm{II})}$. Because the two potentials lead to the same magnetic field, they must be related to each other by a gauge transformation. To find Λ appropriate for this problem we first note that

$$\mathbf{A}^{(\mathrm{II})} - \mathbf{A}^{(\mathrm{I})} = -\left(\frac{2e_M}{r\sin\theta}\right)\hat{\boldsymbol{\phi}}. \tag{2.6.81}$$

Recalling the expression for gradient in spherical coordinates,

$$\nabla\Lambda = \hat{\mathbf{r}}\frac{\partial\Lambda}{\partial r} + \hat{\boldsymbol{\theta}}\frac{1}{r}\frac{\partial\Lambda}{\partial\theta} + \hat{\boldsymbol{\phi}}\frac{1}{r\sin\theta}\frac{\partial\Lambda}{\partial\phi}, \tag{2.6.82}$$

we deduce that

$$\Lambda = -2e_M\phi \tag{2.6.83}$$

will do the job.

Next, we consider the wave function of an electrically charged particle of charge e subjected to magnetic field (2.6.74). As we emphasized earlier, the particular form of the wave function depends on the particular gauge used. In the overlap region where we may use either $\mathbf{A}^{(\mathrm{I})}$ or $\mathbf{A}^{(\mathrm{II})}$, the corresponding wave functions are, according to (2.6.55), related to each other by

$$\psi^{(\mathrm{II})} = \exp\left(\frac{-2iee_M\phi}{\hbar c}\right)\psi^{(\mathrm{I})}. \tag{2.6.84}$$

Wave functions $\psi^{(\mathrm{I})}$ and $\psi^{(\mathrm{II})}$ must each be *single-valued* because once we choose particular gauge, the expansion of the state ket in terms of the position eigenkets must be *unique*. After all, as we have repeatedly emphasized, the wave function is simply an expansion coefficient for the state ket in terms of the position eigenkets.

Let us now examine the behavior of wave function $\psi^{(\mathrm{II})}$ on the equator $\theta = \pi/2$ with some definite radius r, which is a constant. When we increase the azimuthal angle ϕ along the equator and go around once, say from $\phi = 0$ to $\phi = 2\pi$, $\psi^{(\mathrm{II})}$, as well as $\psi^{(\mathrm{I})}$, must return to its original value because each is single-valued. According to (2.6.84), this is possible only if

$$\frac{2ee_M}{\hbar c} = \pm N, \quad N = 0, \pm 1, \pm 2, \ldots. \tag{2.6.85}$$

So we reach a very far-reaching conclusion: The magnetic charges must be *quantized* in units of

$$\frac{\hbar c}{2|e|} \simeq \left(\frac{137}{2}\right)|e|. \tag{2.6.86}$$

The smallest magnetic charge possible is $\hbar c/2|e|$, where e is the electronic charge. It is amusing that once a magnetic monopole is assumed to exist, we can use (2.6.85) backward, so to speak, to explain why the electric charges are quantized, for example, why the proton charge cannot be 0.999972 times $|e|$.*

We repeat once again that quantum mechanics does not require magnetic monopoles to exist. However, it unambiguously predicts that a magnetic charge, if it is ever found in nature, must be quantized in units of $\hbar c/2|e|$. The quantization of magnetic charges in quantum mechanics was first shown in 1931 by P. A. M. Dirac. The derivation given here is due to T. T. Wu and C. N. Yang.

PROBLEMS

1. Consider the spin-precession problem discussed in the text. It can also be solved in the Heisenberg picture. Using the Hamiltonian

$$H = -\left(\frac{eB}{mc}\right)S_z = \omega S_z,$$

write the Heisenberg equations of motion for the time-dependent operators $S_x(t)$, $S_y(t)$, and $S_z(t)$. Solve them to obtain $S_{x,y,z}$ as functions of time.

2. Look again at the Hamiltonian of Chapter 1, Problem 11. Suppose the typist made an error and wrote H as

$$H = H_{11}|1\rangle\langle 1| + H_{22}|2\rangle\langle 2| + H_{12}|1\rangle\langle 2|.$$

What principle is now violated? Illustrate your point explicitly by attempting to solve the most general time-dependent problem using an illegal Hamiltonian of this kind. (You may assume $H_{11} = H_{22} = 0$ for simplicity.)

3. An electron is subject to a uniform, time-independent magnetic field of strength B in the positive z-direction. At $t = 0$ the electron is known to be in an eigenstate of $\mathbf{S}\cdot\hat{\mathbf{n}}$ with eigenvalue $\hbar/2$, where $\hat{\mathbf{n}}$ is a unit vector, lying in the xz-plane, that makes an angle β with the z-axis.
   a. Obtain the probability for finding the electron in the $s_x = \hbar/2$ state as a function of time.
   b. Find the expectation value of S_x as a function of time.
   c. For your own peace of mind show that your answers make good sense in the extreme cases (i) $\beta \to 0$ and (ii) $\beta \to \pi/2$.

4. Let $x(t)$ be the coordinate operator for a free particle in one dimension in the Heisenberg picture. Evaluate

$$[x(t), x(0)].$$

*Empirically the equality in magnitude between the electron charge and the proton charge is established to an accuracy of four parts in 10^{19}.

5. Consider a particle in one dimension whose Hamiltonian is given by
$$H = \frac{p^2}{2m} + V(x).$$
By calculating $[[H, x], x]$ prove
$$\sum_{a'} |\langle a''|x|a'\rangle|^2 (E_{a'} - E_{a''}) = \frac{\hbar^2}{2m},$$
where $|a'\rangle$ is an energy eigenket with eigenvalue $E_{a'}$.
6. Consider a particle in three dimensions whose Hamiltonian is given by
$$H = \frac{\mathbf{p}^2}{2m} + V(\mathbf{x}).$$
By calculating $[\mathbf{x}\cdot\mathbf{p}, H]$ obtain
$$\frac{d}{dt}\langle \mathbf{x}\cdot\mathbf{p}\rangle = \left\langle \frac{\mathbf{p}^2}{m} \right\rangle - \langle \mathbf{x}\cdot\nabla V\rangle.$$
To identify the preceding relation with the quantum-mechanical analogue of the virial theorem it is essential that the left-hand side vanish. Under what condition would this happen?
7. Consider a free-particle wave packet in one dimension. At $t = 0$ it satisfies the minimum uncertainty relation
$$\langle(\Delta x)^2\rangle\langle(\Delta p)^2\rangle = \frac{\hbar^2}{4} \quad (t = 0).$$
In addition, we know
$$\langle x\rangle = \langle p\rangle = 0 \quad (t = 0).$$
Using the Heisenberg picture, obtain $\langle(\Delta x)^2\rangle_t$ as a function of $t\,(t \geq 0)$ when $\langle(\Delta x)^2\rangle_{t=0}$ is given. (*Hint:* Take advantage of the property of the minimum-uncertainty wave packet you worked out in Chapter 1, Problem 18).
8. Let $|a'\rangle$ and $|a''\rangle$ be eigenstates of a Hermitian operator A with eigenvalues a' and a'', respectively $(a' \neq a'')$. The Hamiltonian operator is given by
$$H = |a'\rangle\delta\langle a''| + |a''\rangle\delta\langle a'|,$$
where δ is just a real number.
   a. Clearly, $|a'\rangle$ and $|a''\rangle$ are not eigenstates of the Hamiltonian. Write down the eigenstates of the Hamiltonian. What are their energy eigenvalues?
   b. Suppose the system is known to be in state $|a'\rangle$ at $t = 0$. Write down the state vector in the Schrödinger picture for $t > 0$.

   c. What is the probability for finding the system in $|a''\rangle$ for $t > 0$ if the system is known to be in state $|a'\rangle$ at $t = 0$?
   d. Can you think of a physical situation corresponding to this problem?
9. A box containing a particle is divided into a right and left compartment by a thin partition. If the particle is known to be on the right (left) side with certainty, the state is represented by the position eigenket $|R\rangle$ ($|L\rangle$), where we have neglected spatial variations within each half of the box. The most general state vector can then be written as
$$|\alpha\rangle = |R\rangle\langle R|\alpha\rangle + |L\rangle\langle L|\alpha\rangle,$$
where $\langle R|\alpha\rangle$ and $\langle L|\alpha\rangle$ can be regarded as "wave functions." The particle can tunnel through the partition; this tunneling effect is characterized by the Hamiltonian
$$H = \Delta(|L\rangle\langle R| + |R\rangle\langle L|),$$
where Δ is a real number with the dimension of energy.
   a. Find the normalized energy eigenkets. What are the corresponding energy eigenvalues?
   b. In the Schrödinger picture the base kets $|R\rangle$ and $|L\rangle$ are fixed, and the state vector moves with time. Suppose the system is represented by $|\alpha\rangle$ as given above at $t = 0$. Find the state vector $|\alpha, t_0 = 0; t\rangle$ for $t > 0$ by applying the appropriate time-evolution operator to $|\alpha\rangle$.
   c. Suppose at $t = 0$ the particle is on the right side with certainty. What is the probability for observing the particle on the left side as a function of time?
   d. Write down the coupled Schrödinger equations for the wave functions $\langle R|\alpha, t_0 = 0; t\rangle$ and $\langle L|\alpha, t_0 = 0; t\rangle$. Show that the solutions to the coupled Schrödinger equations are just what you expect from (b).
   e. Suppose the printer made an error and wrote H as
$$H = \Delta|L\rangle\langle R|.$$
By explicitly solving the most general time-evolution problem with this Hamiltonian, show that probability conservation is violated.
10. Using the one-dimensional simple harmonic oscillator as an example, illustrate the difference between the Heisenberg picture and the Schrödinger picture. Discuss in particular how (a) the dynamic variables x and p and (b) the most general state vector evolve with time in each of the two pictures.
11. Consider a particle subject to a one-dimensional simple harmonic oscillator potential. Suppose at $t = 0$ the state vector is given by
$$\exp\left(\frac{-ipa}{\hbar}\right)|0\rangle,$$
where p is the momentum operator and a is some number with dimension of length. Using the Heisenberg picture, evaluate the expectation value $\langle x\rangle$ for $t \geq 0$.

12. a. Write down the wave function (in coordinate space) for the state specified in Problem 11 at $t = 0$. You may use

$$\langle x'|0\rangle = \pi^{-1/4} x_0^{-1/2} \exp\left[-\frac{1}{2}\left(\frac{x'}{x_0}\right)^2\right], \quad \left(x_0 \equiv \left(\frac{\hbar}{m\omega}\right)^{1/2}\right).$$

b. Obtain a simple expression for the probability that the state is found in the ground state at $t = 0$. Does this probability change for $t > 0$?

13. Consider a one-dimensional simple harmonic oscillator.

a. Using

$$\left.\begin{matrix} a \\ a^\dagger \end{matrix}\right\} = \sqrt{\frac{m\omega}{2\hbar}}\left(x \pm \frac{ip}{m\omega}\right), \quad \left.\begin{matrix} a|n\rangle \\ a^\dagger|n\rangle \end{matrix}\right\} = \left\{\begin{matrix} \sqrt{n}\,|n-1\rangle \\ \sqrt{n+1}\,|n+1\rangle, \end{matrix}\right.$$

evaluate $\langle m|x|n\rangle$, $\langle m|p|n\rangle$, $\langle m|\{x, p\}|n\rangle$, $\langle m|x^2|n\rangle$, and $\langle m|p^2|n\rangle$.

b. Check that the virial theorem holds for the expectation values of the kinetic and the potential energy taken with respect to an energy eigenstate.

14. a. Using

$$\langle x'|p'\rangle = (2\pi\hbar)^{-1/2} e^{ip'x'/\hbar} \quad \text{(one dimension)}$$

prove

$$\langle p'|x|\alpha\rangle = i\hbar \frac{\partial}{\partial p'}\langle p'|\alpha\rangle.$$

b. Consider a one-dimensional simple harmonic oscillator. Starting with the Schrödinger equation for the state vector, derive the Schrödinger equation for the *momentum-space* wave function. (Make sure to distinguish the operator p from the eigenvalue p'.) Can you guess the energy eigenfunctions in momentum space?

15. Consider a function, known as the **correlation function**, defined by

$$C(t) = \langle x(t)x(0)\rangle,$$

where $x(t)$ is the position operator in the Heisenberg picture. Evaluate the correlation function explicitly for the ground state of a one-dimensional simple harmonic oscillator.

16. Consider again a one-dimensional simple harmonic oscillator. Do the following algebraically, that is, without using wave functions.

a. Construct a linear combination of $|0\rangle$ and $|1\rangle$ such that $\langle x\rangle$ is as large as possible.

b. Suppose the oscillator is in the state constructed in (a) at $t = 0$. What is the state vector for $t > 0$ in the Schrödinger picture? Evaluate the expectation value $\langle x\rangle$ as a function of time for $t > 0$ using (i) the Schrödinger picture and (ii) the Heisenberg picture.

c. Evaluate $\langle(\Delta x)^2\rangle$ as a function of time using either picture.

17. Show for the one-dimensional simple harmonic oscillator

$$\langle 0|e^{ikx}|0\rangle = \exp\left[-k^2\langle 0|x^2|0\rangle/2\right],$$

where x is the position *operator*.

18. A coherent state of a one-dimensional simple harmonic oscillator is defined to be an eigenstate of the (non-Hermitian) annihilation operator a:

$$a|\lambda\rangle = \lambda|\lambda\rangle,$$

where λ is, in general, a complex number.

a. Prove that

$$|\lambda\rangle = e^{-|\lambda|^2/2} e^{\lambda a^\dagger}|0\rangle$$

is a normalized coherent state.

b. Prove the minimum uncertainty relation for such a state.

c. Write $|\lambda\rangle$ as

$$|\lambda\rangle = \sum_{n=0}^{\infty} f(n)|n\rangle.$$

Show that the distribution of $|f(n)|^2$ with respect to n is of the Poisson form. Find the most probable value of n, hence of E.

d. Show that a coherent state can also be obtained by applying the translation (finite-displacement) operator $e^{-ipl/\hbar}$ (where p is the momentum operator, and l is the displacement distance) to the ground state. (See also Gottfried 1966, 262–64.)

19. Let

$$J_\pm = \hbar a_\pm^\dagger a_\mp, \quad J_z = \frac{\hbar}{2}(a_+^\dagger a_+ - a_-^\dagger a_-), \quad N = a_+^\dagger a_+ + a_-^\dagger a_-$$

where $a_\pm$ and $a_\pm^\dagger$ are the annihilation and creation operators of two *independent* simple harmonic oscillator satisfying the usual simple harmonic oscillator commutation relations. Prove

$$[J_z, J_\pm] = \pm\hbar J_\pm, \quad [\mathbf{J}^2, J_z] = 0, \quad \mathbf{J}^2 = \left(\frac{\hbar^2}{2}\right) N\left[\left(\frac{N}{2}\right) + 1\right].$$

20. Consider a particle of mass m subject to a one-dimensional potential of the following form:

$$V = \begin{cases} \frac{1}{2}kx^2 & \text{for } x > 0 \\ \infty & \text{for } x < 0. \end{cases}$$

a. What is the ground-state energy?

b. What is the expectation value $\langle x^2\rangle$ for the ground state?

21. A particle in one dimension is trapped between two rigid walls:

$$V(x) = \begin{cases} 0, & \text{for } 0 < x < L \\ \infty, & \text{for } x < 0, \quad x > L. \end{cases}$$

At $t = 0$ it is known to be exactly at $x = L/2$ with certainty. What are the *relative* probabilities for the particle to be found in various energy eigenstates? Write down the wave function for $t \geq 0$. (You need not worry about absolute normalization, convergence, and other mathematical subtleties.)

22. Consider a particle in one dimension bound to a fixed center by a δ-function potential of the form

$$V(x) = -v_0\delta(x), \quad (v_0 \text{ real and positive}).$$

Find the wave function and the binding energy of the ground state. Are there excited bound states?

23. A particle of mass m in one dimension is bound to a fixed center by an attractive δ-function potential:

$$V(x) = -\lambda\delta(x), \quad (\lambda > 0).$$

At $t = 0$, the potential is suddenly switched off (that is, $V = 0$ for $t > 0$). Find the wave function for $t > 0$. (Be quantitative! But you need not attempt to evaluate an integral that may appear.)

24. A particle in one dimension $(-\infty < x < \infty)$ is subjected to a constant force derivable from

$$V = \lambda x, \quad (\lambda > 0).$$

a. Is the energy spectrum continuous or discrete? Write down an approximate expression for the energy eigenfunction specified by E. Also sketch it crudely.

b. Discuss briefly what changes are needed if V is replaced by

$$V = \lambda|x|.$$

25. Consider an electron confined to the *interior* of a hollow cylindrical shell whose axis coincides with the z-axis. The wave function is required to vanish on the inner and outer walls, $\rho = \rho_a$ and ρ_b, and also at the top and bottom, $z = 0$ and L.

a. Find the energy eigenfunctions. (Do not bother with normalization.) Show that the energy eigenvalues are given by

$$E_{lmn} = \left(\frac{\hbar^2}{2m_e}\right)\left[k_{mn}^2 + \left(\frac{l\pi}{L}\right)^2\right] \quad (l = 1, 2, 3, \ldots, \ m = 0, 1, 2, \ldots),$$

where k_{mn} is the nth root of the transcendental equation

$$J_m(k_{mn}\rho_b)N_m(k_{mn}\rho_a) - N_m(k_{mn}\rho_b)J_m(k_{mn}\rho_a) = 0.$$

b. Repeat the same problem when there is a uniform magnetic field $\mathbf{B} = B\hat{\mathbf{z}}$ for $0 < \rho < \rho_a$. Note that the energy eigenvalues are influenced by the magnetic field even though the electron never "touches" the magnetic field.

c. Compare, in particular, the ground state of the $B = 0$ problem with that of the $B \neq 0$ problem. Show that if we require the ground-state energy to be unchanged in the presence of B, we obtain "flux quantization"

$$\pi\rho_a^2 B = \frac{2\pi N\hbar c}{e}, \quad (N = 0, \pm 1, \pm 2, \ldots).$$

26. Consider a particle moving in one dimension under the influence of a potential $V(x)$. Suppose its wave function can be written as $\exp[iS(x, t)/\hbar]$. Prove that $S(x, t)$ satisfies the classical Hamilton-Jacobi equation to the extent that $\hbar$ can be regarded as small in some sense. Show how one may obtain the correct wave function for a plane wave by starting with the solution of the classical Hamilton-Jacobi equation with $V(x)$ set equal to zero. Why do we get the exact wave function in this particular case?

27. Using spherical coordinates, obtain an expression for $\mathbf{j}$ for the ground and excited states of the hydrogen atom. Show, in particular, that for $m_l \neq 0$ states, there is a circulating flux in the sense that $\mathbf{j}$ is in the direction of increasing or decreasing ϕ, depending on whether m_l is positive or negative.

28. Derive (2.5.16) and obtain the three-dimensional generalization of (2.5.16).

29. Define the partition function as

$$Z = \int d^3x' K(\mathbf{x}', t; \mathbf{x}', 0)|_{\beta = it/\hbar},$$

as in (2.5.20)–(2.5.22). Show that the ground-state energy is obtained by taking

$$-\frac{1}{Z}\frac{\partial Z}{\partial \beta}, \quad (\beta \to \infty).$$

Illustrate this for a particle in a one-dimensional box.

30. The propagator in momentum space analogous to (2.5.26) is given by $\langle \mathbf{p}'', t|\mathbf{p}', t_0\rangle$. Derive an explicit expression for $\langle \mathbf{p}'', t|\mathbf{p}', t_0\rangle$ for the free-particle case.

31. a. Write down an expression for the classical action for a simple harmonic oscillator for a finite time interval.

b. Construct $\langle x_n, t_n|x_{n-1}, t_{n-1}\rangle$ for a simple harmonic oscillator using Feynman's prescription for $t_n - t_{n-1} = \Delta t$ small. Keeping only terms up to order $(\Delta t)^2$, show that it is in complete agreement with the $t - t_0 \to 0$ limit of the propagator given by (2.5.26).

32. State the Schwinger action principle (see Finkelstein 1973, 155). Obtain the solution for $\langle x_2 t_2|x_1 t_1\rangle$ by integrating the Schwinger principle and compare it with the corresponding Feynman expression for $\langle x_2 t_2|x_1 t_1\rangle$. Describe the classical limits of these two expressions.

33. Show that the wave-mechanics approach to the gravity-induced problem discussed in Section 2.6 also leads to phase-difference expression (2.6.17).
34. a. Verify (2.6.25) and (2.6.27).
 b. Verify continuity equation (2.6.30) with **j** given by (2.6.31).
35. Consider the Hamiltonian of a spinless particle of charge e. In the presence of a static magnetic field, the interaction terms can be generated by

$$\mathbf{p}_{\text{operator}} \to \mathbf{p}_{\text{operator}} - \frac{e\mathbf{A}}{c},$$

where **A** is the appropriate vector potential. Suppose, for simplicity, that the magnetic field **B** is uniform in the positive z-direction. Prove that the above prescription indeed leads to the correct expression for the interaction of the orbital magnetic moment $(e/2mc)\mathbf{L}$ with the magnetic field **B**. Show that there is also an extra term proportional to $B^2(x^2+y^2)$, and comment briefly on its physical significance.
36. An electron moves in the presence of a uniform magnetic field in the z-direction ($\mathbf{B} = B\hat{\mathbf{z}}$).
 a. Evaluate

$$[\Pi_x, \Pi_y],$$

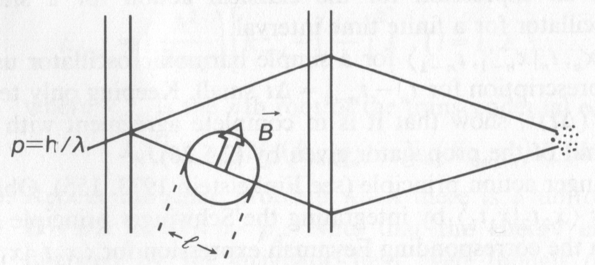
 where

$$\Pi_x \equiv p_x - \frac{eA_x}{c}, \qquad \Pi_y \equiv p_y - \frac{eA_y}{c}.$$

 b. By comparing the Hamiltonian and the commutation relation obtained in (a) with those of the one-dimensional oscillator problem, show how we can immediately write the energy eigenvalues as

$$E_{k,n} = \frac{\hbar^2 k^2}{2m} + \left(\frac{|eB|\hbar}{mc}\right)\left(n + \frac{1}{2}\right),$$

 where $\hbar k$ is the continuous eigenvalue of the p_z operator and n is a nonnegative integer including zero.
37. Consider the neutron interferometer.



Prove that the difference in the magnetic fields that produce two successive maxima in the counting rates is given by

$$\Delta B = \frac{4\pi\hbar c}{|e| g_n \bar{\lambda} l},$$

where g_n $(= -1.91)$ is the neutron magnetic moment in units of $-e\hbar/2m_n c$. [If you had solved this problem in 1967, you could have published it in *Physical Review Letters*!]