# Contents



# 1 Fundamental concepts

## 1.1 Stern Gerlach

## 1.2 Kets, Bras, Operators

## 1.3 Base kets and matrix operations

## 1.4 Measurements, Observables and the Uncertainty Relations

Summary:

1. Measurement changes the state. A measurement modeled by an observable (operator) $\hat{A}$ on a state $|\alpha\rangle$ changes the state to one of the eigenstates of $\hat{A}$:
$$\hat{A}|\alpha\rangle \propto |a\rangle$$
where $\hat{A}|a\rangle = a|a\rangle$.

2. The probability of measuring an eigenvalue $a$ is given by
$$P(a) = |\langle a|\alpha\rangle|^2$$
Notice that this is consistent in two respects. If the state $|\alpha\rangle$ is already the eigenstate $|a\rangle$, then we measure $a$ with certainty,
$$P(a) = |\langle a|a\rangle|^2 = 1$$
and for an arbitrary state, the probability of measuring one of the eigenvalues of $\hat{A}$ is 1, that is, we always get one of the eigenvalues:
$$\begin{aligned} P(some\, a_i) &= \sum_i |\langle a_i|\alpha\rangle|^2 \\ &= \sum_i \langle\alpha|a_i\rangle\langle a_i|\alpha\rangle \\ &= \langle\alpha|\left(\sum_i |a_i\rangle\langle a_i|\right)|\alpha\rangle \\ &= \langle\alpha|1|\alpha\rangle \\ &= \langle\alpha|\alpha\rangle \\ &= 1 \end{aligned}$$
where we use the completeness of the set of eigenvectors.

3. The expectation value of an operator is
$$\langle\alpha|\hat{A}|\alpha\rangle = \langle\alpha|\left(\sum_i |a_i\rangle\langle a_i|\right)\hat{A}\left(\sum_j |a_j\rangle\langle a_j|\right)|\alpha\rangle$$

$$
\begin{aligned}
&= \sum_{i,j} \langle \alpha | a_i \rangle \langle a_i | a_j | a_j \rangle \langle a_j | \alpha \rangle \\
&= \sum_{i,j} a_j \delta_{ij} \langle \alpha | a_i \rangle \langle a_j | \alpha \rangle \\
&= \sum_{i} a_i \langle \alpha | a_i \rangle \langle a_i | \alpha \rangle \\
&= \sum_{i} a_i |\langle a_i | \alpha \rangle|^2 \\
&= \sum_{i} a_i P(a_i)
\end{aligned}
$$

**Examples:** Spin operators

We can construct any operator from its eigenstates and eigenvalues as

$$\hat{A} = \sum_i a_i |a_i\rangle \langle a_i|$$

For $\hat{S}_x$, we know that acting on $S_z$ eigenkets $|\pm\rangle$, we have equal probability of finding $|S_x, \pm\rangle$. Therefore, we must have

$$|S_x, +\rangle = \frac{1}{\sqrt{2}} e^{i\delta_1} |+\rangle + \frac{1}{\sqrt{2}} e^{i\delta_2} |-\rangle$$

for arbitrary phases $\delta_1, \delta_2$. Since the overall phase is arbitrary and nonphysical we can eliminate one of these and write

$$|S_x, +\rangle = \frac{1}{\sqrt{2}} |+\rangle + \frac{1}{\sqrt{2}} e^{i\delta} |-\rangle$$

Since $|S_x, +\rangle$ and $|S_x, -\rangle$ must be orthogonal (since the observable $\hat{S}_x$ must be Hermitian), setting $|S_x, -\rangle = \alpha |+\rangle + \beta |-\rangle$, we have

$$
\begin{aligned}
0 &= \langle S_x, + | S_x, - \rangle \\
&= \frac{1}{\sqrt{2}} \left( \langle +| + \langle -| e^{-i\delta} \right) (\alpha |+\rangle + \beta |-\rangle) \\
&= \frac{1}{\sqrt{2}} \left( \alpha + \beta e^{-i\delta} \right)
\end{aligned}
$$

so $\beta = -e^{i\delta}\alpha$ and the normalized state must be

$$|S_x, -\rangle = \frac{1}{\sqrt{2}} |+\rangle - \frac{1}{\sqrt{2}} e^{i\delta} |-\rangle$$

Then $\hat{S}_x$ is given by

$$
\begin{aligned}
\hat{S}_x &= +\frac{\hbar}{2} |S_x, +\rangle \langle S_x, +| + \left(-\frac{\hbar}{2}\right) |S_x, -\rangle \langle S_x, -| \\
&= \frac{\hbar}{2} \left( \frac{1}{\sqrt{2}} \left( |+\rangle + e^{i\delta} |-\rangle \right) \frac{1}{\sqrt{2}} \left( \langle +| + \langle -| e^{-i\delta} \right) - \frac{1}{\sqrt{2}} \left( |+\rangle - e^{i\delta} |-\rangle \right) \frac{1}{\sqrt{2}} \left( \langle +| - \langle -| e^{-i\delta} \right) \right) \\
&= \frac{\hbar}{4} \left( |+\rangle \langle +| + |+\rangle \langle -| e^{-i\delta} + e^{i\delta} |-\rangle \langle +| + |-\rangle \langle -| - |+\rangle \langle +| + e^{i\delta} |-\rangle \langle +| + |+\rangle \langle -| e^{-i\delta} - |-\rangle \langle -| \right) \\
&= \frac{\hbar}{4} \left( 2e^{-i\delta} |+\rangle \langle -| + 2e^{i\delta} |-\rangle \langle +| \right) \\
&= \frac{\hbar}{2} \left( e^{-i\delta} |+\rangle \langle -| + e^{i\delta} |-\rangle \langle +| \right)
\end{aligned}
$$

We could have made exactly the same arguments for $\hat{S}_y$, so with $\sigma$ some other phase we may also write

$$\hat{S}_y = \frac{\hbar}{2} \left( e^{-i\sigma} |+\rangle \langle -| + e^{i\sigma} |-\rangle \langle +| \right)$$

But we also know that the $x$ and $y$ directions will have the same relationship to one another that they each have with the $z$ direction, for example,

$$\begin{aligned} \left|\left\langle S_x,\pm\left|S_y,+\right.\right\rangle\right| &= \frac{1}{\sqrt{2}} \\ \left|\left\langle S_x,\pm\left|S_y,+\right.\right\rangle\right| &= \frac{1}{\sqrt{2}} \end{aligned}$$

Either of these relations gives:

$$\begin{aligned} \left|\frac{1}{\sqrt{2}}\left(\langle+|\pm\langle-|e^{-i\delta}\right)\frac{1}{\sqrt{2}}\left(|+\rangle+e^{i\sigma}|-\rangle\right)\right| &= \frac{1}{\sqrt{2}} \\ \frac{1}{2}\left|1\pm e^{i(\sigma-\delta)}\right| &= \frac{1}{\sqrt{2}} \end{aligned}$$

which happens if and only if

$$\begin{aligned} e^{i(\sigma-\delta)} &= \pm i \\ (\sigma-\delta) &= \pm\frac{\pi}{2} \end{aligned}$$

The remaining indefiniteness of the phase can be chosen by fixing the overall phase of $|S_x,+\rangle$. It is conventional to choose $\hat{S}_x$ to be real, so that $\delta=0$ and

$$\hat{S}_x=\frac{\hbar}{2}\left(|+\rangle\langle-|+|-\rangle\langle+|\right)=\frac{\hbar}{2}\sigma_x$$

Then $\hat{S}_y$ must be pure imaginary. With $\sigma=\frac{\pi}{2}$ we have

$$\begin{aligned} \hat{S}_y &= \frac{\hbar}{2}\left(e^{-\frac{i\pi}{2}}|+\rangle\langle-|+e^{\frac{i\pi}{2}}|-\rangle\langle+|\right) \\ &= \frac{\hbar}{2}\left(-i|+\rangle\langle-|+i|-\rangle\langle+|\right) \\ &= \frac{\hbar}{2}\sigma_y \end{aligned}$$

### 1.4.1 The algebra of spin

The essential properties of angular momentum are implicit in the products of these spin operators. Most importantly, we have the commutators,

$$\left[\hat{S}_i,\hat{S}_j\right]=i\hbar\varepsilon_{ijk}\hat{S}_k$$

We also have the anticommutators,

$$\left\{\hat{S}_i,\hat{S}_j\right\}=\frac{1}{2}\hbar^2\delta_{ij}$$

It will be useful to define the combinations

$$\hat{S}_\pm=\hat{S}_x\pm i\hat{S}_y$$

and the squared sum,

$$\begin{aligned} \hat{\mathbf{S}}^2 &= \hat{\mathbf{S}}\cdot\hat{\mathbf{S}} \\ &= \hat{S}_x^2+\hat{S}_y^2+\hat{S}_z^2 \\ &= \frac{3}{4}\hbar^2 1 \end{aligned}$$

since each Pauli matrix squares to the identity. Since $\hat{\mathbf{S}}^2$ is proportional to the identity operator, we have

$$\left[\hat{\mathbf{S}}^2,S_i\right]=0$$

### 1.4.2 Quantum vs. classical conditional probability: Bell's Theorem

Consider three measurements, with corresponding operators $A, B$ and $C$, performed in order. Classically, let

$$P_{a\to b}$$

be the probability of the $B$ measurement giving $b$ when the $A$ measurement has given $a$. This is called a *conditional probability*. Then, performing $C$, we have

$$P_{b\to c}$$

for the conditional probability of $c$ given $b$, and the joint probability of measuring $b$ then $c$, given $a$ is the product:

$$P_{a\to(b,c)} = P_{a\to b}P_{b\to c}$$

If we sum over all possible outcomes, $b_i$, for $B$, we must get the conditional probability of $c$, given $a$,

$$P_{a\to c} = \sum_{b_i} P_{a\to b_i}P_{b_i\to c}$$

because we have accounted for all possible intermediate routes from $a$ to $c$.

Quantum mechanically, we may write each of the conditional probabilities as:

$$\begin{aligned}
P_{a\to b} &= |\langle b|a\rangle|^2 \\
P_{b\to c} &= |\langle c|b\rangle|^2 \\
P_{a\to c} &= |\langle c|a\rangle|^2
\end{aligned}$$

Now expand the last,

$$\begin{aligned}
P_{a\to c} &= |\langle c|a\rangle|^2 \\
&= \langle a|c\rangle\langle c|a\rangle \\
&= \sum_i \langle a|b_i\rangle\langle b_i|a\rangle\langle c|a\rangle \\
&= \sum_i \langle a|b_i\rangle\langle b_i|a\rangle\sum_j \langle c|b_j\rangle\langle b_j|a\rangle
\end{aligned}$$

where we have inserted two copies of the identity operator. As in the classical case, the joint probability of measuring $b$ then $c$, given $a$ is now

$$\begin{aligned}
P_{a\to(b,c)} &= |\langle b|a\rangle\langle c|b\rangle|^2 \\
&= (\langle a|b\rangle\langle b|c\rangle)(\langle b|a\rangle\langle c|b\rangle) \\
&= (\langle a|b\rangle\langle b|a\rangle)(\langle c|b\rangle\langle b|c\rangle) \\
&= P_{a\to b}P_{b\to c}
\end{aligned}$$

However, if we sum over all intermediate states $b_i$ we do *not* get $P_{a\to c}$! Instead, we have the *single* sum

$$\sum_i \langle a|b_i\rangle\langle b_i|a\rangle$$

$$\begin{aligned}
\sum_i P_{a\to(b_i,c)} &= \sum_i \langle a|b_i\rangle\langle b_i|a\rangle(\langle c|b_i\rangle\langle b_i|c\rangle) \\
&= \sum_i P_{a\to b_i}P_{b_i\to c}
\end{aligned}$$

This is an extremely important difference between classical and quantum physics!

Let's look at an example. Let the three operators be the spin operators in the $z, x$ and $y$ directions, respectively, and let's work as usual in the $z$ basis. Start with a general spin state,

$$|\psi\rangle = \alpha|+\rangle + \beta|-\rangle$$

where normalization requires $\alpha\bar{\alpha}+\beta\bar{\beta}=1$. Then the probability of measuring spin up in the $x$ and then spin up in the $y$ directions is

$$\begin{aligned}
P_{\psi\to((x+),(y+))} &= \left(\langle\psi|S_x,+\rangle\langle S_x,+|\psi\rangle\right)\left(\langle S_y,+|S_x,+\rangle\langle S_x,+|S_y,+\rangle\right)\\
&= \alpha\bar{\alpha}\left(\frac{1}{\sqrt{2}}\times\frac{1}{\sqrt{2}}\right)\\
&= \frac{\alpha\bar{\alpha}}{2}
\end{aligned}$$

Summing $P_{\psi\to((x\pm),(y+))}$ over both possible intermediate $x$ states gives

$$\begin{aligned}
\sum_{(\pm,x)} P_{\psi\to((x\pm),(y+))} &= \left(\langle\psi|S_x,+\rangle\langle S_x,+|\psi\rangle\right)\left(\langle S_y,+|S_x,+\rangle\langle S_x,+|S_y,+\rangle\right)\\
&\quad+\left(\langle\psi|S_x,-\rangle\langle S_x,-|\psi\rangle\right)\left(\langle S_y,+|S_x,-\rangle\langle S_x,-|S_y,+\rangle\right)\\
&= \frac{\alpha\bar{\alpha}}{2}+\frac{\beta\bar{\beta}}{2}\\
&= \frac{1}{2}
\end{aligned}$$

Now consider

$$\begin{aligned}
P_{a\to c} &= |\langle c|a\rangle|^2\\
&= \left(\langle a|+\rangle\langle +|c\rangle+\langle a|-\rangle\langle -|c\rangle\right)\left(\langle c|+\rangle\langle +|a\rangle+\langle c|-\rangle\langle -|a\rangle\right)\\
&= \left(\bar{\alpha}\gamma+\bar{\beta}\delta\right)\left(\bar{\gamma}\alpha+\bar{\delta}\beta\right)\\
&= \alpha\bar{\alpha}\gamma\bar{\gamma}+\beta\bar{\beta}\delta\bar{\delta}+\bar{\alpha}\beta\gamma\bar{\delta}+\alpha\bar{\beta}\bar{\gamma}\delta
\end{aligned}$$

$$\begin{aligned}
P_{a\to c} &= |\langle c|a\rangle|^2\\
&= \langle\psi|S_y,+\rangle\langle S_y,+|\psi\rangle\\
&= \frac{1}{\sqrt{2}}\left(\bar{\alpha}+i\bar{\beta}\right)\frac{1}{\sqrt{2}}\left(\alpha-i\beta\right)\\
&= \frac{1}{2}\left(\alpha\bar{\alpha}+\beta\bar{\beta}+i\left(\alpha\bar{\beta}-\beta\bar{\alpha}\right)\right)
\end{aligned}$$

Now consider the general spin-$\frac{1}{2}$ case:

$$\begin{aligned}
P_{a\to c} &= |\langle c|a\rangle|^2\\
&= \sum_i \langle a|b_i\rangle\langle b_i|a\rangle\sum_j\langle c|b_j\rangle\langle b_j|a\rangle
\end{aligned}$$

and

$$\sum_i P_{a\to(b_i,c)} = \sum_i\langle a|b_i\rangle\langle b_i|a\rangle\left(\langle c|b_i\rangle\langle b_i|c\rangle\right)$$

Specialize these to spin $\frac{1}{2}$, and let the middle state be in the $z$-basis (since we lose no generality by letting one of the directions be $z$):

$$\begin{aligned}
P_{a\to c} &= |\langle c|a\rangle|^2\\
&= \sum_{\pm}\langle a|b_i\rangle\langle b_i|c\rangle\sum_j\langle c|b_j\rangle\langle b_j|a\rangle\\
&= \left(\langle a|+\rangle\langle +|c\rangle+\langle a|-\rangle\langle -|c\rangle\right)\left(\langle c|+\rangle\langle +|a\rangle+\langle c|-\rangle\langle -|a\rangle\right)
\end{aligned}$$

Set

$$\begin{aligned}
\langle +|a\rangle &= \alpha\\
\langle -|a\rangle &= \beta\\
\langle +|c\rangle &= \gamma\\
\langle -|c\rangle &= \delta
\end{aligned}$$

Then

$$
\begin{aligned}
P_{a\to c} &= |\langle c|a\rangle|^2 \\
&= (\langle a|+\rangle\langle +|c\rangle + \langle a|-\rangle\langle -|c\rangle)(\langle c|+\rangle\langle +|a\rangle + \langle c|-\rangle\langle -|a\rangle) \\
&= \left(\bar{\alpha}\gamma+\bar{\beta}\delta\right)\left(\bar{\gamma}\alpha+\bar{\delta}\beta\right) \\
&= \alpha\bar{\alpha}\gamma\bar{\gamma}+\beta\bar{\beta}\delta\bar{\delta}+\bar{\alpha}\beta\gamma\bar{\delta}+\alpha\bar{\beta}\bar{\gamma}\delta
\end{aligned}
$$

For the sum over intermediate states we have

$$
\begin{aligned}
\sum_i P_{a\to(b_i,c)} &= \sum_i \langle a|b_i\rangle\langle b_i|a\rangle(\langle c|b_i\rangle\langle b_i|c\rangle) \\
&= \langle a|+\rangle\langle +|a\rangle(\langle c|+\rangle\langle +|c\rangle) + \langle a|-\rangle\langle -|a\rangle(\langle c|-\rangle\langle -|c\rangle) \\
&= \alpha\bar{\alpha}\gamma\bar{\gamma}+\beta\bar{\beta}\delta\bar{\delta}
\end{aligned}
$$

So we have the difference,

$$
P_{a\to c}-\sum_i P_{a\to(b_i,c)} = \bar{\alpha}\beta\gamma\bar{\delta}+\alpha\bar{\beta}\bar{\gamma}\delta
$$

Write the complex numbers as

$$
\begin{aligned}
\alpha &= a \\
\beta &= \sqrt{1-a^2}e^{i\varphi} \\
\gamma &= b \\
\delta &= \sqrt{1-b^2}e^{i\theta}
\end{aligned}
$$

where the overall phase freedom allows us to choose $\alpha$ and $\gamma$ real. Then

$$
\begin{aligned}
P_{a\to c}-\sum_i P_{a\to(b_i,c)} &= a\sqrt{1-a^2}e^{i\varphi}b\sqrt{1-b^2}e^{-i\theta}+a\sqrt{1-a^2}e^{-i\varphi}b\sqrt{1-b^2}e^{i\theta} \\
&= ab\sqrt{1-a^2}\sqrt{1-b^2}\left(e^{i(\varphi-\theta)}+e^{i\varphi}e^{-i(\varphi-\theta)}\right) \\
&= 2ab\sqrt{1-a^2}\sqrt{1-b^2}\cos(\varphi-\theta)
\end{aligned}
$$

This is a simple example of Bell's Theorem (Wikipaedia has a good article under "Bell's Theorem").