Irreducible Tensor Operators

April 14, 2010

1 Cartesian Tensors

We know that vectors may be rotated by applying a rotation matrix. Beginning with vectors, we can build other objects that transform simply under rotations by taking outer products:

$$
\begin{aligned}
T &= \vec{v} \otimes \vec{w} \\
T_{ij} &= v_i w_j \\
&= \begin{pmatrix} v_1 w_1 & v_1 w_2 & v_1 w_3 \\ v_2 w_1 & v_2 w_2 & v_2 w_3 \\ v_3 w_1 & v_3 w_2 & v_3 w_3 \end{pmatrix}
\end{aligned}
$$

By adding objects of this sort, we can build arbitrary matrices. By choosing enough different vectors, $v^1, v^2, \ldots, v^n$, we can build any matrix,

$$M_{ij} = v_i^1 v_j^2 + v_i^3 v_j^4 + v_i^5 v_j^6 + \ldots + v_i^{n-1} v_j^n$$

This tells us how matrices transform under rotations. Since each of the vectors rotates according to

$$\tilde{v}_i = \sum_j R_{ij} v_j$$

The matrix M_{ij} rotates to

$$
\begin{aligned}
\tilde{M}_{ij} &= \tilde{v}_i^1 \tilde{v}_j^2 + \ldots + \tilde{v}_i^{n-1} \tilde{v}_j^n \\
&= \left(\sum_m R_{im} v_m^1 \right) \left(\sum_k R_{ik} v_k^2 \right) + \ldots + \left(\sum_m R_{im} v_m^{n-1} \right) \left(\sum_k R_{ik} v_k^n \right) \\
&= \sum_k \sum_m R_{im} R_{ik} \left(v_m^1 v_k^2 + \ldots + v_m^{n-1} v_k^n \right) \\
&= \sum_k \sum_m R_{im} R_{ik} M_{mk}
\end{aligned}
$$

Therefore, M_{ij} transforms with two copies of the rotation matrix R_{ij}, one on each index.

We can build up more general objects in the same way. Taking outer products of n vectors, $\vec{u}, \vec{v}, \ldots, \vec{w}$, we build rank-$n$ tensors:

$$
\begin{aligned}
T &= \vec{u} \otimes \vec{v} \otimes \ldots \otimes \vec{w} \\
T_{ij\ldots k} &= u_i v_j \ldots w_k
\end{aligned}
$$

and these will also transform with one factor of R_{ij} on each index.

A more systematic way to build these tensors is to start with a basis of unit vectors,

$$
\begin{aligned}
\hat{n}_{(1)} &= (1, 0, 0) \\
\hat{n}_{(2)} &= (0, 1, 0) \\
\hat{n}_{(3)} &= (0, 0, 1)
\end{aligned}
$$

These are *vectors*. The labels in parenthesis just tell us *which* vector we're using. Notice that, for example, $\hat{n}_{(1)}$ has components

$$\left[\hat{n}_{(1)}\right]_i = \delta_{1i}$$

We can build a general vector by taking a linear combination of the three basis vectors,

$$v = \sum_i v_i \hat{n}_{(i)}$$

The three numbers $v_i = v_1, v_2, v_3$ are the Cartesian components of v. This works for higher rank tensors as well. Take the nine possible outer products of the basis vectors,

$$\hat{n}_{(i)} \otimes \hat{n}_{(j)}$$

This is a matrix with a 1 in row i, column j, and zeros everywhere else. Then an arbitrary matrix may be written as

$$M = \sum_{i,j} M_{ij} \hat{n}_{(i)} \otimes \hat{n}_{(j)}$$

where M_{ij} is just the number in row i and column j. Think of this object as *rotationally invariant*. It is the components M_{ij} which change under rotation.

For a general rank-n tensor, we have

$$T = \sum_{i,j} T_{ij\ldots k} \hat{n}_{(i)} \otimes \hat{n}_{(j)} \otimes \ldots \otimes \hat{n}_{(k)}$$

and the 3^n numbers $T_{ij\ldots k}$ are the Cartesian components.

2 Irreducible tensors

The Cartesian components of tensors are mixed by the rotation transformations, R_{ij}, but not all components of a given tensor mix with all the others. For example, we have seen how a matrix may be broken into rotationally independent pieces as

$$M_{ij} = \frac{1}{3} tr(M) \delta_{ij} + \frac{1}{2}(M_{ij} - M_{ji}) + \frac{1}{2}\left(M_{ij} + M_{ji} - \frac{1}{3}\delta_{ij} tr(M)\right)$$

where $tr(M) = \sum_k M_{kk}$ is the trace of M. Each of these three parts,

$$\begin{aligned} M_{ij}^{tr} &= \frac{1}{3} tr(M) \delta_{ij} \\ M_{ij}^{A} &= \frac{1}{2}(M_{ij} - M_{ji}) \\ M_{ij}^{S} &= \frac{1}{2}\left(M_{ij} + M_{ji} - \frac{1}{3}\delta_{ij} tr(M)\right) \end{aligned}$$

is preserved by rotations, in the sense that

$$\begin{aligned} tr\left(M'_{ij}\right) &= \sum_{i,k,l} R_{ik} R_{il} M_{kl} \\ &= \sum_{i,k,l} R^t_{ki} R_{il} M_{kl} \\ &= \sum_{i,k,l} R^{-1}_{ki} R_{il} M_{kl} \\ &= \sum_{i,k,l} \delta_{kl} M_{kl} \\ &= \sum M_{kk} \end{aligned}$$

$$
\begin{aligned}
&= (tr(M))' \\
\left(M^A_{ij}\right)' &= \sum R_{ik}R_{jl}\frac{1}{2}(M_{kl}-M_{lk}) \\
&= \frac{1}{2}\sum\left(R_{ik}R_{jl}M_{kl}-R_{ik}R_{jl}M_{lk}\right) \\
&= \frac{1}{2}\left(M'_{ij}-\sum R_{jl}R_{ik}M_{lk}\right) \\
&= \frac{1}{2}\left(M'_{ij}-M'_{ji}\right) \\
&= \left(M'_{ij}\right)^A \\
\left(M^S_{ij}\right)' &= \sum R_{ik}R_{jl}\frac{1}{2}\left(M_{kl}+M_{lk}-\frac{1}{3}\delta_{kl}tr(M)\right) \\
&= \sum\frac{1}{2}\left(R_{ik}R_{jl}M_{kl}+R_{ik}R_{jl}M_{lk}-\frac{1}{3}R_{ik}R_{jl}\delta_{kl}tr(M)\right) \\
&= \frac{1}{2}\left(M'_{ij}+\sum R_{jl}R_{ik}M_{lk}-\frac{1}{3}\sum R_{ik}R^{-1}_{kj}tr(M)\right) \\
&= \frac{1}{2}\left(M'_{ij}+M'_{ji}-\frac{1}{3}\delta_{ij}tr(M)\right) \\
&= \left(M'_{ij}\right)^S
\end{aligned}
$$

For example, the components of the antisymmetric part mix only with the components of the antisymmetric part. These are the irreducible parts of the tensor under rotations.

3 Arbitrary angular momentum

Consider the 2-dimensional space of spin-$\frac{1}{2}$ vectors,

$$\left|\frac{1}{2},m\right\rangle$$

for $m=\pm\frac{1}{2}$. Rotations act on these linearly and homogeneously,

$$e^{\frac{i\varphi}{2}\hat{\mathbf{n}}\cdot\sigma}\left|\frac{1}{2},m\right\rangle$$

To simplify the notation, we just write the two states as

$$|m\rangle=|\pm\rangle \quad = \quad \left|\frac{1}{2},\pm\frac{1}{2}\right\rangle$$

when we know that $j=\frac{1}{2}$.

Just as in the real case, we may take outer products of them,

$$|m_1\rangle\otimes|m_2\rangle\otimes\ldots\otimes|m_k\rangle$$

By taking linear combinations of these, we may write general $SU(2)$ tensors,

$$T=\sum_{m_1\ldots m_k}T_{m_1m_2\ldots m_k}|m_1\rangle\otimes|m_2\rangle\otimes\ldots\otimes|m_k\rangle$$

These objects transform multilinearly and homogeneously, with one transformation matrix, $e^{\frac{i\varphi}{2}\hat{\mathbf{n}}\cdot\sigma}$, on each index.

However, we know that this Cartesian basis is not irreducible. Not only that – we also know how to find the irreducible parts in general. We can write a product of two vectors as a sum of a triplet and a singlet,

$$\begin{aligned}
|1,1\rangle &= |+\rangle\,|+\rangle \\
|1,0\rangle &= \frac{1}{\sqrt{2}}\left(|+\rangle\,|-\rangle + |-\rangle\,|+\rangle\right) \\
|1,-1\rangle &= |-\rangle\,|-\rangle
\end{aligned}$$

and

$$|0,0\rangle = \frac{1}{\sqrt{2}}\left(|+\rangle\,|-\rangle - |-\rangle\,|+\rangle\right)$$

We may invert these relationships,

$$\begin{aligned}
|+\rangle\,|+\rangle &= |1,1\rangle \\
|+\rangle\,|-\rangle &= \frac{1}{\sqrt{2}}\left(|1,0\rangle + |0,0\rangle\right) \\
|-\rangle\,|+\rangle &= \frac{1}{\sqrt{2}}\left(|1,0\rangle - |0,0\rangle\right) \\
|-\rangle\,|-\rangle &= |1,-1\rangle
\end{aligned}$$

Notice that the coefficients are the same. This corresponds to the reality of the Clebsch-Gordon coefficients,

$$\langle j,m|\, j_1, j_2; m_1, m_2\rangle = \langle j_1, j_2; m_1, m_2|\, j,m\rangle$$

Formally, the change of basis may be written as

$$\begin{aligned}
|j,m\rangle &= \sum_{m_1,m_2} |j_1, j_2; m_1, m_2\rangle \langle j_1, j_2; m_1, m_2|\, j,m\rangle \\
|j_1, j_2; m_1, m_2\rangle &= \sum_{j,m} \langle j,m|\, j_1, j_2; m_1, m_2\rangle\, |j,m\rangle
\end{aligned}$$

for any j in the range $|j_1 - j_2| \leq j \leq j_1 + j_2$.

Notice that we have produced a vector basis for the product space. Instead of thinking of the product

$$|j_1, m_1\rangle \otimes |j_2, m_2\rangle$$

as a matrix, we treat it as a single, higher-dimensional vector,

$$|j_1, j_2; m_1, m_2\rangle = |j_1, m_1\rangle \otimes |j_2, m_2\rangle$$

and change the basis to a combination of the $|j,m\rangle$ kets by inserting an identity:

$$\begin{aligned}
|j_1, m_1\rangle \otimes |j_2, m_2\rangle &= |j_1, j_2; m_1, m_2\rangle \\
&= \left(\sum_{j,m} |j,m\rangle \langle j,m|\right) |j_1, j_2; m_1, m_2\rangle \\
&= \sum_{j,m} \langle j,m|\, j_1, j_2; m_1, m_2\rangle\, |j,m\rangle
\end{aligned}$$

We trade off higher numbers of indices for higher-dimensional vectors.

This means that we may write tensors in a more condensed form.

Suppose we have a rank-2 tensor,

$$T = \sum_{m_1 \ldots m_k} T_{m_1 m_2}\, |m_1\rangle \otimes |m_2\rangle$$

Using the change of basis, this becomes

$$\begin{aligned}
T &= T_{m_1 m_2} |m_1\rangle \otimes |m_2\rangle \\
&= \sum_{j,m} \sum_{m_1 \ldots m_k} T_{m_1 m_2} \langle j,m|\ j_1, j_2; m_1, m_2\rangle |j,m\rangle \\
&= \sum_{m_1=-m_2} T_{m_1 m_2} \langle 0,0|\ j_1, j_2; m_1, m_2\rangle |0,0\rangle + \sum_{m} \sum_{m_1 \ldots m_k} T_{m_1 m_2} \langle 1,m|\ j_1, j_2; m_1, m_2\rangle |1,m\rangle
\end{aligned}$$

We still know how these transform under rotations; moreover, the first and second terms in the last line transform independently – they are the irreducible tensors that make up the original T.

It makes sense to treat the irreducible parts separately. The first tensor,

$$T_0^0 = \sum_{m_1=-m_2} T_{m_1 m_2} \langle 0,0|\ j_1, j_2; m_1, m_2\rangle |0,0\rangle$$

is a scalar, invariant under $SU(2)$ rotations. The second is a triplet,

$$T_m^1 = \sum_{m_1,m_2} T_{m_1 m_2} \langle 1,m|\ j_1, j_2; m_1, m_2\rangle |1,m\rangle$$

This transforms with one factor of $e^{\frac{i\varphi}{2}\mathbf{n}\cdot\mathbf{J}}$, but now we use the 3×3 representation of $\mathbf{J}$.

These two tensors are *irreducible spherical tensors.*

Notice that the components $T_{m_1 m_2}$ transform as

$$\tilde{T}_{m_1 m_2} = \sum_{m_3,m_4} \left[e^{\frac{i\varphi}{2}\mathbf{n}\cdot\sigma}\right]_{m_1 m_3} \left[e^{\frac{i\varphi}{2}\mathbf{n}\cdot\sigma}\right]_{m_2 m_4} T_{m_3 m_4}$$

with two copies of the 2-dimensional generators, $J_i = \sigma_i$. By contrast, the irreducible tensors T_0^0, T_m^1 transform as

$$\begin{aligned}
\tilde{T}_0^0 &= \sum_{m_3,m_4} \left[e^{\frac{i\varphi}{2}\mathbf{n}\cdot\mathbf{J}^{(1)}}\right]_{mm'} T_{m'}^0 = 1 T_0^0 \\
\tilde{T}_m^1 &= \sum_{m_3,m_4} \left[e^{\frac{i\varphi}{2}\mathbf{n}\cdot\mathbf{J}^{(3)}}\right]_{mm'} T_{m'}^1
\end{aligned}$$

where $\mathbf{J}^{(1)} = 1$ and $\mathbf{J}^{(3)}$ are the 1×1 and 3×3 representations of the rotation generators, respectively.

We can continue this process for higher rank tensors. In general, an irreducible spherical tensor of rank k transforms linearly under the $(2k+1)$-dimensional representation of $SU(2)$,

$$\tilde{T}_q^k = \sum_{q'=-k}^{k} \left[e^{\frac{i\varphi}{2}\mathbf{n}\cdot\mathbf{J}}\right]_{qq'} T_{q'}^k$$

and this expression is suitable for defining them: any object that transforms in this way is an irreducible spherical tensor. This means that, rather than having more and more indices on our (Cartesian) tensors, we represent higher and higher rank tensors as larger and larger vectors. In a basis, the general rank-k spherical tensor may be written as

$$T^k = \sum_{q=-k}^{k} T_q^k |k,q\rangle$$

which clearly has $2k+1$ independent components.

It is easy to multiply spherical tensors. We have

$$\begin{aligned}
T^{k_1} T^{k_2} &= \sum_{q_1=-k_1}^{k_1} T_{q_1}^{k_1} |k_1, q_1\rangle \otimes \sum_{q_2=-k_2}^{k_2} T_{q_2}^{k_2} |k_2, q_2\rangle \\
&= \sum_{k=|k_1-k_2|}^{k_1+k_2} \sum_{q=-k}^{k} \left(\sum_{q_1,q_2} T_{q_1}^{k_1} T_{q_2}^{k_2} \langle k,q|\ k_1, k_2; q_1, q_2\rangle |k,q\rangle \right)
\end{aligned}$$

so that the product is a sum over spherical tensors of ranks $|k_1 - k_2| \le k \le k_1 + k_2$, where the tensor of any given rank k in that range is given by

$$T^k = \sum_{q_1,q_2} T^{k_1}_{q_1} T^{k_2}_{q_2} \langle k,q|\, k_1,k_2;q_1,q_2\rangle\, |k,q\rangle$$

with components

$$T^k_q = \sum_{q_1,q_2} T^{k_1}_{q_1} T^{k_2}_{q_2} \langle k,q|\, k_1,k_2;q_1,q_2\rangle$$

This is Sakurai's addition theorem, eq.(3.10.27). By working in a basis, we already know that this object transforms as a rank k spherical tensor.

Now we move to the Wigner-Ekhart theorem. We would like to find general matrix elements of spherical tensors, and the goal is to show that the dependence on the z-component of spin can be reduced to a Clebsch-Gordon coefficient.

Consider a state $|\lambda\rangle$ which is an eigenstate of $\mathbf{J}^2$ and J_z, and perhaps other operators as well, so that we may write

$$\begin{aligned} |\lambda\rangle &= |\alpha, j, m\rangle \\ &= |\alpha\rangle \otimes |j,m\rangle \end{aligned}$$

where α represents the eigenvalues of the remaining commuting operators. Then we may define the matrix elements of a tensor operator as

$$\begin{aligned} \left\langle \alpha', j', m' \right| T^k \left| \alpha'', j'', m'' \right\rangle &= \left\langle \alpha', j', m' \right| \left(\sum_{q=-k}^{k} T^k_q |k,q\rangle \right) \otimes \left|\alpha''\right\rangle \otimes \left| j'', m'' \right\rangle \\ &= \left\langle \alpha' \right| \otimes \left\langle j', m' \right| \left(\sum_{q=-k}^{k} T^k_q |k,q\rangle \otimes \left| j'', m'' \right\rangle \right) \otimes \left|\alpha''\right\rangle \\ &= \left\langle \alpha' \right| \otimes \left\langle j', m' \right| \left(\sum_{j=|j"-k|}^{j"+k} \sum_{m=-j}^{j} \sum_{q=-k}^{k} T^k_q \left\langle j,m \right| k, j''; q, m'' \rangle \left| j,m; j'',k \right\rangle \right) \otimes \left|\alpha''\right\rangle \end{aligned}$$

The Clebsch-Gordon coefficient vanishes unless $q = m'' - m$, so $T^k_q = T^k_{m''-m}$ has its dependence on the z component fixed by the Clebsch-Gordon coefficient. Then

$$\begin{aligned} \left\langle \alpha', j', m' \right| T^k \left| \alpha'', j'', m'' \right\rangle &= \left\langle \alpha' \right| \otimes \left\langle j', m' \right| \left(\sum_{j=|j"-k|}^{j"+k} \sum_{m=-j}^{j} \sum_{q=-k}^{k} T^k \left\langle j,m \right| k, j''; q, m'' \rangle \left| j,m; j'',k \right\rangle \right) \otimes \left|\alpha''\right\rangle \\ &= \left\langle \alpha' \right| \otimes \sum_{j=|j"-k|}^{j"+k} \sum_{m=-j}^{j} \left\langle j', m' \right| T^k_{m''-m} \left| j,m; j'',k \right\rangle \left\langle j,m \right| k, j''; q, m'' \rangle \otimes \left|\alpha''\right\rangle \\ &= \left\langle \alpha' \right| \otimes \sum_{j=|j"-k|}^{j"+k} \sum_{m=-j}^{j} \left\langle j' \right| T^k_{m''-m} \left| j, j'',k \right\rangle \delta_{m'm} \left\langle j,m \right| k, j''; q, m'' \rangle \otimes \left|\alpha''\right\rangle \\ &= \sum_{j=|j"-k|}^{j"+k} \left\langle \alpha', j' \right| T^k_{m''-m'} \left| j, j'', k, \alpha'' \right\rangle \left\langle j, m' \right| k, j''; q, m'' \rangle \end{aligned}$$

The crucial point here is that all of the J_z dependence is determined by the Clebsch-Gordon coefficient. All of the remaining part of the matrix element depends only on the extra quantum numbers, α', α'', and on the total angular momentum numbers, j', j'', k. We therefore write the right hand side as simply $\langle \alpha', j' \| T^k \| \alpha'', j'' \rangle$ with a convenient normalization, time the Clebsch-Gordon coefficient,

$$\begin{aligned} \left\langle \alpha', j', m' \right| T^k \left| \alpha'', j'', m'' \right\rangle &= \sum_{j=|j"-k|}^{j"+k} \left\langle \alpha', j' \right| T^k \left| j, j'', k, \alpha'' \right\rangle \left\langle j, m' \right| k, j''; q, m'' \rangle \\ &= \frac{1}{\sqrt{2j'+1}} \left\langle \alpha', j' \right\| T^k \left\| \alpha'', j'' \right\rangle \left\langle j, m' \right| k, j''; q, m'' \rangle \end{aligned}$$

This is the Wigner-Ekhart theorem. The factor $\langle \alpha', j' \| T^k \| \alpha'', j'' \rangle$ is called the reduced matrix element. It can often be easily determined.