Tensor operators and the Wigner-Eckhart theorem

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1 Clebsch-Gordon coefficients

In general, the addition of irreducible momentum states leads to the expansion of the resultant, $|j,m\rangle$ states in terms of the product states,

$$|j_1,m_1\rangle\,|j_2,m_2\rangle$$

If we write this as

$$|j,m\rangle=\alpha_{j,m;j_1j_2m_1m_2}\,|j_1,m_1\rangle\,|j_2,m_2\rangle$$

where the coefficients $\alpha_{j_1j_2m_1m_2}$ are called Clebsch-Gordon coefficients. For a more complete notation, notice that we start with four commuting operators,

$$\mathbf{J}_1^2,\mathbf{J}_2^2,J_{1z},J_{2z}$$

so we may write the product states as

$$|j_1,j_2,m_1,m_2\rangle=|j_1,m_1\rangle\otimes|j_2,m_2\rangle$$

In the same way, the total $\mathbf{J}^2$ leads to a new commuting set,

$$\mathbf{J}^2,\mathbf{J}_1^2,\mathbf{J}_2^2,J_z$$

so we may label the final kets by

$$|j_1,j_2;j,m\rangle=|j,m\rangle$$

Then inserting an identity gives a straightforward change of basis

$$|j_1,j_2;j,m\rangle=\sum_{m_1,m_2}|j_1,j_2,m_1,m_2\rangle\,\langle j_1,j_2,m_1,m_2\,|j_1,j_2;j,m\rangle$$

and we see that the Clebsch-Gordon coefficients have the lengthy label,

$$\alpha_{j,m;j_1j_2m_1m_2}=\langle j_1,j_2,m_1,m_2\,|j_1,j_2;j,m\rangle$$

Here we always have

$$\begin{aligned}m_1+m_2&=m\\|j_1-j_2|&\leq j\leq j_1+j_2\end{aligned}$$

We may also expand in the other direction,

$$|j_1,j_2,m_1,m_2\rangle=\sum_{m_1,m_2}|j_1,j_2;j,m\rangle\,\langle j_1,j_2;j,m\,|j_1,j_2,m_1,m_2\rangle$$

2 Tensors

We have briefly mentioned tensors, but have worked mostly with vectors ("rank-1 tensors") and matrices ("rank-2 tensors"). However, they may occur with any rank. To see this, just consider the outer product of many vectors,

$$T_{ij\ldots k} = u_i v_j \ldots w_k$$

If there are n different indices, $i, j, \ldots, k$ then this represents 3^n different numbers, one for each choice $i = 1, 2, 3; j = 1, 2, 3, \ldots k = 1, 2, 3$. We can make even more general combinations by adding such objects together, so the most general sort of *rank n tensor* will have 3^n independent components.

The central point is that we know how any such object transforms under rotations, because we know how rotations transform each of the vectors, $u_i, v_j, \ldots, w_k$,

$$\tilde{u}_i = R_{ij} u_j$$

where R_{ij} is a rotation matrix. If we rotate each vector in $T_{ij\ldots k}$, then each index gets a factor of R_{ij}, so

$$\tilde{T}_{ij\ldots k} = \sum_{m,n,\ldots,s} R_{im} R_{jn} \ldots R_{ks} T_{mn\ldots s}$$

where even though our usual rules for multiplying matrices and vectors do not generalize, in the worst case we could compute this by writing out all the sums.

Objects like this may be thought of as operators. In the same way that we form a scalar as a dot product of two vectors, or use a matrix to map a vector to another vector, we may use $T_{ij\ldots k}$ to map one, two or more vectors to another tensor of lower rank. If we let $T_{ij\ldots k}$ act on n different vectors, we get a real number,

$$\sum_{i,j,\ldots,k} T_{ij\ldots k} a_i b_j \ldots c_k \in R$$

You can find much more about tensors on the General Relativity pages, http://www.physics.usu.edu/Wheeler/GenRel2013

3 Irreducible tensor operators

We have seen how a matrix can be written in terms of its irreducible parts,

$$M_{ij} \equiv \delta_{ij} tr\left(M\right) + \frac{1}{2}\left(M_{ij} - M_{ji}\right) + \frac{1}{2}\left(M_{ij} + M_{ji} - \frac{2}{3}\delta_{ij} tr\left(M\right)\right)$$

which we may write explicitly in terms of rotatationally invariant parts by writing

$$M_{mm'} \to |1, m\rangle\, |1, m'\rangle$$

and computing the addition of the two $j = 1$ space into spaces of dimensions $5, 3$ and 1:

$$\begin{aligned} |j, m\rangle &= |2, m\rangle \\ |j, m\rangle &= |1, m\rangle \\ |j, m\rangle &= |0, 0\rangle \end{aligned}$$

In the same way, we can write any tensor, $T_{ij\ldots k}$, in terms of irreducible parts.

One problem with this view of a matrix is that, while we have captured its rotational properties elegantly, writing

$$3 \otimes 3 = 1 \oplus 3 \oplus 5$$

we do not immediately see the operator character of M_{ij}. Nonetheless, we can use it as one. Expanding M_{ij} as an operator on the three spaces, $|0,0\rangle, |1,m\rangle, |2,m\rangle$. Let

$$\begin{aligned}
M^{(0)}_{(0)} &= tr\,(M) = M_{ii} \\
M^{(1)}_{(m)} &= \frac{1}{2}\sum_{i,j=1}^{3} M_{ij}\varepsilon_{ijm} \\
M^{(2)}_{(m)} &= c_{mij}\left(M_{ij} + M_{ji} - \frac{2}{3}\delta_{ij} trM\right)
\end{aligned}$$

where the constants c_{mij} are chosen to give the five independent degrees of freedom having the appropriate z-components. We needn't go into the assignment of these. Then expanded in irreducible representations,

$$M = M^{(0)}_{(0)}|0,0\rangle + \sum_{m=1}^{3} M^{(1)}_{(m)}|1,m\rangle + \sum_{m=1}^{5} M^{(2)}_{(m)}|2,m\rangle$$

so that each piece operates only on the corresponding vector space. The three parts, $M^{(j)}_{(m)}$ are *irreducible operators*. Since each piece has definite properties under rotations, their operation is easily seen to satisfy the Wigner-Eckart theorem.

First, we need to generalize the idea of an irreducible tensor operator. We have seen that we may decompose tensors of any rank, $T_{ij\ldots k}$, into irreduciple parts by applying our understanding of the addition of angular momentum to the 3^n product states,

$$\begin{aligned}
T_{ij\ldots k} &\rightarrow |1,m\rangle|1,m'\rangle\ldots|1,m''\rangle \\
3\otimes 3\otimes\ldots\otimes 3 &\rightarrow \oplus_j\,(2j+1)
\end{aligned}$$

where the formal sum, $\oplus_j$, on the right is over the various j values that emerge from the addition procedure. Then we can build a series of irreducible operators out of $T_{ij\ldots k}$ in the same way as we did for M_{ij}. Any one piece of $T_{ij\ldots k}$, of the form

$$T^{(k)}_{(m)} = T^{(k)}|k,m\rangle$$

is an irreducible tensor operator where q may take any half-integer or integer value in the range allowed by the addition of states.

As another example, any real 3-vector, v_i, gives an irreducible representation. Under rotations v_i transforms in the same way as $|1,m\rangle$, so multiplying by the length of v, we may write an irreducible vector operator as

$$\tilde{v}_m = v\,|1,m\rangle$$

Looking at this identification in a coordinate basis, we may relate the components, $\tilde{v}_m$, to the Cartesian components, v_i,

$$\begin{aligned}
\tilde{v}_1 &= \langle\theta,\varphi|\,v\,|1,1\rangle \\
&= vY^1_1 \\
&= -\sqrt{\frac{3}{4\pi}}ve^{i\varphi}\sin\theta \\
&= -\sqrt{\frac{3}{4\pi}}\,(v_x + iv_y) \\
\tilde{v}_0 &= \langle\theta,\varphi|\,v\,|1,0\rangle \\
&= vY^1_0 \\
&= \sqrt{\frac{3}{2\pi}}v\cos\theta
\end{aligned}$$

$$\begin{aligned}
&= \sqrt{\frac{3}{2\pi}} v_z \\
\tilde{v}_{-1} &= \langle \theta, \varphi | \, v \, |1, -1\rangle \\
&= v Y^1_{-1} \\
&= \sqrt{\frac{3}{4\pi}} v e^{-i\varphi} \sin\theta \\
&= \sqrt{\frac{3}{4\pi}} \left(v_x - i v_y\right)
\end{aligned}$$

so we have

$$\begin{aligned}
\sqrt{\frac{2\pi}{3}} \tilde{v}_1 &= -\frac{1}{\sqrt{2}} \left(v_x + i v_y\right) \\
\sqrt{\frac{2\pi}{3}} \tilde{v}_0 &= v_z \\
\sqrt{\frac{2\pi}{3}} \tilde{v}_{-1} &= \frac{1}{\sqrt{2}} \left(v_x - i v_y\right)
\end{aligned}$$

4 The Wigner-Eckart theorem

We may now write the Wigner-Eckart theorem, which gives a reduced form for matrix elements of irreducible tensor operators. Let

$$|\alpha, j, m\rangle = |\alpha\rangle \otimes |j, m\rangle$$

be the state of interest, where α stands for all quantum numbers except the angular momentum ones, j, m. The matrix elements of an irreducible tensor operator,

$$T^{(k)}_{(q)} = T^{(k)} \, |k, q\rangle$$

in this state are given by

$$\begin{aligned}
\langle \alpha_2, j_2, m_2 | \, T^{(k)}_{(q)} \, |\alpha_1, j_1, m_1\rangle &= \langle \alpha_2, j_2, m_2 | \, T^{(k)} \, |k, q\rangle \otimes |\alpha_1, j_1, m_1\rangle \\
&= \langle \alpha_2 | \otimes \langle j_2, m_2 | \, T^{(k)} \, |\alpha_1\rangle \otimes |k, q\rangle \otimes |j_1, m_1\rangle
\end{aligned}$$

We now add the angular momentum kets on the right, $|k, q\rangle \otimes |j_1, m_1\rangle = |k, j_1; q, m_1\rangle$, using the Clebsch-Gordon coefficients

$$\begin{aligned}
|k, j_1; q, m_1\rangle &= \sum_{j=|k-j_1|}^{k+j_1} \sum_{m=-j}^{j} |j_1, q; j, m\rangle \langle q, j_1; j, m \, |k, j_1; q, m_1\rangle \\
&= \sum_{j=|k-j_1|}^{k+j_1} \sum_{m=-j}^{j} |j_1, k; j, m\rangle \langle k, j_1; j, m \, |k, j_1; q, m_1\rangle
\end{aligned}$$

Then we find

$$\begin{aligned}
\langle \alpha_2, j_2, m_2 | \, T^{(k)}_{(q)} \, |\alpha_1, j_1, m_1\rangle &= \langle \alpha_2 | \otimes \langle j_2, m_2 | \, T^{(k)} \, |\alpha_1\rangle \otimes |k, j_1; q, m_1\rangle \\
&= \langle \alpha_2 | \otimes \langle j_2, m_2 | \, T^{(q)} \, |\alpha_1\rangle \otimes \sum_{j=|k-j_1|}^{k+j_1} \sum_{m=-j}^{j} |j_1, k; j, m\rangle \langle k, j_1; j, m \, |k, j_1; q, m_1\rangle \\
&= \sum_{j=|k-j_1|}^{k+j_1} \sum_{m=-j}^{j} \langle k, j_1; j, m \, |k, j_1; q, m_1\rangle \langle \alpha_2 | \otimes \langle j_2, m_2 | \, T^{(k)} \, |\alpha_1\rangle \otimes |j_1, k; j, m\rangle
\end{aligned}$$

Now evaluate

$$
\begin{aligned}
\langle\alpha_2|\otimes\langle j_2,m_2|\,T^{(k)}\,|\alpha_1\rangle\otimes|j_1,k;j,m\rangle &= \langle\alpha_2,j_2,m_2|\,T^{(k)}\,|\alpha_1,j,m\rangle \\
&= \langle\alpha_2,j_2|\,T^{(k)}\,|\alpha_1,j\rangle\,\delta_{m_2m}
\end{aligned}
$$

so that we have

$$
\begin{aligned}
\langle\alpha_2,j_2,m_2|\,T^{(k)}_{(q)}\,|\alpha_1,j_1,m_1\rangle &= \sum_{j=|k-j_1|}^{k+j_1}\sum_{m=-j}^{j}\langle k,j_1;j,m\;|k,j_1;q,m_1\rangle\,\langle\alpha_2,j_2|\,T^{(k)}\,|\alpha_1,j\rangle\,\delta_{m_2m} \\
&= \langle k,j_1;j,m_2\;|k,j_1;q,m_1\rangle\,\langle\alpha_2,j_2|\,T^{(k)}\,|\alpha_1,j\rangle
\end{aligned}
$$

The only dependence of the matrix elements on m_1, m_2 and q is through the Clebsch-Gordon coefficient.