Symmetry

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1 Continuous symmetries in quantum mechanics

Transformations in quantum mechanics are accomplished by unitary transformations, because it is these that preserve the norms of states, hence, probability. Given a state

$$|\alpha\rangle$$

with norm

$$\langle\alpha\,|\alpha\rangle$$

a unitary transformation of $|\alpha\rangle$, $|\alpha'\rangle = \hat{\mathcal{U}}\,|\alpha\rangle$, has norm

$$\begin{aligned} \langle\alpha'\,|\alpha'\rangle &= \langle\alpha|\,\hat{\mathcal{U}}^{\dagger}\hat{\mathcal{U}}\,|\alpha\rangle \\ &= \langle\alpha\,|\alpha\rangle \end{aligned}$$

and probabilities are preserved.

The action of a transformation on the Hamiltonian is

$$\hat{\mathcal{U}}\hat{H}\hat{\mathcal{U}}^{\dagger}$$

If this transformation leaves the Hamiltonian invariant,

$$\hat{\mathcal{U}}\hat{H}\hat{\mathcal{U}}^{\dagger} = \hat{H}$$

then $\mathcal{U}$ gives a *symmetry* of the quantum system. There is a corresponding conserved observable because applying unitarity, $\hat{\mathcal{U}}^{\dagger} = \hat{\mathcal{U}}^{-1}$, to an infinitesimal transformation

$$\begin{aligned} \hat{\mathcal{U}} &= 1 - \frac{i\varepsilon}{\hbar}\hat{G} \\ \hat{\mathcal{U}}^{\dagger} &= 1 + \frac{i\varepsilon}{\hbar}\hat{G}^{\dagger} \\ \hat{\mathcal{U}}^{-1} &= 1 + \frac{i\varepsilon}{\hbar}\hat{G} \end{aligned}$$

implies that G is Hermitian. For a symmetry this leads to

$$\begin{aligned} \left(1 - \frac{i\varepsilon}{\hbar}\hat{G}\right)\hat{H}\left(1 + \frac{i\varepsilon}{\hbar}\hat{G}\right) &= \hat{H} \\ \left[\hat{H}, \hat{G}\right] &= 0 \end{aligned}$$

and therefore

$$\frac{d\hat{G}}{dt} = 0$$

We can use G to form simultaneous eigenkets,

$$\begin{aligned}\hat{H}\left|E,g\right\rangle &= E\left|E,g\right\rangle \\ \hat{G}\left|E,g\right\rangle &= g\left|E,g\right\rangle\end{aligned}$$

and since the time evolution operator is built from the Hamiltonian,

$$\left[\hat{\mathcal{U}}\left(g\right),\hat{\mathcal{U}}\left(t,t_0\right)\right] = 0$$

and the simultaneous eigenkets remain simulataneous eigenkets.

Now suppose that for some energy eigenket, $\left|E\right\rangle$, the transformation gives a distinct state,

$$\hat{\mathcal{U}}\left|E\right\rangle \neq \left|E\right\rangle$$

Then since $\hat{\mathcal{U}}$ commutes with $\hat{H}$, the energy is degenerate.

2 Discrete symmetry: Parity

2.1 Parity in classical physics

For discrete symmetries, we cannot expand infinitesimally, and a different approach is required. We still have a expect unitary symmetry (or in the case of time reversal, as we shall see, antiunitary), and the discrete transformation is still a symmetry if it leaves the Hamiltonian invariant,

$$\hat{\mathcal{U}}\hat{H}\hat{\mathcal{U}}^\dagger = \hat{H}$$

We first consider parity, or space inversion. Classically, parity is the reflection of position vectors through the origin,

$$\pi\mathbf{x} = -\mathbf{x}$$

Any vector which transforms in this way is said to be *odd* under parity. Since time is unchanged by the parity transformation, momentum is also of odd parity,

$$\begin{aligned}\pi\mathbf{p} &= \pi\left(m\frac{d\mathbf{x}}{dt}\right) \\ &= m\frac{d\left(\pi\mathbf{x}\right)}{dt} \\ &= -\mathbf{p}\end{aligned}$$

On the other hand, angular momentum is even,

$$\begin{aligned}\pi\mathbf{L} &= \pi\left(\mathbf{x}\times\mathbf{p}\right) \\ &= \left(-\mathbf{x}\right)\times\left(-\mathbf{p}\right) \\ &= \mathbf{L}\end{aligned}$$

We now need to represent these relations quantum mechanically.

2.2 Parity of quantum operators

Defining a parity operator, $\hat{\pi}$, we require the position operator to transform as

$$\hat{\pi}^\dagger\hat{\mathbf{x}}\hat{\pi} = -\hat{\mathbf{x}}$$

and since $\hat{\pi}$ is unitary, $\hat{\pi}^\dagger = \hat{\pi}^{-1}$ and we have

$$\hat{\mathbf{x}}\hat{\pi} + \hat{\pi}\hat{\mathbf{x}} \equiv \{\hat{\mathbf{x}}, \hat{\pi}\} = 0$$

so that the parity operator and the position operators *anticommute*.

Now, for any eigenket of $\hat{\mathbf{x}}$, we have

$$\begin{aligned}
\hat{\mathbf{x}}\hat{\pi}\left|\mathbf{x}\right\rangle &= -\hat{\pi}\hat{\mathbf{x}}\left|\mathbf{x}\right\rangle \\
&= -\hat{\pi}\mathbf{x}\left|\mathbf{x}\right\rangle \\
&= -\mathbf{x}\hat{\pi}\left|\mathbf{x}\right\rangle
\end{aligned}$$

Therefore, $\hat{\pi}\left|\mathbf{x}\right\rangle$ is also an eigenket of the position operator, with eigenvalue $-\mathbf{x}$

$$\hat{\mathbf{x}}\left(\hat{\pi}\left|\mathbf{x}\right\rangle\right) = -\mathbf{x}\left(\hat{\pi}\left|\mathbf{x}\right\rangle\right)$$

so we identify the transformed ket as

$$\hat{\pi}\left|\mathbf{x}\right\rangle = \left|-\mathbf{x}\right\rangle$$

Next, consider the action of the parity operator $\hat{\pi}$ on momentum. Begin with the translation operator,

$$\hat{\mathcal{T}}(\mathbf{a}) = \exp\left(-\frac{i}{\hbar}\mathbf{a}\cdot\hat{\mathbf{p}}\right)$$

which has the effect

$$\hat{\mathcal{T}}(\mathbf{a})\left|\mathbf{x}\right\rangle = \left|\mathbf{x}+\mathbf{a}\right\rangle$$

Transforming the translation operator with parity, $\hat{\mathcal{T}}(\mathbf{a}) \to \hat{\pi}^\dagger\hat{\mathcal{T}}(\mathbf{a})\hat{\pi}$, consider the action on a position eigenket,

$$\begin{aligned}
\hat{\pi}^\dagger\hat{\mathcal{T}}(\mathbf{a})\hat{\pi}\left|\mathbf{x}\right\rangle &= \hat{\pi}^\dagger\hat{\mathcal{T}}(\mathbf{a})\left|-\mathbf{x}\right\rangle \\
&= \hat{\pi}^\dagger\left|-\mathbf{x}+\mathbf{a}\right\rangle \\
&= \hat{\pi}^\dagger\hat{\pi}\left|\mathbf{x}-\mathbf{a}\right\rangle \\
&= \left|\mathbf{x}-\mathbf{a}\right\rangle
\end{aligned}$$

from which we see that

$$\hat{\pi}^\dagger\hat{\mathcal{T}}(\mathbf{a})\hat{\pi} = \hat{\mathcal{T}}(-\mathbf{a})$$

For an infinitesimal translation, $\hat{\mathcal{T}}(\boldsymbol{\varepsilon}) = \hat{1} - \frac{i}{\hbar}\boldsymbol{\varepsilon}\cdot\hat{\mathbf{p}}$, this becomes

$$\begin{aligned}
\hat{\pi}^\dagger\left(\hat{1} - \frac{i}{\hbar}\boldsymbol{\varepsilon}\cdot\hat{\mathbf{p}}\right)\hat{\pi} &= \hat{1} + \frac{i}{\hbar}\boldsymbol{\varepsilon}\cdot\hat{\mathbf{p}} \\
\hat{1} - \frac{i}{\hbar}\boldsymbol{\varepsilon}\cdot\hat{\pi}^\dagger\hat{\mathbf{p}}\hat{\pi} &= \hat{1} + \frac{i}{\hbar}\boldsymbol{\varepsilon}\cdot\hat{\mathbf{p}} \\
-\boldsymbol{\varepsilon}\cdot\hat{\pi}^\dagger\hat{\mathbf{p}}\hat{\pi} &= \boldsymbol{\varepsilon}\cdot\hat{\mathbf{p}}
\end{aligned}$$

and since $\boldsymbol{\varepsilon}$ is arbitrary, we see that the momentum operator is odd,

$$\hat{\pi}^\dagger\hat{\mathbf{p}}\hat{\pi} = -\hat{\mathbf{p}}$$

Now, writing the angular momentum in components, $\mathbf{L} = \hat{\mathbf{x}}\times\hat{\mathbf{p}}$,

$$\begin{aligned}
\hat{\pi}^\dagger\hat{L}_i\hat{\pi} &= \hat{\pi}^\dagger\left(\varepsilon_{ijk}\hat{x}_j\hat{p}_k\right)\hat{\pi} \\
&= \varepsilon_{ijk}\hat{\pi}^\dagger\hat{x}_j\left(\hat{\pi}\hat{\pi}^\dagger\right)\hat{p}_k\hat{\pi} \\
&= \varepsilon_{ijk}\left(\hat{\pi}^\dagger\hat{x}_j\hat{\pi}\right)\left(\hat{\pi}^\dagger\hat{p}_k\hat{\pi}\right) \\
&= \varepsilon_{ijk}\left(-\hat{x}_j\right)\left(-\hat{p}_k\right) \\
&= L_i
\end{aligned}$$

and we find the as in the classical case, angular momentum is even.

2.3 Parity of the wave function

Now consider a state, in the coordiante basis,

$$\psi(\mathbf{x}) = \langle \mathbf{x} | \psi \rangle$$

If we transform the state

$$\hat{\pi} |\psi\rangle$$

then the wave function becomes

$$P\psi(\mathbf{x}) = \langle \mathbf{x} | \hat{\pi} |\psi\rangle$$

However, since

$$\hat{\pi}^2 |\mathbf{x}\rangle = \hat{\pi} |-\mathbf{x}\rangle = |\mathbf{x}\rangle$$

the parity operator is its own inverse, $\hat{\pi}^2 = \hat{1}$, and by its unitarity, $\hat{\pi}^\dagger = \hat{\pi}$. Therefore,

$$\begin{aligned} \langle \mathbf{x} | \hat{\pi} &= \langle \mathbf{x} | \hat{\pi}^\dagger \\ &= \langle -\mathbf{x} | \end{aligned}$$

and the wave function becomes

$$\begin{aligned} P\psi(\mathbf{x}) &= \langle \mathbf{x} | \hat{\pi} |\psi\rangle \\ &= \langle -\mathbf{x} | \psi \rangle \\ &= \psi(-\mathbf{x}) \end{aligned}$$

If $\hat{\pi}$ commutes with the Hamiltonian then any solution of the stationary state Schrodinger equation

$$\hat{H} |E\rangle = E |E\rangle$$

may be made a simultaneous eigenket of parity, $|E, \pi\rangle$. Since, as noted above, $\hat{\pi}^2 = \hat{1}$, the eigenvalues, π, of parity

$$\hat{\pi} |E, \pi\rangle = \pi |E, \pi\rangle$$

must satisfy $\pi^2 = 1$, so that the possible eigenvalues are ± 1. Let $u_E(\mathbf{x})$ be a stationary state solution with energy E. Then the simultaneous eigenstates are

$$\begin{aligned} u_{E,+}(\mathbf{x}) &= u_E(\mathbf{x}) + u_E(-\mathbf{x}) \\ u_{E,-}(\mathbf{x}) &= u_E(\mathbf{x}) - u_E(-\mathbf{x}) \end{aligned}$$

as we check by applying $\hat{\pi}$,

$$\begin{aligned} \hat{\pi} u_{E,+}(\mathbf{x}) &= \hat{\pi} u_E(\mathbf{x}) + \hat{\pi} u_E(-\mathbf{x}) \\ &= u_E(-\mathbf{x}) + u_E(\mathbf{x}) \\ &= +u_{E,+}(\mathbf{x}) \\ \hat{\pi} u_{E,-}(\mathbf{x}) &= \hat{\pi} u_E(\mathbf{x}) - \hat{\pi} u_E(-\mathbf{x}) \\ &= u_E(-\mathbf{x}) - u_E(\mathbf{x}) \\ &= -u_{E,-}(\mathbf{x}) \end{aligned}$$

3 Time reversal

Our picture of symmetries as unitary transformations runs into a difficulty when we try to formulate time reversal invariance, $\Theta t = -t$.

3.1 Time reversal in classical physics

In classical physics, Newton's second law has this symmetry since it contains two time derivatives

$$\mathbf{F} = m\frac{d^2\mathbf{x}}{dt^2}$$

so for a time-independent force, $\Theta\mathbf{F} = \mathbf{F}$,

$$\begin{aligned}
\Theta\mathbf{F} &= \Theta m\frac{d^2\mathbf{x}}{dt^2} \\
\mathbf{F} &= m\left(-\frac{d}{dt}\right)\left(-\frac{d}{dt}\right)\mathbf{x} \\
&= m\frac{d^2\mathbf{x}}{dt^2}
\end{aligned}$$

and the equation of motion is invariant. For Maxwell's equations,

$$\begin{aligned}
\boldsymbol{\nabla}\cdot\mathbf{E} &= 4\pi\rho \\
\boldsymbol{\nabla}\cdot\mathbf{B} &= 0 \\
\boldsymbol{\nabla}\times\mathbf{B} - \frac{1}{c^2}\frac{\partial\mathbf{E}}{\partial t} &= \frac{4\pi}{c}\mathbf{J} \\
\boldsymbol{\nabla}\times\mathbf{E} + \frac{\partial\mathbf{B}}{\partial t} &= 0
\end{aligned}$$

time reversal changes the equations to

$$\begin{aligned}
\boldsymbol{\nabla}\cdot\Theta\mathbf{E} &= 4\pi\rho \\
\boldsymbol{\nabla}\cdot\Theta\mathbf{B} &= 0 \\
\boldsymbol{\nabla}\times\Theta\mathbf{B} + \frac{1}{c^2}\frac{\partial\Theta\mathbf{E}}{\partial t} &= -\frac{4\pi}{c}\mathbf{J} \\
\boldsymbol{\nabla}\times\Theta\mathbf{E} - \frac{\partial\Theta\mathbf{B}}{\partial t} &= 0
\end{aligned}$$

where we expect that time reversal changes the direction of the current

$$\Theta\mathbf{J} = -\mathbf{J}$$

Gauss's law shows that we need $\Theta\mathbf{E} = \mathbf{E}$, while Ampere's law applied to a current carrying wire show that $\Theta\mathbf{B} = -\mathbf{B}$, since time reversal will reverse the direction of flow of the current and therefore reverse the magnetic field.

3.2 The problem with time reversal operator

For a quantum system, we consider the time evolution of a state, $|\alpha\rangle$. We know that

$$|\alpha, t\rangle = \hat{\mathcal{U}}(t)\,|\alpha, 0\rangle$$

Consider the action on a time reversed state,

$$\hat{\Theta}\hat{\mathcal{U}}(t)\,\hat{\Theta}^{\dagger} = \hat{\mathcal{U}}(-t)$$

If this is true, and Θ is unitary, then for the state,

$$\begin{aligned}
\hat{\Theta}\hat{\mathcal{U}}(t)\,|\alpha, 0\rangle &= \hat{\Theta}\,|\alpha, t\rangle \\
&= |\alpha, -t\rangle \\
&= \hat{\mathcal{U}}(-t)\,|\alpha, 0\rangle \\
&= \hat{\mathcal{U}}(-t)\,|\alpha, 0\rangle
\end{aligned}$$

Setting $|\alpha, 0\rangle = \hat{\Theta}|\alpha, 0\rangle$ in the last line, we have the expected result,

$$\hat{\Theta}\hat{\mathcal{U}}(-t) = \hat{\mathcal{U}}(-t)\hat{\Theta}$$

There is a problem, however. If we expand the time translation operator infinitesimally,

$$\begin{aligned}\hat{\Theta}\hat{\mathcal{U}}(t) &= \hat{\mathcal{U}}(-t)\hat{\Theta} \\ \hat{\Theta}\left(\hat{1} - \frac{i}{\hbar}\hat{H}t\right) &= \left(\hat{1} + \frac{i}{\hbar}\hat{H}t\right)\hat{\Theta} \\ -\hat{\Theta}i\hat{H} &= i\hat{H}\hat{\Theta}\end{aligned}$$

So far, this is correct, but it seems to mean that

$$\hat{\Theta}\hat{H} = -\hat{H}\hat{\Theta}$$

so that for a system with time reversal symmetry, time reversal anticommutes with the Hamiltonian

$$\left\{\hat{\Theta}, \hat{H}\right\} = 0$$

This is the result we found for parity and momentum, but here it means that simultaneous eigenkets give negative energies, for if $|E\rangle$ is an energy eigenket with energy $E > 0$ then

$$\begin{aligned}\hat{H}\hat{\Theta}|E\rangle &= -\hat{\Theta}\hat{H}|E\rangle \\ &= -E\hat{\Theta}|E\rangle\end{aligned}$$

so that the time reversed state is also an energy eigenket, but with energy $-E$.

Negative energies are a problem because quantum systems enter all available states in proportion to their abundance (entropy increases!). Suppose the quantum harmonic oscillator had energy eigenstates, $|n\rangle$, for negative n as well as positive. Then every state $|n\rangle$ would have a probability of a transition to $|n-1\rangle + photon$, and the latter is more abundant because the phase space available to a photon is large, i.e., there are many, many states available to a given photon. This process would continue as the oscillator dropped to lower and lower energies, emitting more and more photons.

To avoid this problem we define *antiunitary* operators. When we introduce time reversal as an antiunitary operator, we avoid the negative energies and, ultimately, make a successful prediction of antiparticles.

3.3 Wigner's theorem

Suppose $\hat{U}$ is a map which preserves all transition probabilities. The probability that $|y\rangle$ is found in the state $|x\rangle$ is given by

$$P(x \to y) = |\langle x | y\rangle|^2$$

Then if U is a symmetry of the quantum system, the same probability must arise from the transformed state,

$$|\langle x | y\rangle|^2 = \left|\langle x| \hat{U}^\dagger\hat{U} |y\rangle\right|^2$$

or equivalently $|\langle x | y\rangle| = \left|\langle x| \hat{U}^\dagger\hat{U} |y\rangle\right|$. Though we have always inferred that $\hat{U}$ should be unitary from this, there are other possibilities. For any phase, $e^{i\varphi}$, we may have

$$\hat{U}^\dagger\hat{U} = e^{i\varphi}\hat{1}$$

where $\mathcal{U}$ is unitary. Suppose also that $\hat{U}$ is a discrete symmetry (such as parity or time reversal) so that applying it twice returns the system to the same state,

$$\hat{U}\hat{U} = \hat{1}$$

It follows that

$$\begin{aligned}\hat{1} &= \hat{U}^\dagger\hat{U}^\dagger\hat{U}\hat{U} \\ &= \hat{U}^\dagger e^{i\varphi}\hat{U}\end{aligned}$$

so that either

$$\begin{aligned}\hat{1} &= \hat{U}^\dagger e^{i\varphi}\hat{U} \\ &= \hat{U}^\dagger\hat{U}e^{i\varphi} \\ &= e^{2i\varphi}\end{aligned}$$

and therefore $\varphi = \pi$, so that

$$\hat{U}^\dagger\hat{U} = -\hat{1}$$

or, that we cannot pull out the phase without altering it, giving

$$e^{i\varphi}\hat{U} = \hat{U}e^{-i\varphi}$$

This is a special case of Wigner's theorem.

Let

$$\begin{aligned}|\tilde{\alpha}\rangle &= \hat{\Theta}\,|\alpha\rangle \\ \left|\tilde{\beta}\right\rangle &= \hat{\Theta}\,|\beta\rangle\end{aligned}$$

Then we define an *antiunitary operator* to be one which satisfies

$$\begin{aligned}\left\langle\tilde{\beta}\right.|\tilde{\alpha}\rangle &= \langle\beta\,|\alpha\rangle^* \\ \hat{\Theta}\left(c_1\,|\alpha\rangle + c_2\,|\beta\rangle\right) &= |\alpha\rangle\,|\alpha\rangle\end{aligned}$$

The use of antiunitary operators is unsatisfying in the bra-ket notation, which is not general enough to handle them elegantly. There are several things to note, which we quote without proof:

1. Every invertible operator which preserves transition probabilities, $|\langle x\,|y\rangle| = |\langle\tilde{x}\,|\tilde{y}\rangle|$ is either unitary or antiunitary (Wigner's Theorem)

2. Every antiunitary operator may be decomposed into the product, $\hat{\mathcal{U}}\hat{K}$, of a unitary operator, $\hat{\mathcal{U}}$, times the complex conjugation operator, $\hat{K}$, where $\hat{K}c = c^*\hat{K}$ for any complex number c.

3. We only define the action of $\hat{K}$ on kets, not bras, $|\tilde{\alpha}\rangle = \hat{K}\,|\alpha\rangle$.

4. The charge conjugation operator does *not* change base kets. Therefore, if we expand $|\alpha\rangle = \sum_a |a\rangle\,\langle a\,|\tilde{\alpha}\rangle = \sum_a \langle a\,|\tilde{\alpha}\rangle\,|a\rangle$, we have

$$\begin{aligned}\hat{K}\,|\alpha\rangle &= \hat{K}\sum_a \langle a\,|\tilde{\alpha}\rangle\,|a\rangle \\ &= \sum_a \langle a\,|\tilde{\alpha}\rangle^*\,\hat{K}\,|a\rangle \\ &= \sum_a \langle a\,|\tilde{\alpha}\rangle^*\,|a\rangle\end{aligned}$$

   This makes the definition of $\hat{K}$ dependent on the basis, since another basis may be defined by a complex linear combination, $|b\rangle = \sum_a c_{ba}\,|a\rangle$. Complex conjugation ,$\hat{K}$, cannot leave both the $|a\rangle$ and $|b\rangle$ basis kets invariant at the same time.

3.4 Time reversal

Now, revisit the infinitesimal time translation operator. We had reached the conclusion that

$$-\hat{\Theta} i \hat{H} = i \hat{H} \hat{\Theta}$$

so that, continuing with $\hat{\Theta}$ antiunitary, we have

$$i \hat{\Theta} \hat{H} = i \hat{H} \hat{\Theta}$$

and the Hamiltonian commutes with time reversal.

With a bit of work, we may classify states by their behavior under time reversal. Let $\hat{A}$ be a linear operator, and

$$\begin{aligned} |\tilde{\alpha}\rangle &= \hat{\Theta} |\alpha\rangle \\ \left|\tilde{\beta}\right\rangle &= \hat{\Theta} |\beta\rangle \end{aligned}$$

Define an intermediate state,

$$|\gamma\rangle = \hat{A}^{\dagger} |\beta\rangle$$

and its dual,

$$\langle\gamma| = \langle\beta| \hat{A}$$

Then

$$\langle\beta| \hat{A} |\alpha\rangle = \langle\gamma |\alpha\rangle$$

and since

$$\langle\gamma |\alpha\rangle = \langle\alpha |\gamma\rangle^{*} = \langle\tilde{\alpha} |\tilde{\gamma}\rangle$$

for an antiunitary operator, we have

$$\begin{aligned} \langle\beta| \hat{A} |\alpha\rangle &= \langle\gamma |\alpha\rangle \\ &= \langle\alpha |\gamma\rangle^{*} \\ &= \langle\tilde{\alpha} |\tilde{\gamma}\rangle \\ &= \langle\tilde{\alpha}| \hat{\Theta} |\gamma\rangle \\ &= \langle\tilde{\alpha}| \hat{\Theta} \hat{A}^{\dagger} |\beta\rangle \\ &= \langle\tilde{\alpha}| \hat{\Theta} \hat{A}^{\dagger} \hat{\Theta}^{-1} \hat{\Theta} |\beta\rangle \\ &= \langle\tilde{\alpha}| \hat{\Theta} \hat{A}^{\dagger} \hat{\Theta}^{-1} \left|\tilde{\beta}\right\rangle \end{aligned}$$

and we have shown the effect of a similarity transformation on an operator, $\hat{A}^{\dagger}$. If $\hat{A}$ is Hermitian, then

$$\langle\beta| \hat{A} |\alpha\rangle = \langle\tilde{\alpha}| \hat{\Theta} \hat{A} \hat{\Theta}^{-1} \left|\tilde{\beta}\right\rangle$$

We define an operator as even or odd under time reversal if

$$\hat{\Theta} \hat{A} \hat{\Theta}^{-1} = \pm \hat{A}$$

Under time reversal, we require the momentum operator to be odd just as for the classical variable:

$$\langle\alpha| \hat{\mathbf{p}} |\alpha\rangle = -\langle\tilde{\alpha}| \hat{\mathbf{p}} |\tilde{\alpha}\rangle$$

It follows from the two relations

$$\begin{aligned} \langle\alpha| \hat{\mathbf{p}} |\alpha\rangle &= \langle\tilde{\alpha}| \hat{\Theta} \hat{\mathbf{p}} \hat{\Theta}^{-1} |\tilde{\alpha}\rangle \\ \langle\alpha| \hat{\mathbf{p}} |\alpha\rangle &= -\langle\tilde{\alpha}| \hat{\mathbf{p}} |\tilde{\alpha}\rangle \end{aligned}$$

that

$$\hat{\Theta}\hat{\mathbf{p}}\hat{\Theta}^{-1} = -\hat{\mathbf{p}}$$

We require expectation values of the position operator to be even,

$$\langle\alpha|\,\hat{\mathbf{x}}\,|\alpha\rangle = -\,\langle\tilde{\alpha}|\,\hat{\mathbf{x}}\,|\tilde{\alpha}\rangle$$

and therefore,

$$\hat{\Theta}\hat{\mathbf{x}}\hat{\Theta}^{-1} = \hat{\mathbf{x}}$$

From these we see that the commutator of $\hat{\mathbf{x}}$ with $\hat{\mathbf{p}}$ satisfies

$$\begin{aligned}
\hat{\Theta}\left[\hat{x}_i, \hat{p}_j\right]\hat{\Theta}^{-1} &= \hat{\Theta}\left(\hat{x}_i\hat{p}_j - \hat{p}_j\hat{x}_i\right)\hat{\Theta}^{-1} \\
&= \hat{\Theta}\hat{x}_i\hat{\Theta}^{-1}\hat{\Theta}\hat{p}_j\hat{\Theta}^{-1} - \hat{\Theta}\hat{p}_j\hat{\Theta}^{-1}\hat{\Theta}\hat{x}_i\hat{\Theta}^{-1} \\
&= -\hat{x}_i\hat{p}_j + \hat{p}_j\hat{x}_i
\end{aligned}$$

which agrees with the right hand side

$$\begin{aligned}
\hat{\Theta}\left[\hat{x}_i, \hat{p}_j\right]\hat{\Theta}^{-1} &= \hat{\Theta}i\hbar\delta_{ij}\hat{\Theta}^{-1} \\
&= -i\hbar\delta_{ij}\hat{\Theta}\hat{\Theta}^{-1} \\
&= -i\hbar\delta_{ij}
\end{aligned}$$

so the commutator is preserved.

By considering the fundamental commutator for angular momentum, we see that

$$\hat{\Theta}\left[\hat{J}_i, \hat{J}_j\right]\hat{\Theta}^{-1} = \hat{\Theta}i\hbar\varepsilon_{ijk}\hat{J}_k\hat{\Theta}^{-1}$$

is only consistent if $\hat{\Theta}\hat{\mathbf{J}}\hat{\Theta}^{-1} = -\hat{\mathbf{J}}$, so $\hat{\mathbf{J}}$ is odd under time reversal.