Spin States

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1 Successive measurements of spin components

We have seen that the three spin operators may be written in terms of the Pauli matrices,

$$\hat{S}_i = \frac{\hbar}{2}\hat{\sigma}_i$$

or, equivalently, in bra-ket notation as

$$\begin{aligned}
\hat{S}_x &= \frac{\hbar}{2}\left(|+\rangle\langle-| + |-\rangle\langle+|\right) \\
\hat{S}_y &= \frac{i\hbar}{2}\left(|-\rangle\langle+| - |+\rangle\langle-|\right) \\
\hat{S}_z &= \frac{\hbar}{2}\left(|+\rangle\langle+| - |-\rangle\langle-|\right)
\end{aligned}$$

Consider the evolution of a beam of electrons, prepared in the normalized state,

$$|A\rangle = \alpha|+\rangle + \beta|-\rangle$$

where $|\alpha|^2 + |\beta|^2 = 1$, as we measure successive components of spin. If we make a measurement of the z-component of spin,

$$\begin{aligned}
\hat{S}_z|A\rangle &= \left(\frac{\hbar}{2}\left(|+\rangle\langle+| - |-\rangle\langle-|\right)\right)\left(\alpha|+\rangle + \beta|-\rangle\right) \\
&= \frac{\alpha\hbar}{2}|+\rangle - \frac{\beta\hbar}{2}|-\rangle
\end{aligned}$$

then the probablility that we measure the value $+\frac{\hbar}{2}$, and therefore find the state to be $|+\rangle$ is

$$\begin{aligned}
|\langle+|A\rangle|^2 &= |\langle+|\left(\alpha|+\rangle - \beta|-\rangle\right)|^2 \\
&= |\alpha|^2
\end{aligned}$$

while the probability of measuring $-\frac{\hbar}{2}$ is

$$|\langle-|A\rangle|^2 = |\beta|^2$$

The expectation value of the spin, essentially the average of many measurements, is

$$\begin{aligned}
\langle A|\hat{S}_z|A\rangle &= \left(\langle+|\alpha^* - \langle-|\beta^*\right)\left(\frac{\alpha\hbar}{2}|+\rangle - \frac{\beta\hbar}{2}|-\rangle\right) \\
&= \frac{\hbar}{2}\alpha^*\alpha - \frac{\hbar}{2}\beta^*\beta
\end{aligned}$$

i.e., $+\frac{\hbar}{2}$ times the probability of measuring spin up, plus $-\frac{\hbar}{2}$ times the probability of measuring spin down.

Suppose we measure a given electron to have z-component of spin $+\frac{\hbar}{2}$. Then the subsequent state must reflect this, and is therefore

$$|A'\rangle = \alpha\,|+\rangle$$

for the resulting spin-up beam. This state no longer has the same normalization, because we have eliminated the spin-down portion of the beam of electrons. If we make another measurement of the z-component of this state, the result is

$$\begin{aligned} \hat{S}_z\,|A'\rangle &= \left(\frac{\hbar}{2}\left(|+\rangle\langle+| \;-\; |-\rangle\langle-|\right)\right)\alpha\,|+\rangle \\ &= \frac{\hbar}{2}\alpha\,|+\rangle \end{aligned}$$

This corresponds to measuring the value $+\frac{\hbar}{2}$ every time, and we see that $|A'\rangle$ is already an eigenstate of $\hat{S}_z$. The state is unchanged by the subsequent measurement of the same observable.

On the other hand, suppose we measure the x-component of spin for $|A'\rangle$,

$$\begin{aligned} \hat{S}_x\,|A'\rangle &= \frac{\hbar}{2}\left(|+\rangle\langle-| \;+\; |-\rangle\langle+|\right)\alpha\,|+\rangle \\ &= \frac{\alpha\hbar}{2}\,|-\rangle \end{aligned}$$

The state is altered by the measurement, so it is not an eigenstate of $\hat{S}_x$. To find the probabilities for measuring the x-component up or down, we need to write $|A'\rangle$ in terms of the eigenstates of $\hat{S}_x$. These are not hard to find. They satisfy

$$\hat{S}_x\left|\hat{S}_x,\lambda\right\rangle \;=\; \lambda\left|\hat{S}_x,\lambda\right\rangle$$

Expanding in terms of the z-basis,

$$\left|\hat{S}_x,\lambda\right\rangle \;=\; a\,|+\rangle \;-\; b\,|-\rangle$$

the eigenvector equation becomes

$$\begin{aligned} \frac{\hbar}{2}\left(|+\rangle\langle-| \;+\; |-\rangle\langle+|\right)\left(a\,|+\rangle \;-\; b\,|-\rangle\right) &= \lambda\left(a\,|+\rangle \;-\; b\,|-\rangle\right) \\ -\frac{\hbar}{2}b\,|+\rangle \;+\; \frac{\hbar}{2}a\,|-\rangle &= \lambda\left(a\,|+\rangle \;-\; b\,|-\rangle\right) \end{aligned}$$

so that, equating like components,

$$\begin{aligned} -\frac{\hbar}{2}b\,|+\rangle &= a\lambda\,|+\rangle \\ \frac{\hbar}{2}a\,|-\rangle &= -\lambda b\,|-\rangle \end{aligned}$$

Solving the second, we have

$$b = -\frac{\hbar}{2\lambda}a$$

and substituting this into the first gives

$$\begin{aligned} \frac{\hbar}{2}\frac{\hbar}{2\lambda}a &= a\lambda \\ \left(\frac{\hbar}{2}\right)^2 &= \lambda^2 \\ \lambda &= \pm\frac{\hbar}{2} \end{aligned}$$

With these values for λ, we find two states, having $b = \mp a$. Normalizing by requiring $a^2 + b^2 = 1$, the eigenstates are

$$\begin{aligned} \left|\hat{S}_x, \frac{\hbar}{2}\right\rangle &= \frac{1}{\sqrt{2}}\left(|+\rangle \ - \ |-\rangle\right) \\ \left|\hat{S}_x, -\frac{\hbar}{2}\right\rangle &= \frac{1}{\sqrt{2}}\left(|+\rangle \ + \ |-\rangle\right) \end{aligned}$$

Now return to the state $|A'\rangle = \alpha\,|+\rangle$. We may write this in terms of the eigenstates of $\hat{S}_x$, as

$$\begin{aligned} |A'\rangle &= \alpha\,|+\rangle \\ &= \frac{\alpha}{\sqrt{2}}\left(\left|\hat{S}_x, +\frac{\hbar}{2}\right\rangle \ + \ \left|\hat{S}_x, -\frac{\hbar}{2}\right\rangle\right) \end{aligned}$$

and now we see that the probability of measuring the x-component of spin to be $+\frac{\hbar}{2}$ is

$$\begin{aligned} \left|\left\langle \hat{S}_x, +\frac{\hbar}{2} \mid A'\right\rangle\right|^2 &= \left|\left\langle \hat{S}_x, +\frac{\hbar}{2}\right| \alpha\,|+\rangle\right|^2 \\ &= \left|\frac{\alpha}{\sqrt{2}}\right|^2 \\ &= \frac{1}{2}\,|\alpha|^2 \end{aligned}$$

The probability of measuring spin down is also $\frac{1}{2}\,|\alpha|^2$, where the factor $|\alpha|^2$ reflects the dimunition of the beam by the original z measurement. Therefore, a state which has its z-component of spin in the spin up state has equal probability of finding the x-component of spin in either the spin up or spin down state.

2 Generic spin state

It is useful to have an expression for the eigenstates of an arbitrary direction, $\mathbf{n}$, of the spin operators, which we may write as the linear combination

$$\begin{aligned} \mathbf{n} \cdot \hat{\boldsymbol{S}} &= n_x \hat{S}_x + n_y \hat{S}_y + n_z \hat{S}_z \\ &= \frac{\hbar}{2}\left(n_x \sigma_x + n_y \sigma_y + n_z \sigma_z\right) \\ &= \frac{\hbar}{2}\begin{pmatrix} n_z & n_x - i n_y \\ n_x + i n_y & -n_z \end{pmatrix} \end{aligned}$$

If we write the unit vector $\mathbf{n}$ in spherical coordinates,

$$\begin{aligned} n_x &= \sin\theta\cos\varphi \\ n_y &= \sin\theta\sin\varphi \\ n_y &= \cos\theta \end{aligned}$$

then

$$\mathbf{n} \cdot \hat{\boldsymbol{S}} = \frac{\hbar}{2}\begin{pmatrix} \cos\theta & e^{-i\varphi}\sin\theta \\ e^{i\varphi}\sin\theta & -\cos\theta \end{pmatrix}$$

The eigenvalues satisfy

$$\det\left(\mathbf{n} \cdot \hat{\boldsymbol{S}} - \lambda \hat{\mathbf{1}}\right) = 0$$

$$
\begin{aligned}
\det\begin{pmatrix} \frac{\hbar}{2}\cos\theta-\lambda & \frac{\hbar}{2}e^{-i\varphi}\sin\theta \\ \frac{\hbar}{2}e^{i\varphi}\sin\theta & -\frac{\hbar}{2}\cos\theta-\lambda \end{pmatrix} &= 0 \\
\lambda^2-\frac{\hbar^2}{4}\cos^2\theta-\frac{\hbar^2}{4}\sin^2\theta &= 0 \\
\lambda &= \pm\frac{\hbar}{2}
\end{aligned}
$$

as expected. Then for $\lambda = +\frac{\hbar}{2}$, the eigenvalue equation is

$$
\frac{\hbar}{2}\begin{pmatrix} \cos\theta & e^{-i\varphi}\sin\theta \\ e^{i\varphi}\sin\theta & -\cos\theta \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \frac{\hbar}{2}\begin{pmatrix} \alpha \\ \beta \end{pmatrix}
$$

so that

$$
\begin{aligned}
\alpha\cos\theta+\beta e^{-i\varphi}\sin\theta &= \alpha \\
\alpha e^{i\varphi}\sin\theta-\beta\cos\theta &= \beta
\end{aligned}
$$

and we solve for β,

$$
\beta = \frac{\alpha e^{i\varphi}\sin\theta}{1+\cos\theta}
$$

Alternatively, as a check, the first equation gives

$$
\begin{aligned}
\beta e^{-i\varphi}\sin\theta &= \alpha\left(1-\cos\theta\right) \\
\beta &= \frac{\alpha\left(1-\cos\theta\right)}{e^{-i\varphi}\sin\theta} \\
&= \frac{\alpha e^{i\varphi}\left(1-\cos\theta\right)}{\sin\theta}
\end{aligned}
$$

These are equal provided

$$
\begin{aligned}
\frac{1-\cos\theta}{\sin\theta} &= \frac{\sin\theta}{1+\cos\theta} \\
1-\cos^2\theta &= \sin^2\theta
\end{aligned}
$$

which clearly holds.

We now use α to normalize the state. It helps to write the trig functions using half angle formulas,

$$
\begin{aligned}
1+\cos\theta &= 2\cos^2\frac{\theta}{2} \\
\sin\theta &= 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}
\end{aligned}
$$

so that

$$
\begin{aligned}
\beta &= \alpha\left(\frac{e^{i\varphi}\sin\theta}{1+\cos\theta}\right) \\
&= \alpha\left(\frac{e^{i\varphi}2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{2\cos^2\frac{\theta}{2}}\right) \\
&= \alpha\left(\frac{e^{i\varphi}\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}}\right)
\end{aligned}
$$

We have

$$\begin{aligned} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} &= \begin{pmatrix} \alpha \\ \frac{\alpha e^{i\varphi}\sin\theta}{1+\cos\theta} \end{pmatrix} \\ &= \alpha \begin{pmatrix} 1 \\ \frac{e^{i\varphi}\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}} \end{pmatrix} \end{aligned}$$

The norm is

$$\begin{aligned} 1 &= \alpha\alpha^* + \beta\beta^* \\ &= \alpha\alpha^* \left(1 + \frac{e^{i\varphi}\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}} \frac{e^{-i\varphi}\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}}\right) \\ &= \alpha\alpha^* \left(1 + \frac{\sin^2\frac{\theta}{2}}{\cos^2\frac{\theta}{2}}\right) \\ &= \frac{\alpha\alpha^*}{\cos^2\frac{\theta}{2}} \left(\cos^2\frac{\theta}{2} + \sin^2\frac{\theta}{2}\right) \\ \cos^2\frac{\theta}{2} &= \alpha\alpha^* \end{aligned}$$

so choosing α real,

$$\alpha = \cos\frac{\theta}{2}$$

Then we have the normalized eigenstate,

$$\begin{pmatrix} \alpha \\ \beta \end{pmatrix}_+ = \begin{pmatrix} \cos\frac{\theta}{2} \\ e^{i\varphi}\sin\frac{\theta}{2} \end{pmatrix}$$

For the negative eigenvalue, the equation becomes

$$\begin{aligned} \frac{\hbar}{2} \begin{pmatrix} \cos\theta & e^{-i\varphi}\sin\theta \\ e^{i\varphi}\sin\theta & -\cos\theta \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} &= -\frac{\hbar}{2} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \\ \begin{pmatrix} \alpha\cos\theta + \beta e^{-i\varphi}\sin\theta \\ \alpha e^{i\varphi}\sin\theta - \beta\cos\theta \end{pmatrix} &= -\begin{pmatrix} \alpha \\ \beta \end{pmatrix} \end{aligned}$$

so that

$$\begin{aligned} \alpha\cos\theta + \beta e^{-i\varphi}\sin\theta &= -\alpha \\ \alpha e^{i\varphi}\sin\theta - \beta\cos\theta &= -\beta \end{aligned}$$

and therefore

$$\beta = -\alpha e^{i\varphi}\frac{\sin\theta}{1-\cos\theta}$$

The state is then

$$\begin{aligned} \begin{pmatrix} \alpha \\ \beta \end{pmatrix}_- &= \alpha \begin{pmatrix} 1 \\ -e^{i\varphi}\frac{\sin\theta}{1-\cos\theta} \end{pmatrix} \\ &= \alpha \begin{pmatrix} 1 \\ -e^{i\varphi}\frac{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{2\sin^2\frac{\theta}{2}} \end{pmatrix} \\ &= \begin{pmatrix} \sin\frac{\theta}{2} \\ -e^{i\varphi}\cos\frac{\theta}{2} \end{pmatrix} \end{aligned}$$

Alternatively, notice that the eigenstate corresponding to the negative eigenvalue, $-\frac{\hbar}{2}$, must be orthogonal to the $-\frac{\hbar}{2}$ state, so if we let

$$\begin{aligned}\begin{pmatrix}\alpha_- \\ \beta_-\end{pmatrix} &= \begin{pmatrix}\sin\frac{\theta}{2} \\ -e^{i\varphi}\cos\frac{\theta}{2}\end{pmatrix} \\ \begin{pmatrix}\alpha_+ \\ \beta_+\end{pmatrix} &= \begin{pmatrix}\cos\frac{\theta}{2} \\ e^{i\varphi}\sin\frac{\theta}{2}\end{pmatrix}\end{aligned}$$

the inner product is

$$\begin{aligned}\alpha_+\alpha_-^* + \beta_+\beta_-^* &= \sin\frac{\theta}{2}\cos\frac{\theta}{2} - e^{i\varphi}e^{-i\varphi}\sin\frac{\theta}{2}\cos\frac{\theta}{2} \\ &= 0\end{aligned}$$

As a check, write $\mathbf{n}\cdot\hat{\mathbf{S}}$ using half angles,

$$\frac{\hbar}{2}\begin{pmatrix}\cos\theta & e^{-i\varphi}\sin\theta \\ e^{i\varphi}\sin\theta & -\cos\theta\end{pmatrix} = \frac{\hbar}{2}\begin{pmatrix}2\cos^2\frac{\theta}{2}-1 & e^{-i\varphi}2\sin\frac{\theta}{2}\cos\frac{\theta}{2} \\ e^{i\varphi}2\sin\frac{\theta}{2}\cos\frac{\theta}{2} & -2\cos^2\frac{\theta}{2}+1\end{pmatrix}$$

Then,

$$\begin{aligned}\frac{\hbar}{2}\begin{pmatrix}\cos\theta & e^{-i\varphi}\sin\theta \\ e^{i\varphi}\sin\theta & -\cos\theta\end{pmatrix}\begin{pmatrix}\alpha \\ \beta\end{pmatrix}_- &= \frac{\hbar}{2}\begin{pmatrix}2\cos^2\frac{\theta}{2}-1 & e^{-i\varphi}2\sin\frac{\theta}{2}\cos\frac{\theta}{2} \\ e^{i\varphi}2\sin\frac{\theta}{2}\cos\frac{\theta}{2} & -2\cos^2\frac{\theta}{2}+1\end{pmatrix}\begin{pmatrix}\sin\frac{\theta}{2} \\ -e^{i\varphi}\cos\frac{\theta}{2}\end{pmatrix} \\ &= \frac{\hbar}{2}\begin{pmatrix}\left(2\cos^2\frac{\theta}{2}-1\right)\sin\frac{\theta}{2} - \left(e^{-i\varphi}2\sin\frac{\theta}{2}\cos\frac{\theta}{2}\right)e^{i\varphi}\cos\frac{\theta}{2} \\ e^{i\varphi}2\sin^2\frac{\theta}{2}\cos\frac{\theta}{2} - e^{i\varphi}\cos\frac{\theta}{2}\left(-2\cos^2\frac{\theta}{2}+1\right)\end{pmatrix} \\ &= \frac{\hbar}{2}\begin{pmatrix}2\cos^2\frac{\theta}{2}\sin\frac{\theta}{2} - \sin\frac{\theta}{2} - 2\sin\frac{\theta}{2}\cos^2\frac{\theta}{2} \\ 2e^{i\varphi}\sin^2\frac{\theta}{2}\cos\frac{\theta}{2} + 2e^{i\varphi}\cos^3\frac{\theta}{2} - e^{i\varphi}\cos\frac{\theta}{2}\end{pmatrix} \\ &= \frac{\hbar}{2}\begin{pmatrix}-\sin\frac{\theta}{2} \\ 2e^{i\varphi}\cos\frac{\theta}{2} - e^{i\varphi}\cos\frac{\theta}{2}\end{pmatrix} \\ &= -\frac{\hbar}{2}\begin{pmatrix}\sin\frac{\theta}{2} \\ -e^{i\varphi}\cos\frac{\theta}{2}\end{pmatrix}\end{aligned}$$

The spin up and spin down eigenkets of $\mathbf{n}\cdot\hat{\mathbf{S}}$ are therefore,

$$\begin{aligned}\left|\mathbf{n}\cdot\hat{\mathbf{S}},+\right\rangle &= \cos\frac{\theta}{2}\left|+\right\rangle + e^{i\varphi}\sin\frac{\theta}{2}\left|-\right\rangle \\ \left|\mathbf{n}\cdot\hat{\mathbf{S}},-\right\rangle &= \sin\frac{\theta}{2}\left|+\right\rangle - e^{i\varphi}\cos\frac{\theta}{2}\left|-\right\rangle\end{aligned}$$

where $|+\rangle$ and $|-\rangle$ are the eigenkets in the z-direction.