Problems in Wave Mechanics

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1. We found that for a Gaussian distribution for k, the normalized, square integrable amplitude is

$$A(k) = \frac{1}{(2\pi\sigma^2)^{1/4}} e^{-\frac{(k-k_0)^2}{4\sigma^2}}$$

Let $p = \hbar k$ so that this becomes a distribution in momentum, with the density

$$|A(p)|^2$$

a normalized Gaussian momentum distribution. Let Δp be the standard deviation of this distribution. Similarly, the wave function turns out to be Gaussian as well,

$$\psi(x) = \left(\frac{2\sigma^2}{\pi}\right)^{1/4} e^{-\sigma^2 x^2} e^{-ik_0 x}$$

with

$$|\psi|^2 = \left(\frac{2\sigma^2}{\pi}\right)^{1/2} e^{-2\sigma^2 x^2}$$

Let Δx be the standard deviation of this Gaussian and compute $\Delta x \Delta p$.

2. The wave function

$$\psi(x) = \left(\frac{2\sigma^2}{\pi}\right)^{1/4} e^{-\sigma^2 x^2} e^{-ik_0 x}$$

may seem like a plane wave $e^{-ik_0 x}$, but it is not – it is a superposition of many plane waves,

$$\psi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left(\frac{1}{(2\pi\sigma^2)^{1/4}} e^{-\frac{(k-k_0)^2}{4\sigma^2}}\right) e^{ikx} dk$$

where the integral extends over all k. Since the time dependence of each plane wave $\frac{1}{\sqrt{2\pi}} e^{ikx}$ is given by

$$\frac{1}{\sqrt{2\pi}} \exp\left(ikx - \frac{i}{\hbar} Et\right) = \frac{1}{\sqrt{2\pi}} \exp\left(ikx - \frac{i}{\hbar}\frac{\hbar^2 k^2}{2m} t\right)$$

the full time-dependent wave function is given by

$$\Psi(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left(\frac{1}{(2\pi\sigma^2)^{1/4}} e^{-\frac{(k-k_0)^2}{4\sigma^2}}\right) e^{ikx - \frac{i}{\hbar}\frac{\hbar^2 k^2}{2m} t} dk$$

(a) Integrate to find the explicit form of $\Psi(x,t)$.

(b) Compute the probability density, $\Psi^*\Psi$ to show that it is a Gaussian centered on the moving point, $x - \frac{\hbar k_0}{m}t$. Comment on the qualitative features of the time evolution of the particle position.

3. Solve the stationary state Schrödinger equation for an attractive δ-function potential,

$$V = -V_0 L \delta(x)$$

for states with energy $-V_0 < E < 0$. You will need to rederive the boundary conditions at $x = 0$, following the general considerations in the Notes.