Problems in perturbation theory

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1 Zeeman Effect

Consider hydrogen atoms held in a uniform magnetic field. The field couples to the magnetic moment of the orbiting electron, both through its intrinsic magnetic moment and its orbital magnetic moment.

The intrinsic magnetic moment of the electron is

$$\boldsymbol{\mu} = \frac{e}{mc}\mathbf{S}$$

so in a constant magnetic field $\mathbf{B}$ the energy is perturbed by

$$-\boldsymbol{\mu}\cdot\mathbf{B} = -\frac{e}{mc}\mathbf{S}\cdot\mathbf{B}$$

For the orbital angular momentum, we may use minimal coupling, replacing the momentum operator $\hat{\mathbf{p}}$ by

$$\hat{\mathbf{p}} - \frac{e}{c}\mathbf{A}$$

where $\mathbf{A}$ is the magnetic vector potential. Choosing the gauge so that $\boldsymbol{\nabla}\cdot\mathbf{A} = 0$, the kinetic term of the Hamiltonian becomes

$$
\begin{aligned}
\frac{\left(\hat{\mathbf{p}} - \frac{e}{c}\mathbf{A}\right)^2}{2m} &= \frac{1}{2m}\left(\hat{\mathbf{p}}^2 - \frac{e}{c}\left(\hat{\mathbf{p}}\cdot\mathbf{A} + \mathbf{A}\cdot\hat{\mathbf{p}}\right) + \frac{e^2}{c^2}\mathbf{A}\cdot\mathbf{A}\right) \\
&\approx \frac{1}{2m}\left(\hat{\mathbf{p}}^2 - \frac{e}{c}\left(\hat{\mathbf{p}}\cdot\mathbf{A} + \mathbf{A}\cdot\hat{\mathbf{p}}\right)\right)
\end{aligned}
$$

In a coordinate basis, the perturbing term acting on the state is therefore

$$
\begin{aligned}
-\frac{e}{2mc}\langle\mathbf{x}|\left(\hat{\mathbf{p}}\cdot\mathbf{A} + \mathbf{A}\cdot\hat{\mathbf{p}}\right)|\psi\rangle &= \frac{i\hbar e}{2mc}\left(\boldsymbol{\nabla}\cdot\left(\mathbf{A}\psi\left(\mathbf{x}\right)\right) + \mathbf{A}\cdot\boldsymbol{\nabla}\psi\left(\mathbf{x}\right)\right) \\
&= \frac{i\hbar e}{2mc}\left(\left(\boldsymbol{\nabla}\cdot\mathbf{A}\right)\psi + 2\mathbf{A}\cdot\boldsymbol{\nabla}\psi\right)
\end{aligned}
$$

We set $\boldsymbol{\nabla}\cdot\mathbf{A} = 0$ and look at the second term. For a magnetic field in the z-direction, let

$$\mathbf{A} = -\frac{1}{2}B\left(y\hat{\mathbf{i}} - x\hat{\mathbf{j}}\right)$$

since

$$
\begin{aligned}
\mathbf{B} &= \boldsymbol{\nabla}\times\mathbf{A} \\
&= \hat{\mathbf{k}}\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right) \\
&= B\hat{\mathbf{k}}
\end{aligned}
$$

Then the perturbing term becomes

$$
\begin{aligned}
\frac{i\hbar e}{2mc}2\mathbf{A}\cdot\boldsymbol{\nabla}\psi &= -\frac{i\hbar e B}{2mc}\left(y\hat{\mathbf{i}}-x\hat{\mathbf{j}}\right)\cdot\boldsymbol{\nabla}\psi \\
&= -\frac{i\hbar e B}{2mc}\left(y\frac{\partial}{\partial x}-x\frac{\partial}{\partial y}\right)\psi \\
&= -\frac{eB}{2mc}\hat{\mathbf{k}}\left(yi\hbar\frac{\partial}{\partial x}-xi\hbar\frac{\partial}{\partial y}\right)\psi \\
&= -\frac{eB}{2mc}\hat{\mathbf{k}}\left(-y\hat{p}_x+x\hat{p}_y\right)\psi \\
&= -\frac{e}{2mc}\mathbf{B}\cdot\mathbf{L}\psi
\end{aligned}
$$

Combining the two effects, we have the perturbation due to the Zeeman effect,

$$
H_Z = -\frac{e}{2mc}\mathbf{B}\cdot(\mathbf{L}+2\mathbf{S})
$$

Using this, for B in the z-direction, compute the change in the energy levels of hydrogen. Neglect fine structure.

2   Stark effect

Consider hydrogen in a uniform electric field. The interaction energy is simply

$$
-\mathbf{E}\cdot\mathbf{d}
$$

where $\mathbf{d}$ is the dipole moment of the electron,

$$
\mathbf{d} = -e\mathbf{r}
$$

Therefore, taking the electric field in the z-direction,

$$
V = eEr\cos\theta
$$

Since $\cos\theta \sim Y_{10}(\theta,\varphi)$, the Stark effect connects states differing by one unit of angular momentum. The lowest order effect is between the $|nlm\rangle = |000\rangle$ state and the $|nlm\rangle = |110\rangle$ state. Since these have different energies, we may use non-degenerate perturbation theory. For $|nlm\rangle = |2lm\rangle$ states, however, there is degeneracy (See Notes on degenerate perturbation theory).

Neglecting spin and fine structure, compute the Stark effect for $n = 0, 1, 2$ and find the first order correction to the state.