Orbital angular momentum and the spherical harmonics

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1 Orbital angular momentum

We compare our result on representations of rotations with our previous experience of angular momentum, defined for a point particle as

$$\mathbf{L} = \mathbf{x} \times \mathbf{p}$$

or, for a quantum system as the operator relationship

$$\hat{\mathbf{L}} = \hat{\mathbf{x}} \times \hat{\mathbf{p}}$$

Notice that since

$$\hat{L}_i = \varepsilon_{ijk} \hat{x}_j \hat{p}_k$$

there is no ordering ambiguity: $\hat{x}_j$ and $\hat{p}_k$ commute as long as $j \neq k$, and the cross product insures this. Computing commutators, we have

$$\begin{aligned}
\left[\hat{L}_i, \hat{L}_m\right] &= \left[\varepsilon_{ijk} \hat{x}_j \hat{p}_k, \varepsilon_{mns} \hat{x}_n \hat{p}_s\right] \\
&= \varepsilon_{ijk} \varepsilon_{mns} \left[\hat{x}_j \hat{p}_k, \hat{x}_n \hat{p}_s\right] \\
&= \varepsilon_{ijk} \varepsilon_{mns} \left(\hat{x}_j \left[\hat{p}_k, \hat{x}_n \hat{p}_s\right] + \left[\hat{x}_j, \hat{x}_n \hat{p}_s\right] \hat{p}_k\right) \\
&= \varepsilon_{ijk} \varepsilon_{mns} \left(\hat{x}_j \left[\hat{p}_k, \hat{x}_n\right] \hat{p}_s + \hat{x}_n \left[\hat{x}_j, \hat{p}_s\right] \hat{p}_k\right) \\
&= \varepsilon_{ijk} \varepsilon_{mns} \left(-i\hbar \delta_{kn} \hat{x}_j \hat{p}_s + i\hbar \delta_{js} \hat{x}_n \hat{p}_k\right) \\
&= i\hbar \left(-\varepsilon_{ijk} \varepsilon_{mks} \hat{x}_j \hat{p}_s + \varepsilon_{ijk} \varepsilon_{mnj} \hat{x}_n \hat{p}_k\right) \\
&= i\hbar \left(\left(\delta_{im} \delta_{js} - \delta_{is} \delta_{jm}\right) \hat{x}_j \hat{p}_s - \left(\delta_{im} \delta_{kn} - \delta_{in} \delta_{km}\right) \hat{x}_n \hat{p}_k\right) \\
&= i\hbar \left(\delta_{im} \hat{x}_j \hat{p}_j - \hat{x}_m \hat{p}_i - \delta_{im} \hat{x}_k \hat{p}_k + \hat{x}_i \hat{p}_m\right) \\
&= i\hbar \left(-\hat{x}_m \hat{p}_i + \hat{x}_i \hat{p}_m\right)
\end{aligned}$$

Therefore, with

$$\begin{aligned}
\varepsilon_{imn} \hat{L}_i &= \varepsilon_{imn} \varepsilon_{ijk} \hat{x}_j \hat{p}_k \\
&= \left(\delta_{mj} \delta_{nk} - \delta_{mk} \delta_{nj}\right) \hat{x}_j \hat{p}_k \\
&= \hat{x}_m \hat{p}_n - \hat{x}_n \hat{p}_m
\end{aligned}$$

this becomes

$$\left[\hat{L}_i, \hat{L}_m\right] = i\hbar \varepsilon_{imn} \hat{L}_n$$

and we see that $\hat{L}_m$ satisfies the angular momentum commutation relations and must therefore admit $|l, m\rangle$ representations satisfying

$$\begin{aligned}
\hat{L}_z \left|l, m\right\rangle &= m\hbar \left|l, m\right\rangle \\
\hat{\mathbf{L}}^2 \left|l, m\right\rangle &= l\left(l+1\right) \hbar^2 \left|l, m\right\rangle
\end{aligned}$$

along with raising and lowering operators, $\hat{L}_\pm$. However, in this case we have an explicit coordinate representation for the operators. For the z-component,

$$\begin{aligned}\langle \mathbf{x}| \hat{L}_3 |\alpha\rangle &= \langle \mathbf{x}| (\hat{x}_1\hat{p}_2 - \hat{x}_2\hat{p}_1) |\alpha\rangle \\ &= -i\hbar\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right)\langle \mathbf{x} |\alpha\rangle\end{aligned}$$

Similarly, the x- and y-components are

$$\begin{aligned}\langle \mathbf{x}| \hat{L}_1 |\alpha\rangle &= -i\hbar\left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}\right)\langle \mathbf{x} |\alpha\rangle \\ \langle \mathbf{x}| \hat{L}_2 |\alpha\rangle &= -i\hbar\left(z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}\right)\langle \mathbf{x} |\alpha\rangle\end{aligned}$$

so we may construct the raising and lowering operators,

$$\begin{aligned}\langle \mathbf{x}| \hat{L}_\pm |\alpha\rangle &= -i\hbar\left[\left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}\right) \pm i\left(z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}\right)\right]\langle \mathbf{x} |\alpha\rangle \\ &= -i\hbar\left[\pm iz\frac{\partial}{\partial x} - z\frac{\partial}{\partial y} \mp i\,(x \pm iy)\frac{\partial}{\partial z}\right]\langle \mathbf{x} |\alpha\rangle\end{aligned}$$

This exposes an essential asymmetry between spinors and vectors. We have seen that 3-vectors may be represented a matrices in a complex, 2-dim spinor representation, there does not exist a similar representation of spinors using 3-dim coordinates. Thus, since orbital angular momentum operators may be written in a coordinate representation, we will see that they only admit integer j representations, so the states $|l,m\rangle$ only exist for integer l. To see this in detail, we need to change from Cartesian to spherical coordinates.

2 Changing to spherical coordinates

Here we rewrite $\hat{L}_z, \hat{L}_\pm$ and $\hat{\mathbf{L}}^2$ in spherical coordinates. The coordinate transformation and its inverse are given by

$$\begin{aligned}r &= \sqrt{x^2+y^2+z^2} \\ \theta &= \tan^{-1}\left(\frac{x^2+y^2}{r^2}\right) \\ \varphi &= \tan^{-1}\left(\frac{y}{x}\right)\end{aligned}$$

and

$$\begin{aligned}x &= r\sin\theta\cos\varphi \\ y &= r\sin\theta\sin\varphi \\ z &= r\cos\theta\end{aligned}$$

We also need the derivative operators, $\frac{\partial}{\partial x^i}$. Using the chain rule, we have

$$\begin{aligned}\frac{\partial}{\partial x} &= \frac{\partial r}{\partial x}\frac{\partial}{\partial r} + \frac{\partial \theta}{\partial x}\frac{\partial}{\partial \theta} + \frac{\partial \varphi}{\partial x}\frac{\partial}{\partial \varphi} \\ \frac{\partial}{\partial y} &= \frac{\partial r}{\partial y}\frac{\partial}{\partial r} + \frac{\partial \theta}{\partial y}\frac{\partial}{\partial \theta} + \frac{\partial \varphi}{\partial y}\frac{\partial}{\partial \varphi} \\ \frac{\partial}{\partial z} &= \frac{\partial r}{\partial z}\frac{\partial}{\partial r} + \frac{\partial \theta}{\partial z}\frac{\partial}{\partial \theta} + \frac{\partial \varphi}{\partial z}\frac{\partial}{\partial \varphi}\end{aligned}$$

Computing the partial derivatives, we start with the differential of r,

$$dr = \frac{x}{r}dx + \frac{y}{r}dy + \frac{z}{r}dz$$

and read off

$$\begin{aligned}
\frac{\partial r}{\partial x} &= \frac{x}{r} \\
\frac{\partial r}{\partial y} &= \frac{y}{r} \\
\frac{\partial r}{\partial z} &= \frac{z}{r}
\end{aligned}$$

Next, for θ, we take the differential of $\tan\theta$,

$$\begin{aligned}
\tan\theta &= \frac{\sqrt{x^2+y^2}}{z} \\
\frac{1}{\cos^2\theta}d\theta &= \frac{1}{\sqrt{x^2+y^2}}\frac{x}{z}dx + \frac{1}{\sqrt{x^2+y^2}}\frac{y}{z}dy - \frac{\sqrt{x^2+y^2}}{z^2}dz
\end{aligned}$$

Then, since

$$\cos^2\theta = \frac{z^2}{r^2}$$

we have

$$d\theta = \frac{1}{\sqrt{x^2+y^2}}\frac{xz}{r^2}dx + \frac{1}{\sqrt{x^2+y^2}}\frac{yz}{r^2}dy - \frac{\sqrt{x^2+y^2}}{r^2}dz$$

and read off the partials,

$$\begin{aligned}
\frac{\partial\theta}{\partial x} &= \frac{1}{\sqrt{x^2+y^2}}\frac{xz}{r^2} \\
\frac{\partial\theta}{\partial y} &= \frac{1}{\sqrt{x^2+y^2}}\frac{yz}{r^2} \\
\frac{\partial\theta}{\partial z} &= -\frac{\sqrt{x^2+y^2}}{r^2}
\end{aligned}$$

Finally, we compute the differential of $\tan\varphi = \frac{y}{x}$, and use $\cos^2\varphi = \frac{x^2}{x^2+y^2}$

$$\begin{aligned}
\frac{1}{\cos^2\varphi}d\varphi &= \frac{1}{x}dy - \frac{y}{x^2}dx \\
d\varphi &= \frac{x}{x^2+y^2}dy - \frac{y}{x^2+y^2}dx
\end{aligned}$$

and once again read off the partials

$$\begin{aligned}
\frac{\partial\varphi}{\partial x} &= -\frac{y}{x^2+y^2} \\
\frac{\partial\varphi}{\partial y} &= \frac{x}{x^2+y^2} \\
\frac{\partial\varphi}{\partial z} &= 0
\end{aligned}$$

Now, returning to the chain rule expansions, we substitute to find

$$
\begin{aligned}
\frac{\partial}{\partial x} &= \frac{x}{r}\frac{\partial}{\partial r} + \frac{1}{\sqrt{x^2+y^2}}\frac{xz}{r^2}\frac{\partial}{\partial\theta} - \frac{y}{x^2+y^2}\frac{\partial}{\partial\varphi} \\
&= \sin\theta\cos\varphi\frac{\partial}{\partial r} + \frac{1}{r}\cos\theta\cos\varphi\frac{\partial}{\partial\theta} - \frac{1}{r}\frac{\sin\varphi}{\sin\theta}\frac{\partial}{\partial\varphi} \\
\frac{\partial}{\partial y} &= \frac{y}{r}\frac{\partial}{\partial r} + \frac{1}{\sqrt{x^2+y^2}}\frac{yz}{r^2}\frac{\partial}{\partial\theta} + \frac{x}{x^2+y^2}\frac{\partial}{\partial\varphi} \\
&= \sin\theta\sin\varphi\frac{\partial}{\partial r} + \frac{1}{r}\cos\theta\sin\varphi\frac{\partial}{\partial\theta} + \frac{1}{r}\frac{\cos\varphi}{\sin\theta}\frac{\partial}{\partial\varphi} \\
\frac{\partial}{\partial z} &= \frac{z}{r}\frac{\partial}{\partial r} - \frac{\sqrt{x^2+y^2}}{r^2}\frac{\partial}{\partial\theta} \\
&= \cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial\theta}
\end{aligned}
$$

and we may substitute into the orbital angular momentum operators.

3 Orbital angular momentum operators in spherical coordiates

Carrying out the coordinate substitutions, for $\hat{L}_3$ we have

$$
\begin{aligned}
-i\hbar\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right) &= -i\hbar r\sin\theta\cos\varphi\left(\sin\theta\sin\varphi\frac{\partial}{\partial r} + \frac{1}{r}\cos\theta\sin\varphi\frac{\partial}{\partial\theta} + \frac{1}{r}\frac{\cos\varphi}{\sin\theta}\frac{\partial}{\partial\varphi}\right) \\
&\quad +i\hbar r\sin\theta\sin\varphi\left(\sin\theta\cos\varphi\frac{\partial}{\partial r} + \frac{1}{r}\cos\theta\cos\varphi\frac{\partial}{\partial\theta} - \frac{1}{r}\frac{\sin\varphi}{\sin\theta}\frac{\partial}{\partial\varphi}\right) \\
&= -i\hbar\frac{\partial}{\partial\varphi}
\end{aligned}
$$

For the raising operator, we have

$$
\begin{aligned}
\frac{1}{\hbar}\hat{L}_+ &= z\frac{\partial}{\partial x} + iz\frac{\partial}{\partial y} - (x+iy)\frac{\partial}{\partial z} \\
&= r\cos\theta\left(\sin\theta\cos\varphi\frac{\partial}{\partial r} + \frac{1}{r}\cos\theta\cos\varphi\frac{\partial}{\partial\theta} - \frac{1}{r}\frac{\sin\varphi}{\sin\theta}\frac{\partial}{\partial\varphi}\right) \\
&\quad +r\cos\theta\left(i\sin\theta\sin\varphi\frac{\partial}{\partial r} + \frac{i}{r}\cos\theta\sin\varphi\frac{\partial}{\partial\theta} + \frac{i}{r}\frac{\cos\varphi}{\sin\theta}\frac{\partial}{\partial\varphi}\right) \\
&\quad -r\sin\theta\left(\cos\varphi + i\sin\varphi\right)\left(\cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\right) \\
&= \left(\cos\varphi + i\sin\varphi - e^{i\varphi}\right) r\cos\theta\sin\theta\frac{\partial}{\partial r} + \cos^2\theta e^{i\varphi}\frac{\partial}{\partial\theta} + e^{i\varphi}\sin^2\theta\frac{\partial}{\partial\theta} \\
&\quad +i\left(\cos\varphi + i\sin\varphi\right)\frac{\cos\theta}{\sin\theta}\frac{\partial}{\partial\varphi} \\
&= e^{i\varphi}\left(\frac{\partial}{\partial\theta} + i\frac{\cos\theta}{\sin\theta}\frac{\partial}{\partial\varphi}\right)
\end{aligned}
$$

while the lowering operator is

$$
\frac{1}{\hbar}\hat{L}_- = -z\frac{\partial}{\partial x} + iz\frac{\partial}{\partial y} + (x-iy)\frac{\partial}{\partial z}
$$

$$
\begin{aligned}
&= -r\cos\theta\left(\sin\theta\cos\varphi\frac{\partial}{\partial r}+\frac{1}{r}\cos\theta\cos\varphi\frac{\partial}{\partial\theta}-\frac{1}{r}\frac{\sin\varphi}{\sin\theta}\frac{\partial}{\partial\varphi}\right)\\
&\quad +r\cos\theta\left(i\sin\theta\sin\varphi\frac{\partial}{\partial r}+\frac{i}{r}\cos\theta\sin\varphi\frac{\partial}{\partial\theta}+\frac{i}{r}\frac{\cos\varphi}{\sin\theta}\frac{\partial}{\partial\varphi}\right)\\
&\quad +r\sin\theta\left(\cos\varphi-i\sin\varphi\right)\left(\cos\theta\frac{\partial}{\partial r}-\frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\right)\\
&= re^{-i\varphi}\sin\theta\cos\theta\frac{\partial}{\partial r}-re^{-i\varphi}\cos\theta\sin\theta\frac{\partial}{\partial r}\\
&\quad -e^{-i\varphi}\cos^2\theta\frac{\partial}{\partial\theta}-e^{-i\varphi}\sin^2\theta\frac{\partial}{\partial\theta}+ie^{-i\varphi}\frac{\cos\theta}{\sin\theta}\frac{\partial}{\partial\varphi}\\
&= -e^{-i\varphi}\frac{\partial}{\partial\theta}+ie^{-i\varphi}\frac{\cos\theta}{\sin\theta}\frac{\partial}{\partial\varphi}
\end{aligned}
$$

Collecting these,

$$
\begin{aligned}
\hat{L}_+ &= \hbar e^{i\varphi}\left(\frac{\partial}{\partial\theta}+i\frac{\cos\theta}{\sin\theta}\frac{\partial}{\partial\varphi}\right)\\
\hat{L}_- &= \hbar e^{-i\varphi}\left(-\frac{\partial}{\partial\theta}+i\frac{\cos\theta}{\sin\theta}\frac{\partial}{\partial\varphi}\right)
\end{aligned}
$$

Finally, since

$$
\hat{L}_+\hat{L}_- = \mathbf{L}^2-L_3^2+\hbar L_3
$$

we have

$$
\begin{aligned}
\mathbf{L}^2 &= \hat{L}_+\hat{L}_- + L_3^2-\hbar L_3\\
&= \hbar e^{i\varphi}\left(\frac{\partial}{\partial\theta}+i\frac{\cos\theta}{\sin\theta}\frac{\partial}{\partial\varphi}\right)\left[\hbar e^{-i\varphi}\left(-\frac{\partial}{\partial\theta}+i\frac{\cos\theta}{\sin\theta}\frac{\partial}{\partial\varphi}\right)\right]+\left(-i\hbar\frac{\partial}{\partial\varphi}\right)^2+i\hbar^2\frac{\partial}{\partial\varphi}\\
&= -\hbar^2\frac{\cos\theta}{\sin\theta}\frac{\partial}{\partial\theta}+i\hbar^2\frac{\cos^2\theta}{\sin^2\theta}\frac{\partial}{\partial\varphi}\\
&\quad -\hbar^2\frac{\partial^2}{\partial\theta^2}-i\hbar^2\frac{\cos\theta}{\sin\theta}\frac{\partial^2}{\partial\varphi\partial\theta}+i\hbar^2\left(-1-\frac{\cos^2\theta}{\sin^2\theta}\right)\frac{\partial}{\partial\varphi}\\
&\quad +i\hbar^2\frac{\cos\theta}{\sin\theta}\frac{\partial^2}{\partial\theta\partial\varphi}-\hbar^2\frac{\cos^2\theta}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\\
&\quad -\hbar^2\frac{\partial^2}{\partial\varphi^2}+i\hbar^2\frac{\partial}{\partial\varphi}\\
&= -\hbar^2\frac{\partial^2}{\partial\theta^2}-\hbar^2\frac{\cos\theta}{\sin\theta}\frac{\partial}{\partial\theta}-\hbar^2\left(1+\frac{\cos^2\theta}{\sin^2\theta}\right)\frac{\partial^2}{\partial\varphi^2}\\
&\quad +i\hbar^2\left(-1-\frac{\cos^2\theta}{\sin^2\theta}+1+\frac{\cos^2\theta}{\sin^2\theta}\right)\frac{\partial}{\partial\varphi}\\
&= -\hbar^2\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)+\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\right)
\end{aligned}
$$

This last equation establishes the relationship between the spherical harmonics and the angular momentum states, because the Laplace equation in spherical coordinates is

$$
\begin{aligned}
\nabla^2 &= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\\
&= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right)+\frac{1}{r^2}\hat{\mathbf{L}}^2
\end{aligned}
$$

and we know that the general solution for $f(r,\theta,\varphi)$ is given in terms of spherical harmonics,

$$f(r,\theta,\varphi) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} A_l(r)\, Y^l_{\ m}(\theta,\varphi)$$

where the spherical harmonics satisfy

$$\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}Y^l_{\ m}(\theta,\varphi)\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}Y^l_{\ m}(\theta,\varphi) + l(l+1)\,Y^l_{\ m}(\theta,\varphi) = 0$$

for integer l and $m = -l, -l+1, \ldots + l$, while the eigenstates of $\hat{\mathbf{L}}^2$ satisfy precisely the same equation,

$$\langle \mathbf{x} | \hat{\mathbf{L}}^2 | l, m\rangle = l(l+1)\hbar^2 \langle \mathbf{x} | l, m\rangle$$

with $\hat{\mathbf{L}}^2$ given above. This shows that orbital angular momentum only describes integer j states.

4 Spherical harmonics

We can now use the quantum formalism to find the spherical harmonics, $Y^l_{\ m}(\theta,\varphi) = \langle \theta,\varphi | l, m\rangle$. For any state $|\alpha\rangle$, we know the effect of $\hat{L}_z$ is given by

$$\begin{aligned}
\langle \theta,\varphi | \hat{L}_z |\alpha\rangle &= -i\hbar\frac{\partial}{\partial\varphi}\langle \theta,\varphi |\alpha\rangle \\
\langle \theta,\varphi | \hat{L}_z | l, m\rangle &= m\hbar \langle \theta,\varphi | l, m\rangle \\
\langle \theta,\varphi | \hat{L}_z | l, m\rangle &= -i\hbar\frac{\partial}{\partial\varphi}\langle \theta,\varphi | l, m\rangle
\end{aligned}$$

so for an eigenstate,

$$-i\hbar\frac{\partial}{\partial\varphi}\langle \theta,\varphi | l, m\rangle = m\hbar\langle \theta,\varphi | l, m\rangle$$

This is trivially integrated to give

$$\langle \theta,\varphi | l, m\rangle = e^{im\varphi}\langle \theta,\varphi | l\rangle$$

Furthermore, we know that the raising operator will anihilate the state with the highest value of m,

$$\hat{L}_+ | l, m = l\rangle = 0$$

In a coordinate basis, this translates to a differential equation,

$$\begin{aligned}
0 &= \langle \theta,\varphi | \hat{L}_+ | l, l\rangle \\
&= \hbar e^{i\varphi}\left(\frac{\partial}{\partial\theta} + i\frac{\cos\theta}{\sin\theta}\frac{\partial}{\partial\varphi}\right) e^{il\varphi}\langle \theta,\varphi | l, l\rangle \\
&= \hbar e^{i\varphi}\left(\frac{\partial}{\partial\theta} + i\frac{\cos\theta}{\sin\theta}\frac{\partial}{\partial\varphi}\right) e^{il\varphi}\langle \theta,\varphi | l\rangle
\end{aligned}$$

Setting $\langle \mathbf{x} | l\rangle = f_l(\theta)$, we have

$$0 = \sin\theta\frac{\partial f_l}{\partial\theta} - l\cos\theta f_l$$

This is solved by $f_l = \sin^l\theta$, so we have, for $m = l$

$$Y^l_{\ l}(\theta,\varphi) = A_{ll}e^{il\varphi}\sin^l\theta$$

Now we can find all other $Y^l{}_m(\theta,\varphi)$ by acting with the lowering operator,

$$\langle\theta,\varphi|\,\hat{L}_-\,|l,m\rangle = \sqrt{l\,(l+1)-m\,(m-1)}\hbar\,\langle\theta,\varphi\,|l,m-1\rangle$$

Inserting the coordinate expression for $\langle\theta,\varphi|\,\hat{L}_-\,|l,m\rangle$ and solving for the next lower state, we have

$$\begin{aligned}\langle\theta,\varphi\,|l,m-1\rangle &= \frac{e^{-i\varphi}}{\sqrt{l\,(l+1)-m\,(m-1)}}\left(-\frac{\partial}{\partial\theta}+i\frac{\cos\theta}{\sin\theta}\frac{\partial}{\partial\varphi}\right)\langle\theta,\varphi\,|l,m\rangle \\ &= -\frac{e^{-i\varphi}e^{im\varphi}}{\sqrt{l\,(l+1)-m\,(m-1)}}\left(\frac{\partial}{\partial\theta}+m\frac{\cos\theta}{\sin\theta}\right)\langle\theta\,|l,m\rangle\end{aligned}$$

thereby defining all $Y^l{}_m(\theta,\varphi)$ recursively.