Modeling measurement

January 26, 2017

In wave mechanics, we found that we could replace dynamical variables such as the momentum or energy with operators. We saw how the quantity

$$\int_{-\infty}^{\infty} dx\psi^* \hat{p}\psi$$

gave the expectation value of momentum by writing, $\hat{p} = -i\hbar\frac{d}{dx}$. We can take this idea further. We have written the wave function as a superposition of plane waves,

$$\psi = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} A(k)\, e^{ikx} dk$$

and this expression is, in fact, expressing a vector in a basis. Linear combinations of functions give other functions, and we can show that the same is true of normalizable functions – linear combinations of square-integrable functions are square integrable functions. We will gain substantial insight by regarding quantum states as elements of a normed vector space, called *Hilbert space*.

We may choose a basis for a vector space; in the case of an infinite dimensional space this basis may be either countable or uncountable. For example, in the case of the infinite square well, we found a countable basis of energy eigenstates,

$$\psi_n(x) = \sqrt{\frac{2}{L}} \sin\frac{n\pi x}{L}, \quad n = 1, 2, 3, \ldots$$

For a free particle, however, the basis is continuous,

$$\psi_k(x) = \frac{1}{\sqrt{2\pi}} e^{ikx}$$

where k may be any real number. The essential fact remains: we may write any physical state as a linear combination of the basis states.

As with finite dimensional vector spaces, the norm of our Hilbert space generalizes to an inner product. If we have two wave functions, $\psi(\mathbf{x}), \chi(\mathbf{x})$, we may define the complex inner product,

$$\langle \psi(\mathbf{x}) | \chi(\mathbf{x}) \rangle \equiv \int_{-\infty}^{\infty} d^3x \psi(\mathbf{x})^* \chi(\mathbf{x})$$

To find a component of this infinite vector, we take the inner product with one particular basis vector. Thus, just as we may dot $\hat{\mathbf{i}}$ into an arbitrary vector to find the x-component,

$$\hat{\mathbf{i}} \cdot \mathbf{v} = v_x$$

we may take the inner product of one member of our plane wave basis, $\frac{1}{\sqrt{2\pi}}e^{iqx}$, with ψ:

$$\begin{aligned}
\langle\psi_k(x)|\psi(x)\rangle &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dx\left(e^{iqx}\right)^*\psi \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} dx e^{-iqx}\int_{-\infty}^{\infty} A(k)\, e^{ikx}dk \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} dk A(k)\int_{-\infty}^{\infty} dx e^{i(k-q)x} \\
&= \int_{-\infty}^{\infty} dk A(k)\,\delta(k-q) \\
&= A(q)
\end{aligned}$$

The inner product selects out the correct component – the coefficient of the basis vector $\frac{1}{\sqrt{2\pi}}e^{iqx}$.

To make a measurement of the momentum of magnitude q, we now include the operator,

$$\begin{aligned}
\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dx\left(e^{iqx}\right)^*\hat{p}\psi &= \frac{-i\hbar}{2\pi}\int_{-\infty}^{\infty} dx e^{-iqx}\frac{d}{dx}\int_{-\infty}^{\infty} A(k)\, e^{ikx}dk \\
&= \frac{-i\hbar}{2\pi}\int_{-\infty}^{\infty} dx\int_{-\infty}^{\infty} ikA(k)\, e^{i(k-q)x}dk \\
&= \frac{-i\hbar}{2\pi}\int_{-\infty}^{\infty} dk ikA(k)\int_{-\infty}^{\infty} dx e^{i(k-q)x} \\
&= \hbar\int_{-\infty}^{\infty} dk A(k)\, k\delta(k-q) \\
&= A(q)\,\hbar q
\end{aligned}$$

and we find the momentum, $\hbar q$, times the corresponding amplitude, $A(q)$. This mathematical operation corresponds to a measurement in which we filter the wave, selecting only the part with momentum q. Naturally, after we accomplish this measurement, the wave is in a state of definite momentum q. In practice, we only measure momentum in a range about q, and the final state is a normalizable wave function centered tightly around q. The important thing to notice is that the measurement reduces our list of possible outcomes, $\psi = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} A(k)\, e^{ikx}dk$, to the momentum eigenstate $A(q)\,\hbar q e^{iqx}$. The fact of measurement changes the list of things we might possibly measure.