Quantum Mechanics: Wheeler: Physics 6210

Problems (some modified) from Sakurai, chapter 1.

W.1 (S.1.2): The Pauli matrices, σ^i, are a triple of 2×2 matrices, $\boldsymbol{\sigma}$,

$$\sigma_i = (\sigma_1, \sigma_2, \sigma_3)$$

given by

$$\begin{aligned} \sigma_1 &= \begin{pmatrix} & 1 \\ 1 & \end{pmatrix} \\ \sigma_2 &= \begin{pmatrix} & -i \\ i & \end{pmatrix} \\ \sigma_3 &= \begin{pmatrix} 1 & \\ & -1 \end{pmatrix} \end{aligned}$$

Let $\mathbf{1}$ stand for the 2×2 identity matrix, and consider the matrix of all complex linear combinations

$$X = a_0 1 + \mathbf{a} \cdot \boldsymbol{\sigma}$$

where a_0 and a_1, a_2, a_3 are complex numbers.

1. Express these numbers in terms of the four traces, $tr\,(X)$ and $tr\,(\sigma_k X)$.
2. Show that X is Hermitian if and only if a_0 and $\mathbf{a}$ are real.
3. Show that X is traceless if and only if $a_0 = 0$.

S.1.3: This problem is not too hard if we first review some facts about determinants and unitary matrices. The determinant of a product is the product of the determinants,

$$\det(AB) = \det(A)\det(B)$$

Now suppose $B = A^{-1}$, so we have $AA^{-1} = 1$. Then

$$\begin{aligned} \det\left(AA^{-1}\right) &= \det(1) \\ \det(A)\det\left(A^{-1}\right) &= 1 \\ \det\left(A^{-1}\right) &= \frac{1}{\det(A)} \end{aligned}$$

Looking at the problem, we see that the result follows if we can show that the matrices $\exp\left(\frac{i\varphi}{2}\mathbf{n} \cdot \boldsymbol{\sigma}\right)$ and $\exp\left(-\frac{i\varphi}{2}\mathbf{n} \cdot \boldsymbol{\sigma}\right)$ are inverse to one another. This follows from a simple theorem relating unitary and Hermitian matrices. Let a matrix U be written as the exponential of i times another matrix, H, so that

$$U = e^{iH}$$

Then U is unitary if and only if H is Hermitian. To prove this, note that unitarity requires $U^\dagger = U^{-1}$ and it is not hard to show that $U^{-1} = e^{-iH}$. Indeed, inserting a parameter λ,

$$\frac{d}{d\lambda}\left(e^{i\lambda H}e^{-i\lambda H}\right) = iHe^{i\lambda H}e^{-i\lambda H} - e^{i\lambda H}iHe^{-i\lambda H}$$

and since H commutes with any function of itself, this vanishes. Therefore, $e^{i\lambda H}e^{-i\lambda H}$ is constant. Setting $\lambda = 0$, shows that the constant must be 1, and $U^{-1} = e^{-iH}$. Therefore, unitarity implies

$$\begin{aligned} U^\dagger &= U^{-1} \\ e^{-iH^\dagger} &= e^{-iH} \end{aligned}$$

The same trick, writing this as $e^{-i\lambda H^\dagger} = e^{-i\lambda H}$ and differentiating with respect to λ shows that

$$-iH^\dagger e^{-i\lambda H^\dagger} = -iHe^{-i\lambda H}$$

and using the original equality with $\lambda = 1$ to remove the exponentials shows that $H^\dagger = H$. Conversely, if $H^\dagger = H$, the result is immediate. With these results at hand, the problem is straightforward.

S.1.4: Practice with Dirac notation.

S.1.6: Don't just show that it works – give a derivation. Set up the eigenket condition and deduce the conditions under which it holds.

S.1.8: These are important relationships, worth checking, and good practice with the Dirac notation. After you are done, find the matrix representations of S_x, S_y, and S_z. They should look familiar.

S.1.10: After you solved the problem, repeat it in the usual matrix notation.

S.1.11: For this problem, it's not hard to find the eigenvalues and eigenkets and the condition that has to hold when $H_{12} = 0$, using standard matrix techniques. It is helpful to recognize that the Hamiltonian

$$\hat{H} = H_{11}\left|1\right\rangle\left\langle 1\right| + H_{22}\left|2\right\rangle\left\langle 2\right| + H_{12}\left(\left|1\right\rangle\left\langle 2\right| + \left|2\right\rangle\left\langle 1\right|\right)$$

is just the matrix

$$\hat{H} = \begin{pmatrix} H_{11} & H_{12} \\ H_{12} & H_{22} \end{pmatrix}$$

Then it is permissible to work with the matrix form. It's not necessary, however – you may just write the eigenket as a general linear combination,

$$\left|E\right\rangle = \alpha\left|1\right\rangle + \beta\left|2\right\rangle$$

and substitute both $\hat{H}$ and $\left|E\right\rangle$ into the eigenvalue equation

$$\hat{H}\left|E\right\rangle = E\left|E\right\rangle$$

S.1.12: Again, Dirac notation or matrices are ok. Remember that spin is a two-state system as well, so you can recast this as a spin problem.

S.1.13: Another good physical problem. Work through step by step.

S.1.14. Worked example problem: a three-state system. We are given the operator

$$A = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

and are asked to find the eigenvalues and normalized eigenvectors. We can find the eigenvalues by solving

$$\begin{aligned}
0 &= \det\left(A - \lambda 1\right) \\
&= \det\left(\frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right) \\
&= \det \begin{pmatrix} -\lambda & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & -\lambda & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & -\lambda \end{pmatrix} \\
&= -\lambda^3 + 0 + 0 - \frac{1}{2}(-\lambda) - \frac{1}{2}(-\lambda) \\
&= -\lambda^3 + \lambda
\end{aligned}$$

so the eigenvalues are

$$\lambda = 0, \pm 1$$

Since these are all different there is no degeneracy. To find the eigenvectors we set $v = (a, b, c)$ and solve

$$Av = \lambda v$$

for each value of λ. For $\lambda = 1$,

$$\begin{aligned}
\frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} &= \lambda \begin{pmatrix} a \\ b \\ c \end{pmatrix} \\
\frac{1}{\sqrt{2}} \begin{pmatrix} b \\ a + c \\ b \end{pmatrix} &= \lambda \begin{pmatrix} a \\ b \\ c \end{pmatrix}
\end{aligned}$$

so we have

$$\begin{aligned}
b &= \sqrt{2}\lambda a = \sqrt{2}\lambda c \\
a + c &= \sqrt{2}\lambda b
\end{aligned}$$

so either $b = 0, c = -a$ when $\lambda = 0$, or $a = c = \frac{\lambda}{\sqrt{2}} b$. This gives the three vectors,

$$b \begin{pmatrix} -\frac{1}{\sqrt{2}} \\ 1 \\ -\frac{1}{\sqrt{2}} \end{pmatrix}, a \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, b \begin{pmatrix} \frac{1}{\sqrt{2}} \\ 1 \\ \frac{1}{\sqrt{2}} \end{pmatrix}$$

corresponding to $-1, 0$ and $+1$ respectively. Notice that each of these vectors is orthogonal to the others. We use the remaining constants, a or b, to normalize. Thus,

$$\begin{aligned} 1 &= b \begin{pmatrix} -\frac{1}{\sqrt{2}} \\ 1 \\ -\frac{1}{\sqrt{2}} \end{pmatrix} \cdot b \begin{pmatrix} -\frac{1}{\sqrt{2}} \\ 1 \\ -\frac{1}{\sqrt{2}} \end{pmatrix} \\ &= b^2 \left(\left(-\frac{1}{\sqrt{2}}\right)\left(-\frac{1}{\sqrt{2}}\right) + 1 + \left(-\frac{1}{\sqrt{2}}\right)\left(-\frac{1}{\sqrt{2}}\right) \right) \\ &= b^2 \left(\frac{1}{2} + 1 + \frac{1}{2} \right) \\ b &= \frac{1}{\sqrt{2}} \end{aligned}$$

The last two cases are similar, so we have the three eigenvectors

$$\begin{aligned} v_{-1} &= \frac{1}{\sqrt{2}} \begin{pmatrix} -\frac{1}{\sqrt{2}} \\ 1 \\ -\frac{1}{\sqrt{2}} \end{pmatrix} \\ v_0 &= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \\ v_1 &= \frac{1}{\sqrt{2}} \begin{pmatrix} \frac{1}{\sqrt{2}} \\ 1 \\ \frac{1}{\sqrt{2}} \end{pmatrix} \end{aligned}$$

For part b, these could correspond to the spin eigenstates of a vector particle, but there's no particular reason to expect you to know that.

S.1.15: The vanishing of an operator, such as $\left[\hat{A}, \hat{B}\right] = 0$, means that it vanishes on every state. Completeness of a basis means that an arbitrary state can be expanded in the basis.

S.1.16: The comment to S.1.15 is relevant here too.

S.1.19: Calculate the expectations and substitute. When you're done with that, try to explain the values you get for part b.

S.1.20: Just remember how to find the extrema of a function. Why does Sakurai ask for the maximum instead of the minimum? What is the minimum value? (You found it in the previous problem.) Does maximizing the left side also maximize the right side?

S.1.22: This is a terrific problem! Ask about it if you don't figure it out!

S.1.23: A 3-state system. Work out all the details here. There's nothing that's not straightforward matrix manipulation, and the problem gives some clear insight into simultaneous eigenkets and degeneracy.

S.1.24: This problem foreshadows chapter three. Thinking about it now will save you effort later. a) Do this by picking a state you know lies in the$+z$ direction: that is, the $|+>$ state. Now apply the operator and if it works it should give you a state that is definitely in the $|y, \pm>$ direction, depending on whether the rotation is clockwise or counterclockwise. To test the operator completely, check what happens to $|x,+>$ and $|y,+>$ states as well. b) Just use the rotator of part a) to rotate S_y. Remember that to rotate an operator you have to do a similarity transformation, that is, if $\hat{U} = \frac{1}{\sqrt{2}}(1 + i\sigma_x)$ and you want to rotate an operator $\hat{\mathcal{O}}$, it is given by $\hat{U}\hat{\mathcal{O}}\hat{U}^\dagger$.

S.1.26: Look at the states you're starting with and the ones you want to end up with.

S.1.27: Good practice with continuum basis changes.

S.1.28: This gives good insight into the canonical quantization procedure. Notice that Sakurai's classical Poisson bracket $[A, B]_{classical}$ is

$$[A, B]_{classical,x,p} = \frac{\partial A}{\partial x}\frac{\partial B}{\partial p} - \frac{\partial A}{\partial p}\frac{\partial B}{\partial x}$$

S.1.33: If this gives you trouble, go back and review the last section.