The Feynman path integral

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1 Heisenberg and Schrödinger pictures

The Schrödinger wave function places the time dependence of a physical system in the state, $|\psi, t\rangle$, where the state is a vector in Hilbert space that moves in time. This Hilbert space can be described in any basis we choose: coordinate, $|\mathbf{x}\rangle$, momentum $|\mathbf{p}\rangle$, or whatever suits our need.

It is also possible to regard the state as fixed and the basis as changing in time. This is the Heisenberg picture.

1.1 Heisenberg operators

Consider an operator acting on a state, then projected onto any other state at time t,

$$\begin{aligned}\langle\chi, t|\hat{A}|\psi, t\rangle &= \left(\langle\chi, t_0|\hat{U}^\dagger(t, t_0)\right)\hat{A}\left(\hat{U}(t, t_0)|\psi, t_0\rangle\right) \\ &= \langle\chi, t_0|\left(\hat{U}^\dagger(t, t_0)\hat{A}\hat{U}(t, t_0)\right)|\psi, t_0\rangle\end{aligned}$$

so if we define a time-dependent Heisenberg operator,

$$\hat{A}_H = \hat{A}(t) \equiv \hat{U}^\dagger(t, t_0)\hat{A}_S\hat{U}(t, t_0)$$

then we get the same prediction by looking at $\hat{A}(t)$ acting on the fixed initial state:

$$\langle\chi, t|\hat{A}_S|\psi, t\rangle = \langle\chi, t_0|\hat{A}_H(t)|\psi, t_0\rangle$$

We may replace the Schrodinger equation with evolution equations for operators. Taking the time derivative where $\hat{U}(t, t_0) = e^{-\frac{i}{\hbar}\hat{H}(t-t_0)}$,

$$\begin{aligned}\frac{d\hat{A}_H}{dt} &= \frac{\partial}{\partial t}\hat{U}^\dagger(t, t_0)\hat{A}_S\hat{U}(t, t_0) + \hat{U}^\dagger(t, t_0)\hat{A}_S\frac{\partial}{\partial t}\hat{U}(t, t_0) \\ &= \frac{i}{\hbar}\hat{H}\hat{U}^\dagger(t, t_0)\hat{A}_S\hat{U}(t, t_0) - \frac{i}{\hbar}\hat{U}^\dagger(t, t_0)\hat{A}_S\hat{U}(t, t_0)\hat{H} \\ &= \frac{i}{\hbar}\hat{H}\hat{A}_H - \frac{i}{\hbar}\hat{A}_H\hat{H}\end{aligned}$$

and we have the Heisenberg equation of motion,

$$\frac{d\hat{A}_H}{dt} = \frac{i}{\hbar}\left[\hat{H}, \hat{A}_H\right]$$

According to Sakurai, this was first written by Dirac.

1.2 Heisenberg basis kets

The Heisenberg picture also requires a change in the basis kets. Since basis kets are eigenkets of particular operators, and the operators are now time-depending, the eigenkets also change. We have

$$\hat{A}_S \left|a\right\rangle_S = a \left|a\right\rangle_S$$

where a state in the Schrödinger basis is given by

$$\psi\left(a, t\right) = \left\langle a \left|\psi, t\right.\right\rangle$$

In the Heisenberg picture, these become

$$\begin{aligned} \hat{A}_H\left(t\right)\left|a, t\right\rangle_H &= a\left|a, t\right\rangle_H \\ \hat{U}^\dagger \hat{A}_S \hat{U}\left|a, t\right\rangle_H &= a\left|a, t\right\rangle_H \\ \hat{A}_S \hat{U}\left|a, t\right\rangle_H &= a\hat{U}\left|a, t\right\rangle_H \end{aligned}$$

so we must have

$$\hat{U}\left|a, t\right\rangle_H = \left|a\right\rangle_S$$

Inverting,

$$\left|a, t\right\rangle_H = \hat{U}^\dagger \left|a\right\rangle_S$$

we see that the Heisenberg basis evolves oppositely to the Schrodinger state to give the same result.

1.3 Transition amplitudes

Given the time-dependence of the basis kets, we may ask for the probability amplitude for a basis ket $\left|a, t_0\right\rangle_H$ at time t_0 to be found in another direction $\left|b, t_0\right\rangle_H$ at time t,

$$\left\langle b, t \left|a, t_0\right.\right\rangle$$

This is called the *transition amplitude*. For example, the transition amplitude for a system to go from $\mathbf{x}'$ at time t_0 to $\mathbf{x}$ at time t is

$$\left\langle \mathbf{x}, t \left|\mathbf{x}', t_0\right.\right\rangle$$

2 Propagators

We have seen that the time evolution of state is given by

$$\left|\psi, t\right\rangle = e^{-\frac{i}{\hbar}\hat{H}(t-t_0)}\left|\psi, t_0\right\rangle$$

when the Hamiltonian is independent of time. Inserting an identity in terms of an energy basis,

$$\begin{aligned} \left|\psi, t\right\rangle &= e^{-\frac{i}{\hbar}\hat{H}(t-t_0)}\sum_a \left|E_a\right\rangle\left\langle E_a \left|\psi, t_0\right.\right\rangle \\ &= \sum_a e^{-\frac{i}{\hbar}E_a(t-t_0)}\left|E_a\right\rangle\left\langle E_a \left|\psi, t_0\right.\right\rangle \end{aligned}$$

Now view the state in a coordinate basis,

$$\begin{aligned} \left\langle \mathbf{x} \left|\psi, t\right.\right\rangle &= \sum_a e^{-\frac{i}{\hbar}E_a(t-t_0)}\left\langle \mathbf{x} \left|E_a\right.\right\rangle\left\langle E_a \left|\psi, t_0\right.\right\rangle \\ \psi\left(\mathbf{x}, t\right) &= \sum_a e^{-\frac{i}{\hbar}E_a(t-t_0)}\left\langle \mathbf{x} \left|E_a\right.\right\rangle\left\langle E_a \left|\psi, t_0\right.\right\rangle \end{aligned}$$

Inserting one more identity in the coordinate basis, we have

$$\begin{aligned}
\psi(\mathbf{x},t) &= \int d^3x' \sum_a e^{-\frac{i}{\hbar}E_a(t-t_0)} \langle \mathbf{x} | E_a \rangle \langle E_a | \mathbf{x}' \rangle \langle \mathbf{x}' | \psi, t_0 \rangle \\
&= \int d^3x' \sum_a e^{-\frac{i}{\hbar}E_a(t-t_0)} \langle \mathbf{x} | E_a \rangle \langle E_a | \mathbf{x}' \rangle \psi(\mathbf{x}', t_0)
\end{aligned}$$

Now define the *propagator*

$$K(\mathbf{x},t;\mathbf{x}',t_0) \equiv \sum_a \langle \mathbf{x} | E_a \rangle \langle E_a | \mathbf{x}' \rangle e^{-\frac{i}{\hbar}E_a(t-t_0)}$$

so that we have

$$\psi(\mathbf{x},t) = \int d^3x' K(\mathbf{x},t;\mathbf{x}',t_0)\, \psi(\mathbf{x}',t_0)$$

Identifying the propagator for a given problem separates the initial wave function from the potential, allowing a formal solution for the wave function at a later time and arbitrary position. Holding $(\mathbf{x}', t_0)$ fixed, $u_a(x)$ is the stationary state wave function, and $e^{-\frac{i}{\hbar}E_a t}$ is its time dependence, so $K(\mathbf{x},t;\mathbf{x}',t_0)$ satisfies the time-dependent Schrödinger equation. Also,

$$\lim_{t\to t_0} K(\mathbf{x},t;\mathbf{x}',t_0) = \delta^3(\mathbf{x}-\mathbf{x}')$$

Moreover, the propagator is essentially a Green's function that includes the time evolution, giving the probability amplitude for a particle initially at $\mathbf{x}'$ at t_0 to be found at $\mathbf{x}$ at the later time t. In this way, the propagator is the transition amplitude for the system. We can make this explicit:

$$\begin{aligned}
K(\mathbf{x},t;\mathbf{x}',t_0) &\equiv \sum_a \langle \mathbf{x} | E_a \rangle \langle E_a | \mathbf{x}' \rangle e^{-\frac{i}{\hbar}E_a(t-t_0)} \\
&= \langle \mathbf{x} | e^{-\frac{i}{\hbar}Ht} \sum_a | E_a \rangle \langle E_a | e^{\frac{i}{\hbar}Ht_0} | \mathbf{x}' \rangle \\
&= \langle \mathbf{x} | \hat{U}(t,0)\, \hat{U}^\dagger(t_0,0) | \mathbf{x}' \rangle
\end{aligned}$$

so removing the identity $1 = \sum_a |E_a\rangle\langle E_a|$ and identifying the Heisenberg basis states, $\hat{U}^\dagger(t_0,0)|\mathbf{x}'\rangle = |\mathbf{x}', t_0\rangle_H$ and $\hat{U}^\dagger(t,0)|\mathbf{x}\rangle = |\mathbf{x},t\rangle_H$ we have the transition amplitude:

$$K(\mathbf{x},t;\mathbf{x}',t_0) = \langle \mathbf{x},t | \mathbf{x}',t_0 \rangle$$

Transition amplitudes, or propagators, have a *composition property*. If we insert the identity operator in the form

$$1 = \int d^3x'' |\mathbf{x}'',t_1\rangle \langle \mathbf{x}'',t_1|$$

where $t_0 < t_1 < t$, into the transition amplitude, it becomes an integral over a product of transition amplitudes:

$$\langle \mathbf{x},t | \mathbf{x}',t_0 \rangle = \int d^3x'' \langle \mathbf{x},t | \mathbf{x}'',t_1 \rangle \langle \mathbf{x}'',t_1 | \mathbf{x}',t_0 \rangle$$

This shows that the probability amplitude for going from $(\mathbf{x}',t_0)$ to $(\mathbf{x},t)$ is the product of the probability amplitudes for going from $(\mathbf{x}',t_0)$ to an intermediate state at time t_1 and the probability of going from that state to $(\mathbf{x},t)$, summed over all possible intermediate positions. This is just like the composition of conditional probabilities:

$$P_{A\,given\,B} = \sum_C P_{A\,given\,C} P_{C\,given\,B}$$

but it is significant that it applies to probability *amplitudes* instead of probabilities. This fact underlies Bell's theorem.

3 The Feynman path integral

We consider a particle with Hamiltonian of the form $\hat{H} = \frac{\hat{\mathbf{p}}^2}{2m} + V(\hat{\mathbf{x}})$.

Applying the composition property $N-1$ times in going from (x_0, t_0) to (x_N, t_N),

$$\langle \mathbf{x}_N, t_N | \mathbf{x}_0, t_0 \rangle = \int \cdots \int \prod_{i=1}^{N-1} d^3x_i \, \langle \mathbf{x}_N, t_N | \mathbf{x}_{N-1}, t_{N-1} \rangle \cdots \langle \mathbf{x}_1, t_1 | \, \mathbf{x}_0, t_0 \rangle$$

Now look at one of the transition amplitudes,

$$\begin{aligned} \langle \mathbf{x}_{i+i}, t_{i+1} | \, \mathbf{x}_i, t_i \rangle &= \langle \mathbf{x}_{i+i}, t_i | \, e^{-\frac{i}{\hbar}\hat{H}(t_{i+1}-t_i)} \, | \mathbf{x}_i, t_i \rangle \\ &= \langle \mathbf{x}_i, t_i | \, e^{\frac{i}{\hbar}\hat{\mathbf{p}}\cdot(\mathbf{x}_{i+1}-\mathbf{x}_i)} e^{-\frac{i}{\hbar}\hat{H}(t_{i+1}-t_i)} \, | \mathbf{x}_i, t_i \rangle \end{aligned}$$

Let N be sufficiently large that $t_{i+1} - t_i = \Delta t$ becomes infinitesimal. To evaluate the translation operator and the Hamiltonian, we insert a momentum basis,

$$\begin{aligned} \langle \mathbf{x}_{i+i}, t_{i+1} | \, \mathbf{x}_i, t_i \rangle &= \int d^3p_i \, \langle \mathbf{x}_{i+1}, t_i | \, | \mathbf{p}_i, t_i \rangle \langle \mathbf{p}_i, t_i | \, e^{-\frac{i}{\hbar}\hat{H}(t_{i+1}-t_i)} \, | \mathbf{x}_i, t_i \rangle \\ &= \int d^3p_i \, \langle \mathbf{x}_{i+1}, t_i \, | \mathbf{p}_i, t_i \rangle \langle \mathbf{p}_i, t_i | \left(1 - \frac{i}{\hbar}\hat{H}\Delta t \right) | \mathbf{x}_i, t_i \rangle \\ &= \int d^3p_i \, \langle \mathbf{x}_{i+1}, t_i \, | \mathbf{p}_i, t_i \rangle \left(1 - \frac{i}{\hbar}\frac{\mathbf{p}_i^2}{2m}\Delta t - \frac{i}{\hbar}V(\mathbf{x}_i)\Delta t \right) \langle \mathbf{p}_i, t_i \, | \mathbf{x}_i, t_i \rangle \end{aligned}$$

Now, using

$$\langle \mathbf{p}_i, t_i \, | \mathbf{x}_i, t_i \rangle = \frac{1}{(2\pi\hbar)^{3/2}} e^{-\frac{i}{\hbar}\mathbf{p}_i\cdot\mathbf{x}_i}$$

the infinitesimal transition amplitude becomes

$$\begin{aligned} \langle \mathbf{x}_{i+i}, t_{i+1} | \, \mathbf{x}_i, t_i \rangle &= \int d^3p_i \, \langle \mathbf{x}_{i+1}, t_i | \, | \mathbf{p}_i, t_i \rangle \langle \mathbf{p}_i, t_i | \, e^{-\frac{i}{\hbar}\hat{H}(t_{i+1}-t_i)} \, | \mathbf{x}_i, t_i \rangle \\ &= \int d^3p_i \, \langle \mathbf{x}_{i+1}, t_i \, | \mathbf{p}_i, t_i \rangle \langle \mathbf{p}_i, t_i | \left(1 - \frac{i}{\hbar}\hat{H}\Delta t \right) | \mathbf{x}_i, t_i \rangle \\ &= \frac{1}{(2\pi\hbar)^3} \int d^3p_i e^{\frac{i}{\hbar}\mathbf{p}_i\cdot\mathbf{x}_{i+1}} \left(1 - \frac{i}{\hbar}\frac{\mathbf{p}_i^2}{2m}\Delta t - \frac{i}{\hbar}V(\mathbf{x}_i)\Delta t \right) e^{-\frac{i}{\hbar}\mathbf{p}_i\cdot\mathbf{x}_i} \\ &= \frac{1}{(2\pi\hbar)^3} \int d^3p_i e^{\frac{i}{\hbar}\mathbf{p}_i\cdot(\mathbf{x}_{i+1}-\mathbf{x}_i)} e^{-\frac{i}{\hbar}\frac{\mathbf{p}_i^2}{2m}\Delta t - \frac{i}{\hbar}V(\mathbf{x}_i)\Delta t} \\ &= \frac{1}{(2\pi\hbar)^3} \int d^3p_i \exp \frac{i}{\hbar} \left[\mathbf{p}_i \cdot (\mathbf{x}_{i+1} - \mathbf{x}_i) - \frac{\mathbf{p}_i^2}{2m}\Delta t - V(\mathbf{x}_i)\Delta t \right] \\ &= \frac{1}{(2\pi\hbar)^3} \int d^3p_i \exp \frac{i}{\hbar} \left[\mathbf{p}_i \cdot \frac{d\mathbf{x}_i}{dt} - H \right] dt \\ &= \frac{1}{(2\pi\hbar)^3} \int d^3p_i \exp \frac{i}{\hbar} L(\mathbf{p}_i, \mathbf{x}_i) \, dt \end{aligned}$$

where we find the Hamiltonian replaced by the Lagrangian,

$$L(\mathbf{p}_i, \mathbf{x}_i) \, dt = (\mathbf{p}_i \cdot \dot{\mathbf{x}}_i - H) \, dt$$

Notice that all operators have been replaced by eigenvalues.

Now reassemble the full, finite transition amplitude:

$$\begin{aligned}
\langle \mathbf{x}_N, t_N \left| \mathbf{x}_0, t_0 \right\rangle &= \int \cdots \int \frac{1}{(2\pi\hbar)^{3N}} \prod_{i=1}^{N-1} d^3x_i d^3p_i \prod_{i=1}^{N-1} \left(\exp \frac{i}{\hbar} L(\mathbf{p}_i, \mathbf{x}_i)\, dt \right) \\
&= \int \cdots \int \frac{1}{(2\pi\hbar)^{3(N-1)}} \prod_{i=1}^{N-1} d^3x_i d^3p_i \left(\exp \frac{i}{\hbar} \sum_{i=1}^{N-1} L(\mathbf{p}_i, \mathbf{x}_i)\, dt \right)
\end{aligned}$$

and replacing the sum of infinitesimals by an integral,

$$\begin{aligned}
\exp \frac{i}{\hbar} \sum_{i=1}^{N-1} L(\mathbf{p}_i, \mathbf{x}_i)\, dt &= \exp \frac{i}{\hbar} \int_{t_0}^{t_N} L(\mathbf{p}_i, \mathbf{x}_i)\, dt \\
&= \exp \frac{i}{\hbar} S[\mathbf{x}(t), \mathbf{p}(t)]
\end{aligned}$$

where $S[\mathbf{x}(t), \mathbf{p}(t)]$ is the action functional in terms of both position and momentum.

Finally, we define the functional integral to be the sum over all intervening paths, here in both configuration and momentum spaces:

$$\begin{aligned}
\int \mathcal{D}[\mathbf{x}(t)] &\equiv \int \cdots \int \frac{1}{(2\pi\hbar)^{3(N-1)/2}} \prod_{i=1}^{N-1} d^3x_i \\
\int \mathcal{D}[\mathbf{p}(t)] &\equiv \int \cdots \int \frac{1}{(2\pi\hbar)^{3(N-1)/2}} \prod_{i=1}^{N-1} d^3p_i
\end{aligned}$$

With this notation, the transition amplitude, or propagator, is given by

$$\langle \mathbf{x}_N, t_N | \, \mathbf{x}_0, t_0 \rangle = \int \mathcal{D}[\mathbf{x}(t)] \int \mathcal{D}[\mathbf{p}(t)] \exp \frac{i}{\hbar} S[\mathbf{x}(t), \mathbf{p}(t)]$$

This is the *Feynman path integral.* Notice again that the action here is written as an independent functional of position and momentum.

The infinite products of intermediate integrals may be interpreted as meaning that the phase $\exp \frac{i}{\hbar} S[\mathbf{x}(t)]$ is to be summed over every value of position and momentum. As we shall see from examples, the result involves some curious normalizations, but the formulation is very powerful because it may be immediately generalized to field theory. Any theory of fields Φ having an action functional may be quantized by averaging $\exp \frac{i}{\hbar} S[\Phi]$ over all field configurations.

$$\begin{aligned}
\langle \Phi(\mathbf{x}, t_f) | \, \Phi(\mathbf{x}, t_i) \rangle &= \int \mathcal{D}[\Phi(\mathbf{x}, t)] \exp \frac{i}{\hbar} S[\Phi(\mathbf{x}, t), \Pi(\mathbf{x}, t)] \\
S[\Phi(\mathbf{x}, t)] &= \int_{t_i}^{t_f} \mathcal{L}(\Phi(\mathbf{x}, t), \Pi(\mathbf{x}, t))\, d^4x
\end{aligned}$$

Here, the position and time are simply parameters, while the field and its conjugate mometum are the dynamical variables.

The most important advantage of the path integral formulation is that it allows for a systematic perturbation theory. If we write the particle Lagrangian as

$$L = L_0 + V$$

and expand the exponential

$$\langle \mathbf{x}_N, t_N | \, \mathbf{x}_0, t_0 \rangle = \int \mathcal{D}[\mathbf{x}(t)] \int \mathcal{D}[\mathbf{p}(t)] \left(\exp \frac{i}{\hbar} \int_{t_0}^{t_N} L_0 dt \right) \left(1 + \frac{i}{\hbar} \int_{t_0}^{t_N} V dt + \cdots \right)$$

it is possible to evaluate the potential terms order by order. The same expansion applies to field theory,

$$\langle \Phi(\mathbf{x}, t_f) | \Phi(\mathbf{x}, t_i) \rangle = \int \mathcal{D}[\Phi(\mathbf{x}, t)] \left(\exp \frac{i}{\hbar} S_0 [\Phi(\mathbf{x}, t)] \right) \left(1 + \frac{i}{\hbar} \int_{t_0}^{t_N} V dt + \cdots \right)$$

allowing term by term approximation. Ultimately, each term in the expansion involves different powers of the potential. We keep track of the large number of required integrals by sets of *Feynman diagrams*, each diagram corresponding to a particular set of integrals.

Typically, equivalence to other methods holds, but is not demanded. The path integral is an independent model for quantization.

4 The momentum integrals

For the form of Hamiltonian we have chosen, $\hat{H} = \frac{\hat{\mathbf{p}}^2}{2m} + V(\hat{\mathbf{x}})$, it is possible to do all of the momentum integrals. Each one is simply a Gaussian:

$$\begin{aligned}
\frac{1}{(2\pi\hbar)^3} \int d^3 p_i \exp \frac{i}{\hbar} L(\mathbf{p}_i, \mathbf{x}_i) dt &= \frac{1}{(2\pi\hbar)^3} \int d^3 p_i \exp \frac{i}{\hbar} \left(\mathbf{p}_i \cdot \frac{(\mathbf{x}_{i+1} - \mathbf{x}_i)}{\Delta t} - \frac{\mathbf{p}_i^2}{2m} - V(\mathbf{x}_i) \right) dt \\
&= \frac{1}{(2\pi\hbar)^3} \int d^3 p_i \exp \frac{i}{\hbar} \left(-\frac{1}{2m} \left(\mathbf{p}_i - m \frac{d\mathbf{x}_i}{dt} \right)^2 + \frac{m}{2} (\mathbf{x}_{i+1} - \mathbf{x}_i)^2 - V(\mathbf{x}_i) \right) dt \\
&= \frac{1}{(2\pi\hbar)^3} \exp \frac{i}{\hbar} \left(\frac{m}{2} \mathbf{v}_i^2 - V(\mathbf{x}_i) \right) dt \int d^3 p_i \exp \left(-\frac{i}{2m\hbar} (\mathbf{p}_i - m\mathbf{v}_i)^2 \right)
\end{aligned}$$

Letting $\mathbf{y} = \mathbf{p}_i - m(\mathbf{x}_{i+1} - \mathbf{x}_i)$, the integral becomes

$$\int d^3 y \exp \left(-\frac{i}{2m\hbar} \mathbf{y}^2 \right)$$

The imaginary unit does not really cause any problem. Adding an infinitesimal part for convergence we have

$$\int d^3 y \exp \left(-\frac{i}{2m\hbar} (1 - i\varepsilon) \mathbf{y}^2 \right) = \int d^3 y \exp \left(-\frac{\varepsilon + i}{2m\hbar} \mathbf{y}^2 \right)$$

Each of the three Gaussians gives

$$\int dy \exp(-\alpha y^2) = \sqrt{\frac{\pi}{\alpha}}$$

so

$$\begin{aligned}
\lim_{\varepsilon \to 0} \int d^3 y \exp \left(-\frac{i}{2m\hbar} (1 - i\varepsilon) \mathbf{y}^2 \right) &= \lim_{\varepsilon \to 0} \left(\frac{2m\hbar\pi}{\varepsilon + i} \right)^{3/2} \\
&= (-2\pi i m\hbar)^{3/2}
\end{aligned}$$

The full i^{th} integral is therefore,

$$\frac{1}{(2\pi\hbar)^3} \int d^3 p_i \exp \frac{i}{\hbar} L(\mathbf{p}_i, \mathbf{x}_i) dt = \left(\frac{m}{2\pi i\hbar} \right)^{3/2} \exp \frac{i}{\hbar} \left(\frac{1}{2} m \mathbf{v}_i^2 - V(\mathbf{x}_i) \right) dt$$

Combining these in the full path integral, we have

$$\begin{aligned}
\langle \mathbf{x}_N, t_N | \mathbf{x}_0, t_0 \rangle &= \int \cdots \int \frac{1}{(2\pi\hbar)^{3(N-1)}} \prod_{i=1}^{N-1} d^3 x_i d^3 p_i \left(\exp \frac{i}{\hbar} \sum_{i=1}^{N-1} L(\mathbf{p}_i, \mathbf{x}_i) dt \right) \\
&= \int \cdots \int \prod_{i=1}^{N-1} d^3 x_i \left(-\frac{im}{(2\pi\hbar)^3} \right)^{3N/2} \exp \frac{i}{\hbar} \int_{t_0}^{t_N} \left(\frac{1}{2} m \mathbf{v}_i^2 - V(\mathbf{x}_i) \right) dt
\end{aligned}$$

and replacing the sum of infinitesimals by an integral, and defining the functional integral measure to be

$$\int \mathcal{D}\left[\mathbf{x}\left(t\right)\right] \equiv \int \cdots \int \prod_{i=1}^{N-1} d^3 x_i \left(-\frac{im}{\left(2\pi\hbar\right)^3}\right)^{3N/2}$$

the transition amplitude is

$$\begin{aligned}\left\langle \mathbf{x}_N, t_N \left| \mathbf{x}_0, t_0 \right.\right\rangle &= \int \mathcal{D}\left[\mathbf{x}\left(t\right)\right] \exp \frac{i}{\hbar} \int_{t_0}^{t_N} L\left(\mathbf{x}, \dot{\mathbf{x}}\right) dt \\ &= \int \mathcal{D}\left[\mathbf{x}\left(t\right)\right] \exp \frac{i}{\hbar} S\left[\mathbf{x}\left(t\right)\right]\end{aligned}$$

where $S\left[\mathbf{x}\left(t\right)\right]$ is now the usual configuration space action. This is the usual form of the Feynman path integral.