Change of basis

January 27, 2013

We know that the eigenvectors of Hermitian operators may be chosen to form a complete, orthonormal set of basis states. Let

$$S = \{|a'\rangle \mid a' \in K\}$$

where K is some (finite or infinite) index set, be such a complete orthonormal basis, with

$$\hat{\mathcal{O}}\,|a'\rangle = a'\,|a'\rangle$$

for some Hermitian operator, $\hat{\mathcal{O}}$. There is a converse to this: if we have a complete, orthonormal set, then we can build operators:

$$\hat{\mathcal{O}}' \equiv \sum_{a' \in K} \alpha(a')\,|a'\rangle\langle a'|$$

and the complex numbers $\alpha(a')$ will be its eigenvalues, since

$$\begin{aligned}
\hat{\mathcal{O}}'\,|a''\rangle &= \sum_{a' \in K} \alpha(a')\,|a'\rangle\langle a'|\,a''\rangle \\
&= \sum_{a' \in K} \alpha(a')\,|a'\rangle\,\delta_{a'a''} \\
&= \alpha(a'')\,|a''\rangle
\end{aligned}$$

By choosing $\alpha(a')$ real for each a', we insure that $\hat{\mathcal{O}}'$ is Hermitian. Therefore, we may find (many) Hermitian operators with a given complete orthonormal set of eigenstates.

If we have any quantum state, it may be expanded in term of any complete basis:

$$\begin{aligned}
|\psi\rangle &= \hat{1}\,|\psi\rangle \\
&= \sum_{a' \in K} |a'\rangle\langle a'|\,\psi\rangle
\end{aligned}$$

so the complex numbers $\langle a'|\,\psi\rangle$ represent the *state* $|\psi\rangle$ in the *basis* $|a'\rangle$.

If we wish to change the basis, consider the relationship between two bases, $|a'\rangle$ and $|b'\rangle$. Because these span the same vector space, there is are equal numbers of indices, a' and b' and we put these in $1-1$, but otherwise arbitrary, correspondence, $a_1 \leftrightarrow b_1, a_2 \leftrightarrow b_2, \ldots$. In general, let $a' \leftrightarrow b'$. Now, since the basis ket $|b'\rangle$ is also a state, it may be expanded in terms of the $|a'\rangle$,

$$|b'\rangle = \sum_{a'' \in K} |a''\rangle\langle a''|\,b'\rangle$$

We may view this as a matrix $\hat{U}$, with components $\langle a''|\,b'\rangle$, acting on $|a''\rangle$ to give $|b'\rangle$. The matrix is the sum over corresponding pairs,

$$\hat{U} = \sum_{m} |b_m\rangle\langle a_m|$$

since then

$$
\begin{aligned}
\hat{U}\,|a_k\rangle &= \sum_m |b_m\rangle\,\langle a_m|\,a_k\rangle \\
&= \sum_m \delta_{mk}\,|b_m\rangle \\
&= |b_k\rangle
\end{aligned}
$$

Using this relation we see that this operator indeed has the right matrix components

$$
\langle a_l|\,\hat{U}\,|a_k\rangle = \langle a_l|\,b_k\rangle
$$

Notice that $\hat{U}$ is unitary, since, with $\hat{U}^\dagger = \sum_m |a_m\rangle\,\langle b_m|$, we have

$$
\begin{aligned}
\hat{U}^\dagger\hat{U} &= \left(\sum_m |a_m\rangle\,\langle b_m|\right)\left(\sum_n |b_n\rangle\,\langle a_n|\right) \\
&= \sum_{m,n} |a_m\rangle\,\langle b_m|\,b_n\rangle\,\langle a_n| \\
&= \sum_{m,n} \delta_{mn}\,|a_m\rangle\,\langle a_n| \\
&= \sum_m |a_m\rangle\,\langle a_m| \\
&= \hat{1}
\end{aligned}
$$