# Bell's Theorem and the EPR paradox

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**Abstract**

The following discussion of Bell's theorem and the EPR paradox relies heavily on Bell's original paper [1] and the example in section 5 on a discussion by David S. Merman [2]. Both papers are accessible and highly recommended. The Wikipedia article [3] is also excellent.

Bell's theorem states: *No local hidden variables theory can ever reproduce the predictions of quantum mechanics.*

By a "local hidden variables" theory, what we mean is that there is some classical property of the system, for example physical electron spin, that evolves in a classical way that is, as yet, unknown.

Suppose an electron is produced at point $A$ and travels to point $B$, where we measure its spin to be up. It is tempting to say that its must have been in the spin up orientation en route from $A$ to $B$, or at least evolving in some predictable way that lead to spin up at $B$. Bell's theorem shows that this is *incorrect*. Specifically, if the spin of the electron definitely pointed in some direction at the initial time, and evolved continuously but randomly, Bell's theorem shows that it could not reproduce the probability distribution expected within quantum mechanics. The predictions of Bell's theorem have been measured in numerous experiments and found to agree with quantum mechanics.

First, let's look at a general probability argument showing that there is an essential difference between classical and quantum probabilities. Then we turn to a discussion of the Einstein-Podolsky-Rosen problem, followed by an example of Bell's theorem where we can clearly see the the theorem at work.

## 1 Basic probability argument

Consider the conditional probability, $P_{A\to B}$, of an event $B$ given that an event $A$ has occured, and the conditional probability, $P_{B\to C}$, that $C$ occurs, given $B$. Classically, the probability, $P_{A\to C}$, is then the product of these summed over all possible intermetiate states $B$:

$$P_{A\to C} = \sum_B P_{A\to B} P_{B\to C}$$

In quantum physics, this is computed differently, using probability amplitudes. Let $\psi_{A\to B}$ be the probability amplitude for event $B$ to occur given that $A$ has occured, and let $\psi_{B\to C}$ be the probability amplitude for event $C$ to occur, given $B$. Then the probabilitiy amplitude for $A$ leading to $C$ is again the sum over all intermediate states of the product,

$$\psi_{A\to C} = \sum_B \psi_{A\to B} \psi_{B\to C}$$

This means that the *quantum probability* for $C$ given $A$ is the absolute *square of the sum*,

$$\begin{aligned} P^{quant}_{A\to C} &= |\psi_{A\to C}|^2 \\ &= \left|\sum_B \psi_{A\to B}\psi_{B\to C}\right|^2 \\ &= \sum_B \psi_{A\to B}\psi_{B\to C} \sum_{B'} \psi^*_{A\to B'}\psi^*_{B'\to C} \end{aligned}$$

We can rewrite this in terms of the classical probabilities, plus extra terms, by separating out the $B = B'$ terms from the $B \neq B'$ terms:

$$\begin{aligned}
P^{quant}_{A\to C} &= \sum_B \sum_{B'} \psi_{A\to B}\psi^*_{A\to B'}\psi_{B\to C}\psi^*_{B'\to C} \\
&= \sum_{B=B'} \left(\psi_{A\to B}\psi^*_{A\to B}\right)\left(\psi_{B\to C}\psi^*_{B\to C}\right) + \sum_B \sum_{B'\neq B} \psi_{A\to B}\psi^*_{A\to B'}\psi_{B\to C}\psi^*_{B'\to C} \\
&= \sum_B P_{A\to B}P_{B\to C} + \sum_B \sum_{B'\neq B} \psi_{A\to B}\psi^*_{A\to B'}\psi_{B\to C}\psi^*_{B'\to C} \\
&= P^{cl}_{A\to C} + \sum_B \sum_{B'\neq B} \psi_{A\to B}\psi^*_{A\to B'}\psi_{B\to C}\psi^*_{B'\to C}
\end{aligned}$$

Both of the probabilities here lie between 0 and 1 for any normalized wave function. However, the conditional probability for the quantum transition from $A$ to $C$ is different from the classical probability. This means that in some physical systems, there may be a greater chance of certain events than would be predicted classically. We now explore a simple, yet disturbing, example.

## 2 Einstein-Podolsky-Rosen (EPR)

Einstein argued that there must be an underlying reality to our experience, in the sense that, if a particle travels from $a$ to $b$, and it has a property $p$ at $a$, and if no process alters that property during its travel time, then it will be measured to have property $p$ at $b$. Einstein, Podolsky and Rosen explored this idea and subsequent authors have honed the arguments. In the simplest version, we consider a particle pair, created with equal but opposite spin, and use the conservation of angular momentum to consider what happens. If Einstein's view is right, the particles have some property (angular momentum), which each particle carries along from its creation to its subsequent detection.

Consider the decay of a single, uncharged, scalar particle (i.e., spin 0, charge 0, angular momentum state $|0,0\rangle$) into an electron-positron pair. If the pair is emitted with no orbital angular momentum, then the final angular momentum is described by the combination of the two spins of the emitted electrons, and from conservation of addition of angular momentum, we must have

$$\begin{aligned}
|0,0\rangle &= \frac{1}{\sqrt{2}}\left(\left|\frac{1}{2},-\frac{1}{2}\right\rangle_{e^-}\left|\frac{1}{2},\frac{1}{2}\right\rangle_{e^+} - \left|\frac{1}{2},\frac{1}{2}\right\rangle_{e^-}\left|\frac{1}{2},-\frac{1}{2}\right\rangle_{e^+}\right) \\
&= \frac{1}{\sqrt{2}}\left(|-\rangle_{e^-}|+\rangle_{e^+} - |+\rangle_{e^-}|-\rangle_{+}\right)
\end{aligned}$$

that is, the electron-positron state must either be a spin down electron $|-\rangle_{e^-}$ with a spin up positron $|+\rangle_{e^+}$, which we write as the product state $|-\rangle_{e^-}|+\rangle_{e^+}$, or it must be a spin up electron with a spin down positron, $|+\rangle_{e^-}|-\rangle_{e^+}$. These occur with equal probability.

Let the emitted particles fly apart for a time, after which observer A measures the electron, then moments later observer B measures the positron. Suppose A measures the component of the electron spin in the $z$-direction. Then A has a $50-50$ chance of getting spin up or spin down. However, if B subsequently measures the spin of the positron, B necessarily gets the opposite result.

This conclusion is seen from the quantum rules. Mathematically, if the first measurement by A shows that the electron has spin up, then the electron-positron pair must now be described by the second term only. Normalizing that revised state of the particle pair,

$$-|+\rangle_{e^-}|-\rangle_{e^+}$$

and the probability for the B to also measure spin up is zero,

$$P_{e^-,up\to e^+,up} = \left|\left(\langle+|_{e^-}\langle+|_{e^+}\right)\left(-|+\rangle_{e^-}|-\rangle_{e^+}\right)\right|^2$$

$$
\begin{aligned}
&= \left|\left(-\langle + \mid +\rangle_{e^-} \langle + \mid -\rangle_{e^+}\right)\right|^2 \\
&= \left|(-1\cdot 0)\right|^2 \\
&= 0
\end{aligned}
$$

The probability that B measures the the second electron to have spin down is one,

$$
\begin{aligned}
P_{e^-,up\to e^+,down} &= \left|\left(\langle +|_{e^-} \langle -|_{e^+}\right)\left(- |+\rangle_{e^-} |-\rangle_{e^+}\right)\right|^2 \\
&= \left|\left(-\langle + \mid +\rangle_{e^-} \langle - \mid -\rangle_{e^+}\right)\right|^2 \\
&= \left|(-1\cdot 1)\right|^2 \\
&= 1
\end{aligned}
$$

So far, there is no obvious conceptual problem. However, suppose just before making the measurement, A decides to measure the $x$-component of spin instead and finds it to be up. $\left|x,+\frac{1}{2}\right\rangle_{e^-}$. Then, since we may write the $\left|x,+\frac{1}{2}\right\rangle_{e^-}$ state in terms of the $z$-basis as $\frac{1}{\sqrt{2}}\left(|+\rangle_{e^-} + |-\rangle_{e^-}\right)$, the normalized state of the system after A's measurement is

$$
\begin{aligned}
|After\ A\rangle &= \frac{1}{\sqrt{2}}\left(|x,+\rangle_{e^-} |+\rangle_{e^+} - |x,+\rangle_{e^-} |-\rangle_{e^+}\right) \\
&= \frac{1}{2}\left(\left(|+\rangle_{e^-} + |-\rangle_{e^-}\right)|+\rangle_{e^+} - \left(|+\rangle_{e^-} + |-\rangle_{e^-}\right)|-\rangle_{e^+}\right) \\
&= \frac{1}{2}\left(|+\rangle_{e^-} |+\rangle_{e^+} + |-\rangle_{e^-} |+\rangle_{e^+} - |+\rangle_{e^-} |-\rangle_{e^+} - |-\rangle_{e^-} |-\rangle_{e^+}\right)
\end{aligned}
$$

This time, if B measures the $z$-component of spin, the probabilities are

$$
\begin{aligned}
P_{e^-,x\,up\to e^+,z\,up} &= \left|\left(\langle x;+|_{e^-} \langle +|_{e^+}\right)|After\ A\rangle\right|^2 \\
&= \left|\left(\langle x;+|_{e^-} \langle +|_{e^+}\right)\frac{1}{2}\left(|+\rangle_{e^-} |+\rangle_{e^+} + |-\rangle_{e^-} |+\rangle_{e^+} - |+\rangle_{e^-} |-\rangle_{e^+} - |-\rangle_{e^-} |-\rangle_{e^+}\right)\right|^2 \\
&= \frac{1}{4}\left|\langle x;+|_{e^-}\left(|+\rangle_{e^-} + |-\rangle_{e^-}\right)\right|^2 \\
&= \frac{1}{4}\left|\frac{1}{\sqrt{2}}\left(\langle +|_{e^-} + \langle -|_{e^-}\right)\left(|+\rangle_{e^-} + |-\rangle_{e^-}\right)\right|^2 \\
&= \frac{1}{8}\left|1+1\right|^2 \\
&= \frac{1}{2}
\end{aligned}
$$

and

$$
\begin{aligned}
P_{e^-,x\,up\to e^+,z\,down} &= \left|\left(\langle x;+|_{e^-} \langle -|_{e^+}\right)|After\ A\rangle\right|^2 \\
&= \left|\left(\langle x;+|_{e^-} \langle -|_{e^+}\right)\frac{1}{2}\left(|+\rangle_{e^-} |+\rangle_{e^+} + |-\rangle_{e^-} |+\rangle_{e^+} - |+\rangle_{e^-} |-\rangle_{e^+} - |-\rangle_{e^-} |-\rangle_{e^+}\right)\right|^2 \\
&= \frac{1}{4}\left|\langle x;+|_{e^-}\left(- |+\rangle_{e^-} - |-\rangle_{e^-}\right)\right|^2 \\
&= \frac{1}{4}\left|\frac{1}{\sqrt{2}}\left(\langle +|_{e^-} + \langle -|_{e^-}\right)\left(|+\rangle_{e^-} + |-\rangle_{e^-}\right)\right|^2 \\
&= \frac{1}{8}\left|1+1\right|^2 \\
&= \frac{1}{2}
\end{aligned}
$$

This is truly strange because the probability for B to measure a given spin depends on which component A decides to measure, even up to the last moment. Since this measurement can take place an instant before the electron arrives at A's location, there is no time for a signal to propagate from A to B to change the second electron's state. This is not something that can happen in a classical system with finite propagation speed.

Einstein, Podolsky and Rosen argue that since no message could have been conveyed from A to B in the time between A deciding what to measure and B making a measurement, that the property A measures must have existed throughout the experiment. They went on to argue that the result means that quantum mechanics is incomplete and that there must be some additional "hidden variables" which characterize that property.

To make this last statement clearer, suppose there is some classical, but unknown, physical property of the electron that determines the outcomes of both A's and B's measurements – a hidden variable. We may even suppose that it is statistical in origin, so that (1) the hidden variable takes its various allowed values with certain probabilities, and (2) the probability of the various possible outcomes of A's and B's measurements depends on which value the hidden variable takes on. Bell's theorem proves an inequality that applies to the distribution of outcomes associated with any such hidden variables picture. We now examine this inequality, and some experiments that test it.

Einstein wrote to Bohr (quoted in Mermin, Physics Today, April 1985),

> That which really exists in B should . . . not depend on what kind of measurement is carried out in part of space A; it should also be independent of whether or not any measurement at all is carried out in space A. If one adheres to this program, one can hardly consider the quantum-theoretical description as a complete representation of the physically real. If one tries to do so in spite of this, one has to assume that the physically real in B suffers a sudden change as a result of a measurement in A. My instinct for physics bristles at this.

# 3 Bell's inequality

Often, the clearest presentation of an idea is found in the original work. In the present discussion, we turn to Bell's original 1964 article, originally published in Physics, 1, 195-200 (1964), and follow the argument there closely.

Consider the EPR experiment with a Stern-Gerlach device at each of two positions, A and B, measuring the spin in directions $\vec{a}$ and $\vec{b}$, respectively. Furthermore, suppose there are one or more hidden variables. Bell writes:

> Let this more complete specification be effected by means of parameters $\lambda$. It is a matter of indifference in the following whether $\lambda$ denotes a single variable or a set, or even a set of functions, and whether the variables are discrete or continuous. However, we write as if $\lambda$ were a single continuous parameter.

Denote by $A(\vec{a},\lambda) = \pm 1$ the result spin up or spin down when the direction $\vec{a}$ is measured at A and by $B\left(\vec{b},\lambda\right) = \pm 1$ the result spin up or spin down when the direction $\vec{b}$ is measured at B. As described by Bell,

> The vital assumption . . . is that the result $B$ for particle 2 does not depend on the setting $\vec{a}$, of the magnet for particle 1, nor $A$ on $\vec{b}$.

Now let $\rho(\lambda)$ be the probability distribution for $\lambda$, certain combinatoins of the hidden variables possibly being more heavily weighted than others. Then we may write the expectation value of the product of the outcomes as the weighted sum and/or integral over all $\lambda$,

$$\langle AB\rangle = E_{AB}\left(\vec{a},\vec{b}\right) = \int \rho(\lambda)\, A(\vec{a},\lambda)\, B\left(\vec{b},\lambda\right) d\lambda$$

We know that the probability distribution $\rho(\lambda)$ is normalized,

$$\int d\lambda \rho(\lambda) = 1$$

and since $A(\vec{a},\lambda)$ and $B\left(\vec{b},\lambda\right)$ take only values$\pm 1$, the expectation must lie in the range $-1 \leq E_{AB}\left(\vec{a},\vec{b}\right) \leq 1$, for all $\vec{a},\vec{b}$. When both observers measure in the same direction $\vec{a}$, the only way the value $E_{AB}\left(\vec{a},\vec{b}\right) = -1$ can be achieved is when

$$A(\vec{a},\lambda) = -B(\vec{a},\lambda)$$

This perfect anticorrelation thereofore exists at least in this special circumstance. We make no assumption about whether the value $-1$ occurs for any other situation. The anticorrelation relation above is enough to show that the distributions $A(\vec{a},\lambda)$ and $-B(\vec{a},\lambda)$ are the same for all $(\vec{a},\lambda)$.

We may therefore replace $B\left(\vec{b},\lambda\right) = -A\left(\vec{b},\lambda\right)$ in the expectation,

$$\begin{aligned} \langle AB\rangle &= \int d\lambda \rho(\lambda) A(\vec{a},\lambda) B\left(\vec{b},\lambda\right) \\ &= -\int d\lambda \rho(\lambda) A(\vec{a},\lambda) A\left(\vec{b},\lambda\right) \end{aligned}$$

Now let $\vec{c}$ be a third orientation of the Stern-Gerlach device and consider the difference

$$E_{AB}\left(\vec{a},\vec{b}\right) - E_{AB}(\vec{a},\vec{c}) = -\int d\lambda \rho(\lambda) A(\vec{a},\lambda) A\left(\vec{b},\lambda\right) + \int d\lambda \rho(\lambda) A(\vec{a},\lambda) A(\vec{c},\lambda)$$

Since $A^2\left(\vec{b},\lambda\right) = (\pm 1)^2 = 1$, we may write this as

$$\begin{aligned} E_{AB}\left(\vec{a},\vec{b}\right) - E_{AB}(\vec{a},\vec{c}) &= \int d\lambda \rho(\lambda) A(\vec{a},\lambda) A^2\left(\vec{b},\lambda\right) A(\vec{c},\lambda) - \int d\lambda \rho(\lambda) A(\vec{a},\lambda) A\left(\vec{b},\lambda\right) \\ &= \int d\lambda \rho(\lambda) A(\vec{a},\lambda) A\left(\vec{b},\lambda\right)\left(A\left(\vec{b},\lambda\right) A(\vec{c},\lambda) - 1\right) \end{aligned}$$

We know that

$$\begin{aligned} -A(\vec{a},\lambda) A\left(\vec{b},\lambda\right) &= \pm 1 \\ \left|-A(\vec{a},\lambda) A\left(\vec{b},\lambda\right)\right| &= 1 \end{aligned}$$

Now look at

$$\begin{aligned} E_{AB}\left(\vec{a},\vec{b}\right) - E_{AB}(\vec{a},\vec{c}) &= \int d\lambda \rho(\lambda) A(\vec{a},\lambda) A\left(\vec{b},\lambda\right)\left(A\left(\vec{b},\lambda\right) A(\vec{c},\lambda) - 1\right) \\ &= \int d\lambda \rho(\lambda)(-1)(\pm 1)\left(1 - A\left(\vec{b},\lambda\right) A(\vec{c},\lambda)\right) \leq \int d\lambda \rho(\lambda)\left(1 - A\left(\vec{b},\lambda\right) A(\vec{c},\lambda)\right) \end{aligned}$$

This means that

$$E_{AB}\left(\vec{a},\vec{b}\right) - E_{AB}(\vec{a},\vec{c}) \leq \int d\lambda \rho(\lambda)\left(1 - A\left(\vec{b},\lambda\right) A(\vec{c},\lambda)\right)$$

We may write the same inequality, interchanging the vectors $\vec{b}$ and $\vec{c}$:

$$\begin{aligned} E_{AB}(\vec{a},\vec{c}) - E_{AB}\left(\vec{a},\vec{b}\right) &= \int d\lambda \rho(\lambda) A(\vec{a},\lambda) A^2(\vec{c},\lambda) A\left(\vec{b},\lambda\right) - \int d\lambda \rho(\lambda) A(\vec{a},\lambda) A(\vec{c},\lambda) \\ &= \int d\lambda \rho(\lambda) A(\vec{a},\lambda) A(\vec{c},\lambda)\left(A\left(\vec{b},\lambda\right) A(\vec{c},\lambda) - 1\right) \\ &\leq \int d\lambda \rho(\lambda)\left(1 - A\left(\vec{b},\lambda\right) A(\vec{c},\lambda)\right) \end{aligned}$$

Therefore,

$$\left|E_{AB}\left(\vec{a},\vec{b}\right)-E_{AB}\left(\vec{a},\vec{c}\right)\right|\leq\int d\lambda\rho\left(\lambda\right)\left(1-A\left(\vec{b},\lambda\right)A\left(\vec{c},\lambda\right)\right)$$

Finally, the right side becomesDropping this product under the integral, then reversing the sign on the right and taking the absolute value on the left,

$$\begin{aligned}\int d\lambda\rho\left(\lambda\right)\left(1-A\left(\vec{b},\lambda\right)A\left(\vec{c},\lambda\right)\right) &= \int d\lambda\rho\left(\lambda\right)-\int d\lambda\rho\left(\lambda\right)A\left(\vec{b},\lambda\right)A\left(\vec{c},\lambda\right)\\ &= 1+E_{AB}\left(\vec{b},\vec{c}\right)\end{aligned}$$

and we arrive at Bell's inequality,

$$1+E_{AB}\left(\vec{b},\vec{c}\right)\geq\left|E_{AB}\left(\vec{a},\vec{b}\right)-E_{AB}\left(\vec{a},\vec{c}\right)\right|$$

# 4 Measurement of spin

To model the measurement of spin, we need a Hermitian operator with eigenvalues $\pm\frac{\hbar}{2}$. If we work in the $z$ basis, the eigenvectors will be

$$\begin{aligned}\left|+\right\rangle &= \begin{pmatrix}1\\0\end{pmatrix}\\ \left|-\right\rangle &= \begin{pmatrix}0\\1\end{pmatrix}\end{aligned}$$

and the Hermitian operator will be diagonal,

$$\hat{S}_z\equiv\frac{\hbar}{2}\sigma_z=\frac{\hbar}{2}\begin{pmatrix}1&0\\0&-1\end{pmatrix}$$

We see immediately that

$$\hat{S}_z\left|\pm\right\rangle=\pm\frac{\hbar}{2}\left|\pm\right\rangle$$

as required.

For a state measured in the $x$direction (still expressed in the $z$basis), we define

$$\hat{S}_x\equiv\frac{\hbar}{2}\sigma_x=\frac{\hbar}{2}\begin{pmatrix}0&1\\1&0\end{pmatrix}$$

Since we know that

$$\begin{aligned}\left|x;+\right\rangle &= \frac{1}{\sqrt{2}}\left(\left|+\right\rangle+\left|-\right\rangle\right)=\frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}\\ \left|x;-\right\rangle &= \frac{1}{\sqrt{2}}\left(\left|+\right\rangle-\left|-\right\rangle\right)=\frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix}\end{aligned}$$

we easily check that

$$\begin{aligned}\hat{S}_x\left|x;+\right\rangle &= \frac{\hbar}{2}\begin{pmatrix}0&1\\1&0\end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}\\ &= \frac{\hbar}{2}\sqrt{2}\begin{pmatrix}1\\1\end{pmatrix}\\ &= \frac{\hbar}{2}\left|x;+\right\rangle\end{aligned}$$

and

$$\begin{aligned}
\hat{S}_x \left| x; - \right\rangle &= \frac{\hbar}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \\
&= \frac{\hbar}{2} \sqrt{2} \begin{pmatrix} -1 \\ 1 \end{pmatrix} \\
&= -\frac{\hbar}{2} \left| x; - \right\rangle
\end{aligned}$$

**Exercise:**

Find the operator with eigenvectors

$$\begin{aligned}
\left| y; + \right\rangle &= \frac{1}{\sqrt{2}} \left( \left| + \right\rangle + i \left| - \right\rangle \right) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix} \\
\left| y; - \right\rangle &= \frac{1}{\sqrt{2}} \left( \left| + \right\rangle - i \left| - \right\rangle \right) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -i \end{pmatrix}
\end{aligned}$$

and eigenvalues $\pm\frac{\hbar}{2}$.

We may define a general spin operator, in direction $\hat{\mathbf{n}}$, by seeking all $2 \times 2$ Hermitian matrices with eigenvalues $\pm\frac{\hbar}{2}$. In a basis which diagonalizes this operator, it must take the form

$$\hat{\mathbf{S}}_{\hat{\mathbf{n}}\, basis} \equiv \frac{\hbar}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

and in addition to being Hermitian it must therefore also be traceless with determinant $-1$. We find the most general traceless, Hermitian $2 \times 2$. Working in the $z$basis, let

$$\hat{\mathbf{S}} \left( \hat{\mathbf{n}} \right) = \frac{\hbar}{2} \begin{pmatrix} \alpha & \beta \\ \mu & -\alpha \end{pmatrix}$$

where we have imposed tracelessness. Then Hermiticity implies

$$\begin{aligned}
\hat{\mathbf{S}}^\dagger \left( \hat{\mathbf{n}} \right) &= \hat{\mathbf{S}}^\dagger \left( \hat{\mathbf{n}} \right) \\
\frac{\hbar}{2} \begin{pmatrix} \alpha^* & \mu^* \\ \beta^* & -\alpha^* \end{pmatrix} &= \frac{\hbar}{2} \begin{pmatrix} \alpha & \beta \\ \mu & \alpha \end{pmatrix}
\end{aligned}$$

We immediately have

$$\begin{aligned}
\alpha^* &= \alpha = c \\
\beta &= \mu^*
\end{aligned}$$

Letting $\mu = a + ib$ with $a, b, c$ real, we have

$$\hat{\mathbf{S}} \left( \hat{\mathbf{n}} \right) = \frac{\hbar}{2} \begin{pmatrix} c & a - ib \\ a + ib & -c \end{pmatrix} = \frac{\hbar}{2} \left( a\sigma_x + b\sigma_y + c\sigma_z \right) = \frac{\hbar}{2} \left( a, b, c \right) \cdot \left( \sigma_x, \sigma_y, \sigma_z \right)$$

The determinant condition now shows that

$$-a^2 - b^2 - c^2 = 1$$

so that $(a,b,c)$ is a unit vector. We define the label of the operator to be $\hat{\mathbf{n}} = (a,b,c)$ ane write

$$\hat{\mathbf{S}}(\hat{\mathbf{n}}) = \frac{\hbar}{2}\hat{\mathbf{n}}\cdot\boldsymbol{\sigma}$$

Now we may define the eigenstates, $|\hat{\mathbf{n}};\pm\rangle$. Let

$$|\hat{\mathbf{n}};+\rangle = \alpha|+\rangle + \beta|-\rangle$$

for complex numbers $\alpha,\beta$, with $|\alpha|^2+|\beta|^2=1$. It follows that the orthogonal state is

$$|\hat{\mathbf{n}};-\rangle = \beta|+\rangle - \frac{\beta}{\beta^*}\alpha^*|-\rangle$$

Asking for these to be the eigenstates of $\hat{\mathbf{S}}(\hat{\mathbf{n}})$ we require

$$\begin{aligned}
\hat{\mathbf{S}}(\hat{\mathbf{n}})|\hat{\mathbf{n}};+\rangle &= \frac{\hbar}{2}|\hat{\mathbf{n}};+\rangle \\
\frac{\hbar}{2}\hat{\mathbf{n}}\cdot\boldsymbol{\sigma}(\alpha|+\rangle+\beta|-\rangle) &= \frac{\hbar}{2}(\alpha|+\rangle+\beta|-\rangle)
\end{aligned}$$

Writing this in matrix notation we find two equations,

$$\begin{aligned}
\begin{pmatrix} c & a-ib \\ a+ib & -c \end{pmatrix}\begin{pmatrix}\alpha\\ \beta\end{pmatrix} &= \begin{pmatrix}\alpha\\ \beta\end{pmatrix} \\
c\alpha + (a-ib)\beta &= \alpha \\
(a+ib)\alpha - c\beta &= \beta
\end{aligned}$$

These give

$$\begin{aligned}
\alpha &= \frac{a-ib}{1-c}\beta \\
\beta &= \frac{a+ib}{1+c}\alpha
\end{aligned}$$

These are equivalent if and only if $(a,b,c)$ is a unit vector. The spin up state is now

$$|\hat{\mathbf{n}};+\rangle = \alpha\left(|+\rangle + \frac{a+ib}{1+c}|-\rangle\right)$$

Normalization requires

$$\begin{aligned}
1 &= \alpha^*\alpha\left(1+\frac{a^2+b^2}{(1+c)^2}\right) \\
&= \alpha^*\alpha\left(\frac{1+2c+c^2+a^2+b^2}{(1+c)^2}\right) \\
&= \alpha^*\alpha\frac{2(1+c)}{(1+c)^2} \\
&= \alpha^*\alpha\frac{2}{1+c} \\
\alpha^*\alpha &= \frac{1+c}{2}
\end{aligned}$$

Taking $\alpha$ real, the spin up and spin down states are therefore

$$\begin{aligned}
|\hat{\mathbf{n}};+\rangle &= \sqrt{\frac{1+c}{2}}\left(|+\rangle + \frac{a+ib}{1+c}|-\rangle\right) \\
&= \frac{1}{\sqrt{2}}\left(\sqrt{1+c}\,|+\rangle + \frac{a+ib}{\sqrt{1+c}}|-\rangle\right) \\
|\hat{\mathbf{n}};-\rangle &= \beta\,|+\rangle - \frac{\beta}{\beta^*}\alpha^*\,|-\rangle \\
&= \frac{a+ib}{1+c}\alpha\,|+\rangle - \frac{\frac{a+ib}{1+c}\alpha}{\frac{a-ib}{1+c}\alpha^*}\alpha^*\,|-\rangle \\
&= \sqrt{\frac{1+c}{2}}\left(\frac{a+ib}{1+c}|+\rangle - \frac{a+ib}{a-ib}|-\rangle\right)
\end{aligned}$$

**Exercise:**

Check orthogonality and normalization of

$$\begin{aligned}
|\hat{\mathbf{n}};+\rangle &= \sqrt{\frac{1+c}{2}}\left(|+\rangle + \frac{a+ib}{1+c}|-\rangle\right) \\
|\hat{\mathbf{n}};-\rangle &= \sqrt{\frac{1+c}{2}}\left(\frac{a+ib}{1+c}|+\rangle - \frac{a+ib}{a-ib}|-\rangle\right)
\end{aligned}$$

In the next section, we will find a more convenient representation for these states.

# 5 Violation of Bell's inequality

Now consider the predictions of quantum mechanics and Bell's theorem.

We know that the normalized quantum expectation value for the two measurements is $\langle\alpha|\,(\boldsymbol{\sigma}_B\cdot\mathbf{b})\,(\boldsymbol{\sigma}_A\cdot\mathbf{a})\,|\alpha\rangle$ and since the particles must be anticorrelated and spin 0,

$$\begin{aligned}
|\alpha\rangle &= \frac{1}{\sqrt{2}}\left(|-\rangle_A\,|+\rangle_B - |+\rangle_A\,|-\rangle_B\right) \\
\langle\alpha| &= \frac{1}{\sqrt{2}}\left(\langle-|_A\,\langle+|_B - \langle+|_A\,\langle-|_B\right)
\end{aligned}$$

Writing this state as a vector,

$$|\alpha\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

Write a general unit vector in spherical coordinates,

$$\begin{aligned}
\hat{\mathbf{a}} &= (a_x, a_y, a_z) \\
&= (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)
\end{aligned}$$

Then a general spin operator may be written as

$$
\begin{aligned}
\frac{\hbar}{2}\hat{\mathbf{a}}\cdot\boldsymbol{\sigma}_A &= \frac{\hbar}{2}\begin{pmatrix} \cos\theta & \sin\theta\cos\varphi - i\sin\theta\sin\varphi \\ \sin\theta\cos\varphi + i\sin\theta\sin\varphi & -\cos\theta \end{pmatrix} \\
&= \frac{\hbar}{2}\begin{pmatrix} \cos\theta & e^{-i\varphi}\sin\theta \\ e^{i\varphi}\sin\theta & -\cos\theta \end{pmatrix}
\end{aligned}
$$

and similarly for $\frac{\hbar}{2}\hat{\mathbf{b}}\cdot\boldsymbol{\sigma}_B$. we have

$$
E_{AB}\left(\hat{\mathbf{a}},\hat{\mathbf{b}}\right) = \frac{\hbar^2}{4}\langle\alpha|\left(\hat{\mathbf{b}}\cdot\boldsymbol{\sigma}_B\right)(\hat{\mathbf{a}}\cdot\boldsymbol{\sigma}_A)|\alpha\rangle
$$

The meaning here is that the $A$ operators act on the $A$ kets, and the $B$ on the $B$. Thus,

$$
\begin{aligned}
E_{AB}\left(\hat{\mathbf{a}},\hat{\mathbf{b}}\right) &= \frac{\hbar^2}{4}\langle\alpha|\left(\hat{\mathbf{b}}\cdot\boldsymbol{\sigma}_B\right)(\hat{\mathbf{a}}\cdot\boldsymbol{\sigma}_A)|\alpha\rangle \\
&= \frac{\hbar^2}{4}\frac{1}{\sqrt{2}}\left(\langle -|_A\langle +|_B - \langle +|_A\langle -|_B\right)\left(\hat{\mathbf{b}}\cdot\boldsymbol{\sigma}_B\right)(\hat{\mathbf{a}}\cdot\boldsymbol{\sigma}_A)\left(\frac{1}{\sqrt{2}}\left(|-\rangle_A|+\rangle_B - |+\rangle_A|-\rangle_B\right)\right) \\
&= \frac{\hbar^2}{8}\left(\langle -|_A\langle +|_B - \langle +|_A\langle -|_B\right)\left((\hat{\mathbf{a}}\cdot\boldsymbol{\sigma}_A)|-\rangle_A\left(\hat{\mathbf{b}}\cdot\boldsymbol{\sigma}_B\right)|+\rangle_B - (\hat{\mathbf{a}}\cdot\boldsymbol{\sigma}_A)|+\rangle_A\left(\hat{\mathbf{b}}\cdot\boldsymbol{\sigma}_B\right)|-\rangle_B\right) \\
&= \frac{\hbar^2}{8}\left(\langle -|_A(\hat{\mathbf{a}}\cdot\boldsymbol{\sigma}_A)|-\rangle_A\langle +|_B\left(\hat{\mathbf{b}}\cdot\boldsymbol{\sigma}_B\right)|+\rangle_B - \langle +|_A(\hat{\mathbf{a}}\cdot\boldsymbol{\sigma}_A)|-\rangle_A\langle -|_B\left(\hat{\mathbf{b}}\cdot\boldsymbol{\sigma}_B\right)|+\rangle_B\right) \\
&\quad -\frac{\hbar^2}{8}\left(\langle -|_A(\hat{\mathbf{a}}\cdot\boldsymbol{\sigma}_A)|+\rangle_A\langle +|_B\left(\hat{\mathbf{b}}\cdot\boldsymbol{\sigma}_B\right)|-\rangle_B - \langle +|_A(\hat{\mathbf{a}}\cdot\boldsymbol{\sigma}_A)|+\rangle_A\langle -|_B\left(\hat{\mathbf{b}}\cdot\boldsymbol{\sigma}_B\right)|-\rangle_B\right)
\end{aligned}
$$

The terms, $\langle -|_A(\hat{\mathbf{a}}\cdot\boldsymbol{\sigma}_A)|-\rangle_A$, simply select out the corresponding components of the matrix $\hat{\mathbf{a}}\cdot\boldsymbol{\sigma}_A$, so we have

$$
\begin{aligned}
E_{AB}\left(\hat{\mathbf{a}},\hat{\mathbf{b}}\right) &= \frac{\hbar^2}{8}\left(-\cos\theta_A\cos\theta_B - e^{-i\varphi_A}\sin\theta_A e^{i\varphi_B}\sin\theta_B\right) \\
&\quad -\frac{\hbar^2}{8}\left(e^{i\varphi_A}\sin\theta_A e^{-i\varphi_B}\sin\theta_B + \cos\theta_A\cos\theta_B\right) \\
&= \frac{\hbar^2}{8}\left(-2\cos\theta_A\cos\theta_B - e^{-i(\varphi_A-\varphi_B)}\sin\theta_A\sin\theta_B - e^{i(\varphi_A-\varphi_B)}\sin\theta_A\sin\theta_B\right) \\
&= -\frac{\hbar^2}{4}\left(\sin\theta_A\sin\theta_B\cos(\varphi_A-\varphi_B) - \cos\theta_A\cos\theta_B\right)
\end{aligned}
$$

Noticing that

$$
\begin{aligned}
\hat{\mathbf{a}}\cdot\hat{\mathbf{b}} &= (\sin\theta_A\cos\varphi_A, \sin\theta_A\sin\varphi_A, \cos\theta_A)\cdot(\sin\theta_B\cos\varphi_B, \sin\theta_B\sin\varphi_B, \cos\theta_B) \\
&= \sin\theta_A\cos\varphi_A\sin\theta_B\cos\varphi_B + \sin\theta_A\sin\varphi_A\sin\theta_B\sin\varphi_B + \cos\theta_A\cos\theta_B \\
&= \sin\theta_A\sin\theta_B(\cos\varphi_A\cos\varphi_B + \sin\varphi_A\sin\varphi_B) + \cos\theta_A\cos\theta_B \\
&= \sin\theta_A\sin\theta_B\cos(\varphi_A-\varphi_B) + \cos\theta_A\cos\theta_B
\end{aligned}
$$

we have

$$
E_{AB}\left(\hat{\mathbf{a}},\hat{\mathbf{b}}\right) = -\frac{\hbar^2}{4}\hat{\mathbf{a}}\cdot\hat{\mathbf{b}}
$$

or, normalizing by $\frac{\hbar^2}{4}$, simply

$$
E_{AB}\left(\hat{\mathbf{a}},\hat{\mathbf{b}}\right) = -\hat{\mathbf{a}}\cdot\hat{\mathbf{b}}
$$

Now, again following Bell, let the directions **a**,**b** and **c** be at angles $0, \theta$ and $2\theta$, respectively, with $\theta \ll 1$. Then the correlation probabilities are

$$\begin{aligned}
E_{AB}(\mathbf{a},\mathbf{b}) &= -\cos\theta \\
&\approx -1+\frac{1}{2}\theta^2 \\
E_{AB}(\mathbf{b},\mathbf{c}) &= -\cos\theta \\
&\approx -1+\frac{1}{2}\theta^2 \\
E_{AB}(\mathbf{a},\mathbf{c}) &= -\cos 2\theta \\
&\approx -1+2\theta^2
\end{aligned}$$

and substituting into Bell's inequality,

$$\begin{aligned}
1+E_{AB}\left(\vec{b},\vec{c}\right) &\geq \left|E_{AB}\left(\vec{a},\vec{b}\right)-E_{AB}\left(\vec{a},\vec{c}\right)\right| \\
1-1+\frac{1}{2}\theta^2 &\geq \left|-1+\frac{1}{2}\theta^2-\left(-1+2\theta^2\right)\right| \\
\frac{1}{2}\theta^2 &\geq \frac{3}{2}\theta^2
\end{aligned}$$

giving a clear contradiction in a quantum mechanical system. Bell goes on to show that the quantum violation of the Bell inequality is not always infinitesimal. It is sufficient, however to consider the angles 0, $\frac{\pi}{6}$ and $\frac{\pi}{3}$, for then

$$\begin{aligned}
E_{AB}(\mathbf{a},\mathbf{b}) &= -\cos\frac{\pi}{6}=-\frac{\sqrt{3}}{2} \\
E_{AB}(\mathbf{b},\mathbf{c}) &= -\frac{\sqrt{3}}{2} \\
E_{AB}(\mathbf{a},\mathbf{c}) &= -\cos\frac{\pi}{3}=-\frac{1}{2}
\end{aligned}$$

from which we find the finite violation of the inequality

$$\begin{aligned}
1+E_{AB}\left(\vec{b},\vec{c}\right) &\geq^? \left|E_{AB}\left(\vec{a},\vec{b}\right)-E_{AB}\left(\vec{a},\vec{c}\right)\right| \\
1-\frac{\sqrt{3}}{2} &\geq^? \left|-\frac{1}{2}+\frac{\sqrt{3}}{2}\right| \\
\frac{2-\sqrt{3}}{2} &\ngeq \frac{\sqrt{3}-1}{2} \\
.134 &\ngeq .366
\end{aligned}$$

# 6 Mermin's example

In his Physics Today article, D. Mermin presents a non-quantum example of the violation of the Bell inequality. The set-up is simple, and corresponds the bare essentials of what is seen quantum mechanically. There are two remote observers, $A$ and $B$, and each can choose one of three things to measure, $1, 2, 3$. Whatever they choose, there are two possible outcomes, $R, G$. Therefore, a complete run consists of a set such as $12RG$, where the first number, 1 in this example, is the setting observer $A$ chooses, the second number, 2, is what $B$ chooses, then $A$ gets the result $R$ and $B$ gets the result $G$. The experiment is repeated many times, each run yielding a sequence of two numbers and two letters, $32RR, 33GG, 13GR, \ldots$.

Mermin makes it clear that we should picture some physical thing traveling from the central apparatus, out to $A$ and $B$, arguing that their devices respond only when that apparatus is triggered, and they respond after a time proportional to their distance from the center. No signal occurs if the line-of-sight is blocked.

There are two features of the results that lead to the conflict with a simple picture of what is happening. Both of the following occur:

1. If both $A$ and $B$ choose the same setting, $11, 22$ or $33$, they get the same result every time, either $RR$ or $GG$.

2. If we ignore the settings of the devices and just look at the outcomes, they are completely random.

The first means that one of the following always occurs if the settings agree:

$$\begin{array}{ccc} 11GG & 11RR & 22GG \\ 22RR & 33GG & 33RR \end{array}$$

while the second condition means that $RR, RG, GR, GG$ are equally probable.

Now, the hidden variables hypothesis says that because of the perfect correlation of point 1, the physical emanation from the center must carry a corresponding property that determines which outcome occurs for each setting, for example, $1R2R3G$, meaning if 1 or 2 is selected by either observer the outcome will be $R$, if 3 is set the outcome will be $G$. This triple applies to both observers. If this were not the case there is no guarantee (without "spukhafte Fernwirkungen") that if $A$ and $B$ both choose 2 they will both get $R$.

This "property" of the "particle" has to be specified for all three settings and every run, because $A$ and $B$ can choose their settings anytime before the particle reaches them. The production of the particle occurs before the choice is made.

Now compare probabilities. There are eight different outcome properties the particle may carry:

$$\begin{array}{c} 1G2G3G \\ 1R2G3G \\ 1G2R3G \\ 1G2G3R \\ 1R2R3G \\ 1R2G3R \\ 1G2R3R \\ 1R2R3R \end{array}$$

These insure that if the same property is measured by $A$ and $B$, the same result occurs. However, suppose the property is, for example, $1R2G3G$. Then for the various settings $A$ and $B$ choose, we get the outcomes

$$\begin{array}{c} 11RR \\ 12RG \\ 13RG \\ 22GG \\ 23GG \\ 24GG \end{array}$$

so that for this property, $RR$ and $GG$ are more probable than $RG$ and $GR$. The same is true for every set of properties, so the results cannot be random.

Let's do the full counting. There are six possible settings of the measurement devices, $11, 12, 13, 22, 23, 33$, and eight possible sets of properties. For either $1G2G3G$ or $1R2R3R$ we always get either $GG$ or $RR$, for all six setttings, giving 12 measurement settings where we always get $GG$ or $RR$. For any of the remaining six property sets, there are four ways to get $RR$ or $GG$ and only two to get $RG$ or $GR$, so the tally is:

$$\begin{aligned}(RR \textit{ or } GG) &= 6 + 6 * 4 + 6 = 36 \\ (RG \textit{ or } GR) &= 0 + 6 * 2 + 0 = 12\end{aligned}$$

instead of 24 for both. Different colors occur only 1/4 of the time, the same colors 3/4. The results are inconsistent with the particles carrying property sets.

## 7 The difference between classical and quantum descriptions

The use of hidden variables is the principal difference between the classical and quantum descriptions of nature. In the classical description of a particle trajectory, for example, we assume that the particle follows a unique trajectory from its initial to its final points. This trajectory – the one given by Newton's second law – is often not actually measured, although in many cases the (approximate!) path may be directly visible to us. However, especially when we no longer deal with the macroscopic realm, we only *assume* such a unique path. This is an example of using a *hidden variable* – we assume the particle has definite physical properties (certain values of position, momentum, energy, angular momentum, etc.) throughout its trajectory, even though we do not *know* these, where by "knowing" we mean measuring. In Mermin's example, we assume the particle or emanation, has one of the sets of properties $1G2G3G - 1R2R3R$ listed above, and this conflicts with EPR results. In the same way, we are tempted by our classical thinking to assume that an electron has a definite spin vector, even though we have measured only, say, the $z$-component. This also turns out to be wrong.

In the quantum mechanical picture, the state provides a list that perfectly characterizes what we do and do not know about the system. Each item in the list is constrained by any measurements we have previously performed, or conservation laws which we know hold. At the same time, there are multiple items in the list. These multiple items reflect the fact that we do *not* know which of various things will be found. Thus, in the state for the electron-positron pair in an EPR experiment, where we concern ourselves only with spin variables, the state is the singlet,

$$\begin{aligned}|0,0\rangle &= \frac{1}{\sqrt{2}}\left(\chi^{-}_{e^-}\chi^{+}_{e^+} - \chi^{+}_{e^-}\chi^{-}_{e^+}\right) \\ &= \frac{1}{\sqrt{2}}\left(\left|\frac{1}{2},-\frac{1}{2}\right\rangle_{e^-}\left|\frac{1}{2},\frac{1}{2}\right\rangle_{e^+} - \left|\frac{1}{2},\frac{1}{2}\right\rangle_{e^-}\left|\frac{1}{2},-\frac{1}{2}\right\rangle_{e^+}\right)\end{aligned}$$

where $\chi^{+}_{e^-} = \left|\frac{1}{2},\frac{1}{2}\right\rangle_{e^-}$ is the state of a spin-up electron, and so on. Each term in this sum is constrained by conservation of angular momentum, $s_{e^-,z} + s_{e^+,z} = 0$, since we know this must hold. We then have a sum of all possible terms satisfying this one known constraint, $\chi^{-}_{e^-}\chi^{+}_{e^+}$ and $\chi^{+}_{e^-}\chi^{-}_{e^+}$, but not the terms $\chi^{+}_{e^-}\chi^{+}_{e^+}$ or $\chi^{-}_{e^-}\chi^{-}_{e^+}$. The *relative* phase, $e^{i\pi} = -1$, insures that the total angular momentum ($j$) is zero, while we know that the *overall* phase is arbitrary, i.e., we could equally well write

$$|0,0\rangle = \frac{1}{\sqrt{2}}e^{i\varphi}\left(\chi^{-}_{e^-}\chi^{+}_{e^+} - \chi^{+}_{e^-}\chi^{-}_{e^+}\right)$$

for any angle $\varphi$. The rules for making predictions from quantum mechanical states insure that this factor will not contribute to predictions.

The path integral approach to quantum mechanics or quantum field theory makes this clear as well. To compute the probability amplitude for a system to evolve from a state A to a state B, this formalism tells us to sum over all possible internal configurations, inserting appropriate phases for each possibility. This is written as

$$\int \mathcal{D}\left[x\left(t\right)\right]\exp\left(\frac{i}{\hbar}S\left[x\left(t\right)\right]\right)$$

where $S[x(t)]$ is the action functional for the system and $\int \mathcal{D}[x(t)]$ means that we sum or integrate over *all* paths $x(t)$ that satisfy the initial and final conditions. Again, we see that we enforce the measured conditions at the endpoints while summing over all of the unmeasured paths. Recall that the classical paths are those which make the variation of the action vanish, $\delta S = 0$, and it is this which makes the classical picture emerge from the quantum one. The phase in the exponential, having $\hbar$ in the denominator, will oscillate rapidly if $S$ is changing. Near the stable points of the action, however, the phases add coherently, so the principal contribution to a path integral comes from paths very close to the classical one. Symmetries of the action, which lead to conservation laws by Noether's theorem, contribute to the result strongly as well, since $S$ does not change when we perform a symmetry transformation.

# References


[1] John S. Bell, *On the Einstein Podolsky Rosen paradox,* Physics 1, (1964) pp 195-200.

[2] N. David Mermin, *Is the moon there when nobody looks? Reality and the quantum theory*, Physics Today (April 1985) pp 38-47.

[3] Wikipedia, *Bell's Theorem.*