Angular Momentum

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1 Angular momentum

Sakurai makes an important point when he notes that what we call "angular momentum" is not just $\mathbf{r}\times\mathbf{p}$. Instead, we will see that $\mathbf{r}\times\mathbf{p}$ only describes orbital angular momentum, when a mass moves in a circle. But we must also include *intrinsic spin* which cannot be cast in this form.

To define what we mean by angular momentum, we generalize the algebra of rotations. The situation is similar to what happens with translations and momentum. In that case, we found that translations could be described by a unitary operator. The translation operator, acting on an eigenstate of position, gives a position eigenstate with a different position eigenvalue. Because the translation operator is unitary, it has an infinitesimal generator which is Hermitian. This Hermitian operator is an observable which we identify with momentum. Thus, we see a direct relationship between the symmetry – translations – and the conserved physical property – momentum.

A similar relationship holds for angular momentum. We will find a set of unitary operators to describe rotations, and the generators of these operators will be our angular momentum operators. We shall show that this procedure leads to different kinds of states, some with orbital angular momentum, some with intrinsic spin, and some with both.

2 Rotations

We begin with ordinary 3-dimensional rotations. Rotations around the x, y and z axes are given by

$$\begin{pmatrix}1&0&0\\0&\cos\theta&-\sin\theta\\0&\sin\theta&\cos\theta\end{pmatrix},\begin{pmatrix}\cos\theta&0&\sin\theta\\0&1&0\\-\sin\theta&0&\cos\theta\end{pmatrix},\begin{pmatrix}\cos\theta&\sin\theta&0\\\sin\theta&\cos\theta&0\\0&0&1\end{pmatrix}$$

Any rotation of a 3-vector may be specified by an axis of rotation, $\mathbf{n}$, and an angle, φ.

Rotations preserve lengths, but the length we are concerned with in quantum mechanis is the hermitian norm of a state:

$$\langle\alpha\,|\alpha\rangle$$

Now let a rotation of the state $|\alpha\rangle$ be denoted by

$$|\tilde{\alpha}\rangle=\mathscr{D}\left(\mathbf{n},\varphi\right)|\alpha\rangle$$

so that

$$\langle\tilde{\alpha}|=\langle\alpha|\,\mathscr{D}^{\dagger}\left(\mathbf{n},\varphi\right)$$

Then preservation of the norm amounts to

$$\begin{aligned}\langle\alpha\,|\alpha\rangle &= \langle\tilde{\alpha}\,|\tilde{\alpha}\rangle\\ &= \langle\alpha|\,\mathscr{D}^{\dagger}\left(\mathbf{n},\varphi\right)\mathscr{D}\left(\mathbf{n},\varphi\right)|\alpha\rangle\end{aligned}$$

We conclude that $\mathscr{D}(\mathbf{n},\varphi)$ must be unitary,

$$\mathscr{D}^{\dagger}(\mathbf{n},\varphi)\,\mathscr{D}(\mathbf{n},\varphi)=\hat{1}$$

This means that $\mathscr{D}^{\dagger}(\mathbf{n},\varphi)=\mathscr{D}^{-1}(\mathbf{n},\varphi)=\mathscr{D}(\mathbf{n},-\varphi)$.

Next, we find the infinitesimal generators. Expanding for a small angle, φ,

$$\mathscr{D}(\mathbf{n},\varphi)=\hat{1}-\frac{i\varphi}{\hbar}\mathbf{n}\cdot\hat{\mathbf{J}}$$

we require the three operators, $\hat{\mathbf{J}}=\left(\hat{J}_1,\hat{J}_2,\hat{J}_3\right)$ to be hermitian. Expanding, infinitesimally, to second order

$$\begin{aligned}\mathscr{D}(\mathbf{n},\varphi) &= \left(\hat{1}-\frac{i\varphi}{\hbar}\mathbf{n}\cdot\hat{\mathbf{J}}+\frac{1}{2}\left(-\frac{i\varphi}{\hbar}\mathbf{n}\cdot\hat{\mathbf{J}}\right)^2+\ldots\right)\\ &= \hat{1}-\frac{i\varphi}{\hbar}\mathbf{n}\cdot\hat{\mathbf{J}}-\frac{\varphi^2}{2\hbar^2}\left(\mathbf{n}\cdot\hat{\mathbf{J}}\right)^2+\ldots\end{aligned}$$

These operators must satisfy certain basic properties of rotations. In particular, we know that any two rotations are equivalent to some third rotation,

$$\mathscr{D}(\mathbf{n},\varphi)\,\mathscr{D}(\mathbf{m},\theta)=\mathscr{D}(\mathbf{l},\chi)$$

It follows that a finite sequence of rotations is also equivalent to a single rotation. Consider the combination

$$\mathscr{D}^{\dagger}(\mathbf{n},\varphi)\,\mathscr{D}^{\dagger}(\mathbf{m},\theta)\,\mathscr{D}(\mathbf{n},\varphi)\,\mathscr{D}(\mathbf{m},\theta)=\mathscr{D}(\mathbf{l},\chi)$$

This must hold for some $\mathbf{l},\chi$. Keeping terms to second order,

$$\begin{aligned}\hat{1}-\frac{i\chi}{\hbar}\mathbf{l}\cdot\hat{\mathbf{J}} &= \left(\hat{1}+\frac{i\varphi}{\hbar}\mathbf{n}\cdot\hat{\mathbf{J}}-\frac{\varphi^2}{2\hbar^2}\left(\mathbf{n}\cdot\hat{\mathbf{J}}\right)^2\right)\left(\hat{1}+\frac{i\theta}{\hbar}\mathbf{m}\cdot\hat{\mathbf{J}}-\frac{\theta^2}{2\hbar^2}\left(\mathbf{m}\cdot\hat{\mathbf{J}}\right)^2\right)\\ &\quad\times\left(\hat{1}-\frac{i\varphi}{\hbar}\mathbf{n}\cdot\hat{\mathbf{J}}-\frac{\varphi^2}{2\hbar^2}\left(\mathbf{n}\cdot\hat{\mathbf{J}}\right)^2\right)\left(\hat{1}-\frac{i\theta}{\hbar}\mathbf{m}\cdot\hat{\mathbf{J}}-\frac{\theta^2}{2\hbar^2}\left(\mathbf{m}\cdot\hat{\mathbf{J}}\right)^2\right)\\ &= \left(\hat{1}+\frac{i\theta}{\hbar}\mathbf{m}\cdot\hat{\mathbf{J}}+\frac{i\varphi}{\hbar}\mathbf{n}\cdot\hat{\mathbf{J}}-\frac{\theta^2}{2\hbar^2}\left(\mathbf{m}\cdot\hat{\mathbf{J}}\right)^2+\frac{i\varphi}{\hbar}\mathbf{n}\cdot\hat{\mathbf{J}}\frac{i\theta}{\hbar}\mathbf{m}\cdot\hat{\mathbf{J}}-\frac{\varphi^2}{2\hbar^2}\left(\mathbf{n}\cdot\hat{\mathbf{J}}\right)^2\right)\\ &\quad\times\left(\hat{1}-\frac{i\theta}{\hbar}\mathbf{m}\cdot\hat{\mathbf{J}}-\frac{i\varphi}{\hbar}\mathbf{n}\cdot\hat{\mathbf{J}}-\frac{\theta^2}{2\hbar^2}\left(\mathbf{m}\cdot\hat{\mathbf{J}}\right)^2+\frac{i\varphi}{\hbar}\mathbf{n}\cdot\hat{\mathbf{J}}\frac{i\theta}{\hbar}\mathbf{m}\cdot\hat{\mathbf{J}}-\frac{\varphi^2}{2\hbar^2}\left(\mathbf{n}\cdot\hat{\mathbf{J}}\right)^2\right)\\ &= \hat{1}-\frac{i\theta}{\hbar}\mathbf{m}\cdot\hat{\mathbf{J}}-\frac{i\varphi}{\hbar}\mathbf{n}\cdot\hat{\mathbf{J}}-\frac{\theta^2}{2\hbar^2}\left(\mathbf{m}\cdot\hat{\mathbf{J}}\right)^2+\frac{i\varphi}{\hbar}\mathbf{n}\cdot\hat{\mathbf{J}}\frac{i\theta}{\hbar}\mathbf{m}\cdot\hat{\mathbf{J}}-\frac{\varphi^2}{2\hbar^2}\left(\mathbf{n}\cdot\hat{\mathbf{J}}\right)^2\\ &\quad+\left(\frac{i\theta}{\hbar}\mathbf{m}\cdot\hat{\mathbf{J}}+\frac{i\varphi}{\hbar}\mathbf{n}\cdot\hat{\mathbf{J}}\right)\left(\hat{1}-\frac{i\theta}{\hbar}\mathbf{m}\cdot\hat{\mathbf{J}}-\frac{i\varphi}{\hbar}\mathbf{n}\cdot\hat{\mathbf{J}}\right)\\ &\quad-\frac{\theta^2}{2\hbar^2}\left(\mathbf{m}\cdot\hat{\mathbf{J}}\right)^2+\frac{i\varphi}{\hbar}\mathbf{n}\cdot\hat{\mathbf{J}}\frac{i\theta}{\hbar}\mathbf{m}\cdot\hat{\mathbf{J}}-\frac{\varphi^2}{2\hbar^2}\left(\mathbf{n}\cdot\hat{\mathbf{J}}\right)^2\\ &= \hat{1}-\frac{i\theta}{\hbar}\mathbf{m}\cdot\hat{\mathbf{J}}+\frac{i\theta}{\hbar}\mathbf{m}\cdot\hat{\mathbf{J}}+\frac{i\varphi}{\hbar}\mathbf{n}\cdot\hat{\mathbf{J}}-\frac{i\varphi}{\hbar}\mathbf{n}\cdot\hat{\mathbf{J}}\\ &\quad+\left(\frac{\theta^2}{\hbar^2}-\frac{\theta^2}{2\hbar^2}-\frac{\theta^2}{2\hbar^2}\right)\left(\mathbf{m}\cdot\hat{\mathbf{J}}\right)^2\\ &\quad+\left(\frac{\varphi^2}{\hbar^2}-\frac{\varphi^2}{2\hbar^2}-\frac{\varphi^2}{2\hbar^2}\right)\left(\mathbf{n}\cdot\hat{\mathbf{J}}\right)^2\\ &\quad-\frac{i\varphi}{\hbar}\mathbf{n}\cdot\hat{\mathbf{J}}\frac{i\theta}{\hbar}\mathbf{m}\cdot\hat{\mathbf{J}}+\frac{i\varphi}{\hbar}\mathbf{n}\cdot\hat{\mathbf{J}}\frac{i\theta}{\hbar}\mathbf{m}\cdot\hat{\mathbf{J}}+\frac{i\varphi}{\hbar}\mathbf{n}\cdot\hat{\mathbf{J}}\frac{i\theta}{\hbar}\mathbf{m}\cdot\hat{\mathbf{J}}-\frac{i\theta}{\hbar}\mathbf{m}\cdot\hat{\mathbf{J}}\frac{i\varphi}{\hbar}\mathbf{n}\cdot\hat{\mathbf{J}}\\ &= \hat{1}-\frac{\varphi\theta}{\hbar^2}\left(\mathbf{n}\cdot\hat{\mathbf{J}}\mathbf{m}\cdot\hat{\mathbf{J}}-\mathbf{m}\cdot\hat{\mathbf{J}}\mathbf{n}\cdot\hat{\mathbf{J}}\right)\end{aligned}$$

we may identify

$$i\hbar\chi l_k \hat{J}_k = \varphi\theta n_i m_j \left[\hat{J}_i, \hat{J}_j\right]$$

This must hold for all $\varphi, \theta, n_i, m_j$. Introducing coefficients, c_{ijk}, such that

$$\left[\hat{J}_i, \hat{J}_j\right] = i\hbar \sum_{k=1}^{3} c_{ijk} \hat{J}_k$$

we have

$$\chi l_k = c_{ijk} \varphi\theta n_i m_j$$

for the parameters. We may find the coefficients c_{ijk} by computing in any particular case. For the real, 3-dimensional rotations above, we expand infinitesimally. With $\cos\theta \approx 1, \sin\theta \approx \theta$, we find,

$$\begin{aligned}
\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix} &\approx \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \theta \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \\
\begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix} &\approx \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \theta \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} \\
\begin{pmatrix} \cos\theta & \sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} &\approx \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \theta \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}
\end{aligned}$$

Defining

$$\begin{aligned}
\bar{J}_x &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \\
\bar{J}_y &= \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} \\
\bar{J}_z &= \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}
\end{aligned}$$

we easily compute

$$\left[\bar{J}_x, \bar{J}_y\right] = \bar{J}_z$$

and in general,

$$\begin{aligned}
\left[\bar{J}_i, \bar{J}_j\right] &= \sum_{k=1}^{3} \varepsilon_{ijk} \bar{J}_k \\
&= \varepsilon_{ijk} \bar{J}_k
\end{aligned}$$

for i, j, k each ranging over all three indices. This shows that c_{ijk} is proportional to the Levi-Civita tensor, so with a suitable normalization of our general generators, J_i, we have

$$\left[\hat{J}_i, \hat{J}_j\right] = i\hbar\varepsilon_{ijk} \hat{J}_k$$

where we include a factor of i to make the real, anti-symmetric matrices $\bar{J}$ hermitian. This final relationship holds for *any linear representation* of rotations. Given any three objects, J_i, satisfying this set of commutation relations, we may use them to generate any finite rotation by taking the limit

$$\begin{aligned}\mathscr{D}(\mathbf{n},\varphi) &= \lim_{n\to\infty}\left(\hat{1}-\frac{i\varepsilon}{\hbar}\mathbf{n}\cdot\hat{\mathbf{J}}\right)^n \\ &= \exp\left(-\frac{i\varphi}{\hbar}\mathbf{n}\cdot\hat{\mathbf{J}}\right)\end{aligned}$$

where $\varphi = \lim_{n\to\infty} n\varepsilon$.

3 Examples

3.1 Spin-1/2

We have already seen that the Pauli matrices satisfy

$$[\sigma_i,\sigma_j] = 2i\varepsilon_{ijk}\sigma_k$$

Therefore, if we set

$$\tau_i = \frac{\hbar}{2}\sigma_i$$

then

$$[\tau_i,\tau_j] = i\hbar\varepsilon_{ijk}\sigma_k$$

are suitable generators and a finite rotation is given by

$$\begin{aligned}\mathscr{D}(\mathbf{n},\varphi) &= \exp\left(-\frac{i\varphi}{\hbar}\mathbf{n}\cdot\boldsymbol{\tau}\right) \\ &= \exp\left(-\frac{i\varphi}{2}\mathbf{n}\cdot\boldsymbol{\sigma}\right)\end{aligned}$$

This 2-dim matrix is easily found from the expansion for the exponential,

$$\exp\left(-\frac{i\varphi}{2}\mathbf{n}\cdot\boldsymbol{\sigma}\right) = \sum_{k=0}^{\infty}\frac{1}{k!}\left(-\frac{i\varphi}{2}\right)^k(\mathbf{n}\cdot\boldsymbol{\sigma})^k$$

Powers of $\mathbf{n}\cdot\boldsymbol{\sigma}$ are given by first computing

$$\begin{aligned}(\mathbf{n}\cdot\boldsymbol{\sigma})^2 &= n_i\sigma_i n_j\sigma_j \\ &= n_i n_j(\sigma_i\sigma_j) \\ &= n_i n_j(\delta_{ij}\mathbf{1}+i\varepsilon_{ijk}\sigma_k) \\ &= (\mathbf{n}\cdot\mathbf{n})\mathbf{1}+i(\mathbf{n}\times\mathbf{n})\cdot\boldsymbol{\sigma} \\ &= 1\end{aligned}$$

Then

$$\begin{aligned}(\mathbf{n}\cdot\boldsymbol{\sigma})^3 &= (\mathbf{n}\cdot\boldsymbol{\sigma})^2(\mathbf{n}\cdot\boldsymbol{\sigma}) \\ &= (\mathbf{n}\cdot\boldsymbol{\sigma})\end{aligned}$$

and iterating we have

$$\begin{aligned}(\mathbf{n}\cdot\boldsymbol{\sigma})^{2n} &= \mathbf{1} \\ (\mathbf{n}\cdot\boldsymbol{\sigma})^{2n+1} &= \mathbf{n}\cdot\boldsymbol{\sigma}\end{aligned}$$

The power series for the exponential is therefore

$$\begin{aligned}
\exp\left(-\frac{i\varphi}{2}\mathbf{n}\cdot\boldsymbol{\sigma}\right) &= \sum_{k=0}^{\infty}\frac{1}{k!}\left(-\frac{i\varphi}{2}\right)^{k}(\mathbf{n}\cdot\boldsymbol{\sigma})^{k} \\
&= \sum_{m=0}^{\infty}\frac{1}{k!}\left(-\frac{i\varphi}{2}\right)^{2m}(\mathbf{n}\cdot\boldsymbol{\sigma})^{2m}+\sum_{m=0}^{\infty}\frac{1}{k!}\left(-\frac{i\varphi}{2}\right)^{2m+1}(\mathbf{n}\cdot\boldsymbol{\sigma})^{2m+1} \\
&= \mathbf{1}\sum_{m=0}^{\infty}\frac{(-1)^{m}}{k!}\left(\frac{\varphi}{2}\right)^{2m}-i\mathbf{n}\cdot\boldsymbol{\sigma}\sum_{m=0}^{\infty}\frac{(-1)^{m}}{k!}\left(\frac{\varphi}{2}\right)^{2m+1} \\
&= \mathbf{1}\cos\frac{\varphi}{2}-i\mathbf{n}\cdot\boldsymbol{\sigma}\sin\frac{\varphi}{2}
\end{aligned}$$

To rotate a spinor, $\chi=\begin{pmatrix}\alpha\\ \beta\end{pmatrix}$, we act with this $\mathscr{D}(\mathbf{n},\varphi)$,

$$\chi'=\mathscr{D}(\mathbf{n},\varphi)\,\chi$$

Thus, starting with an up ket in the z-direction and rotating first around y by θ and then around z by φ,

$$\begin{aligned}
\chi(\theta,\varphi) &= \mathscr{D}(\mathbf{k},\varphi)\,\mathscr{D}(\mathbf{j},\theta)\begin{pmatrix}1\\0\end{pmatrix} \\
&= \mathscr{D}(\mathbf{k},\varphi)\left(1\cos\frac{\theta}{2}-i\sigma_{y}\sin\frac{\theta}{2}\right)\begin{pmatrix}1\\0\end{pmatrix} \\
&= \mathscr{D}(\mathbf{k},\varphi)\begin{pmatrix}\cos\frac{\theta}{2}\\ \sin\frac{\theta}{2}\end{pmatrix} \\
&= \left(1\cos\frac{\varphi}{2}-i\sigma_{z}\sin\frac{\varphi}{2}\right)\begin{pmatrix}\cos\frac{\theta}{2}\\ \sin\frac{\theta}{2}\end{pmatrix} \\
&= \begin{pmatrix}\cos\frac{\theta}{2}\\ \sin\frac{\theta}{2}\end{pmatrix}\cos\frac{\varphi}{2}-\begin{pmatrix}\cos\frac{\theta}{2}\\ -\sin\frac{\theta}{2}\end{pmatrix}i\sin\frac{\varphi}{2} \\
&= \begin{pmatrix}e^{-\frac{i\varphi}{2}}\cos\frac{\theta}{2}\\ e^{\frac{i\varphi}{2}}\sin\frac{\theta}{2}\end{pmatrix} \\
&= e^{-\frac{i\varphi}{2}}\begin{pmatrix}\cos\frac{\theta}{2}\\ e^{i\varphi}\sin\frac{\theta}{2}\end{pmatrix}
\end{aligned}$$

This a phase times the general $\mathbf{n}\cdot\hat{\mathbf{S}}$ eigenket,

$$\begin{aligned}
\chi(\theta,\varphi) &= e^{-\frac{i\varphi}{2}}\left|\mathbf{n}\cdot\hat{\mathbf{S}},+\right\rangle \\
&= e^{-\frac{i\varphi}{2}}\left[\cos\frac{\theta}{2}\left|+\right\rangle+e^{i\varphi}\sin\frac{\theta}{2}\left|-\right\rangle\right]
\end{aligned}$$

3.2 Real 3-vectors in a complex representation

We may represent a real 3-vector, $\mathbf{a}=(a_1,a_2,a_3)$ as a 2×2, traceless, hermitian matrix,

$$\mathbf{a}\cdot\boldsymbol{\sigma}=\begin{pmatrix}a_3 & a_1-ia_2\\ a_1+ia_2 & -a_3\end{pmatrix}$$

This is a 2-dim complex representation of a real, 3-dim vector. The length is given by

$$(\mathbf{a}\cdot\boldsymbol{\sigma})^{2} = a_i\sigma_i a_j\sigma_j$$

$$
\begin{aligned}
&= a_i a_j (\sigma_i \sigma_j) \\
&= a_i a_j (\delta_{ij} + i\varepsilon_{ijk}\sigma_k) \\
&= \mathbf{a}\cdot\mathbf{a}
\end{aligned}
$$

Since it is a matrix, it rotates with two copies of the rotation matrix,

$$
(\mathbf{b}\cdot\boldsymbol{\sigma}) = \mathscr{D}(\mathbf{n},\varphi)(\mathbf{a}\cdot\boldsymbol{\sigma})\mathscr{D}^{\dagger}(\mathbf{n},\varphi)
$$

Checking the norm, we have

$$
\begin{aligned}
\mathbf{b}\cdot\mathbf{b} &= (\mathbf{b}\cdot\boldsymbol{\sigma})^2 \\
&= \mathscr{D}(\mathbf{n},\varphi)(\mathbf{a}\cdot\boldsymbol{\sigma})\mathscr{D}^{\dagger}(\mathbf{n},\varphi)\mathscr{D}(\mathbf{n},\varphi)(\mathbf{a}\cdot\boldsymbol{\sigma})\mathscr{D}^{\dagger}(\mathbf{n},\varphi)
\end{aligned}
$$

and since $\mathscr{D}(\mathbf{n},\varphi)$ is unitary,

$$
\begin{aligned}
\mathbf{b}\cdot\mathbf{b} &= \mathscr{D}(\mathbf{n},\varphi)(\mathbf{a}\cdot\boldsymbol{\sigma})^2\mathscr{D}^{\dagger}(\mathbf{n},\varphi) \\
&= (\mathbf{a}\cdot\mathbf{a})\mathscr{D}(\mathbf{n},\varphi)\mathscr{D}^{\dagger}(\mathbf{n},\varphi) \\
&= \mathbf{a}\cdot\mathbf{a}
\end{aligned}
$$

Since $\mathbf{b}$ and $\mathbf{a}$ have the same length, they are related by a rotation. Carrying out the actual rotations shows that $\mathbf{b}$ is equal to $\mathbf{a}$ rotated by φ about $\mathbf{n}$.

The actual rotation is accomplished as follows. Let

$$
\mathbf{b}\cdot\boldsymbol{\sigma} = \mathscr{D}(\mathbf{n},\varphi)(\mathbf{a}\cdot\boldsymbol{\sigma})\mathscr{D}^{\dagger}(\mathbf{n},\varphi)
$$

where $\mathbf{b}$ is the rotated version of the 3-vector $\mathbf{a}$. Then, expanding,

$$
\begin{aligned}
\mathbf{b}\cdot\boldsymbol{\sigma} &= \left(e^{-\frac{i\varphi}{2}\mathbf{n}\cdot\boldsymbol{\sigma}}\right)(\mathbf{a}\cdot\boldsymbol{\sigma})\left(e^{\frac{i\varphi}{2}\mathbf{n}\cdot\boldsymbol{\sigma}}\right) \\
&= \left(\mathbf{1}\cos\frac{\varphi}{2} - i\mathbf{n}\cdot\boldsymbol{\sigma}\sin\frac{\varphi}{2}\right)(\mathbf{a}\cdot\boldsymbol{\sigma})\left(\mathbf{1}\cos\frac{\varphi}{2} + i\mathbf{n}\cdot\boldsymbol{\sigma}\sin\frac{\varphi}{2}\right) \\
&= (\mathbf{a}\cdot\boldsymbol{\sigma})\cos^2\frac{\varphi}{2} + i(\mathbf{a}\cdot\boldsymbol{\sigma})(\mathbf{n}\cdot\boldsymbol{\sigma})\sin\frac{\varphi}{2}\cos\frac{\varphi}{2} - i(\mathbf{n}\cdot\boldsymbol{\sigma})(\mathbf{a}\cdot\boldsymbol{\sigma})\sin\frac{\varphi}{2}\cos\frac{\varphi}{2} \\
&\quad + (\mathbf{n}\cdot\boldsymbol{\sigma})(\mathbf{a}\cdot\boldsymbol{\sigma})(\mathbf{n}\cdot\boldsymbol{\sigma})\sin^2\frac{\varphi}{2} \\
&= (\mathbf{a}\cdot\boldsymbol{\sigma})\cos^2\frac{\varphi}{2} + i\left[(\mathbf{a}\cdot\boldsymbol{\sigma}),(\mathbf{n}\cdot\boldsymbol{\sigma})\right]\sin\frac{\varphi}{2}\cos\frac{\varphi}{2} \\
&\quad + (\mathbf{n}\cdot\boldsymbol{\sigma})(\mathbf{a}\cdot\boldsymbol{\sigma})(\mathbf{n}\cdot\boldsymbol{\sigma})\sin^2\frac{\varphi}{2}
\end{aligned}
$$

Working out the commutator and the triple product, we have

$$
\begin{aligned}
\left[(\mathbf{a}\cdot\boldsymbol{\sigma}),(\mathbf{n}\cdot\boldsymbol{\sigma})\right] &= a_i n_j [\sigma_i,\sigma_j] \\
&= a_i n_j (2i\varepsilon_{ijk}\sigma_k) \\
&= 2i(\mathbf{a}\times\mathbf{n})\cdot\boldsymbol{\sigma}
\end{aligned}
$$

and

$$
\begin{aligned}
(\mathbf{n}\cdot\boldsymbol{\sigma})(\mathbf{a}\cdot\boldsymbol{\sigma})(\mathbf{n}\cdot\boldsymbol{\sigma}) &= (\mathbf{n}\cdot\boldsymbol{\sigma})a_i n_j \sigma_i \sigma_j \\
&= (\mathbf{n}\cdot\boldsymbol{\sigma})a_i n_j (\delta_{ij} + i\varepsilon_{ijk}\sigma_k) \\
&= (\mathbf{n}\cdot\boldsymbol{\sigma})(\mathbf{a}\cdot\mathbf{n} + i(\mathbf{a}\times\mathbf{n})_k\sigma_k) \\
&= (\mathbf{a}\cdot\mathbf{n})\mathbf{n}\cdot\boldsymbol{\sigma} + in_m(\mathbf{a}\times\mathbf{n})_k\sigma_m\sigma_k \\
&= (\mathbf{a}\cdot\mathbf{n})\mathbf{n}\cdot\boldsymbol{\sigma} + in_m(\mathbf{a}\times\mathbf{n})_k(\delta_{mk} + i\varepsilon_{mkn}\sigma_n) \\
&= (\mathbf{a}\cdot\mathbf{n})\mathbf{n}\cdot\boldsymbol{\sigma} + i\mathbf{n}\cdot(\mathbf{a}\times\mathbf{n}) - (\mathbf{n}\times(\mathbf{a}\times\mathbf{n}))\cdot\boldsymbol{\sigma} \\
&= (\mathbf{a}\cdot\mathbf{n})\mathbf{n}\cdot\boldsymbol{\sigma} - ((\mathbf{n}\cdot\mathbf{n})\mathbf{a} - (\mathbf{a}\cdot\mathbf{n})\mathbf{n})\cdot\boldsymbol{\sigma} \\
&= (2(\mathbf{a}\cdot\mathbf{n})\mathbf{n} - \mathbf{a})\cdot\boldsymbol{\sigma}
\end{aligned}
$$

Substituting,

$$\begin{aligned}
\mathbf{b}\cdot\boldsymbol{\sigma} &= (\mathbf{a}\cdot\boldsymbol{\sigma})\cos^2\frac{\varphi}{2} - 2(\mathbf{a}\times\mathbf{n})\cdot\boldsymbol{\sigma}\sin\frac{\varphi}{2}\cos\frac{\varphi}{2} \\
&\quad + (2(\mathbf{a}\cdot\mathbf{n})\mathbf{n}-\mathbf{a})\cdot\boldsymbol{\sigma}\sin^2\frac{\varphi}{2} \\
&= \left[\mathbf{a}\cos^2\frac{\varphi}{2} - 2(\mathbf{a}\times\mathbf{n})\sin\frac{\varphi}{2}\cos\frac{\varphi}{2} + (2(\mathbf{a}\cdot\mathbf{n})\mathbf{n}-\mathbf{a})\sin^2\frac{\varphi}{2}\right]\cdot\boldsymbol{\sigma}
\end{aligned}$$

Notice that the right hand side comes out as a real 3-vector dotted into $\boldsymbol{\sigma}$. Equating $\mathbf{b}$ to the coefficient on the right, collecting terms and using trig identitites,

$$\begin{aligned}
\mathbf{b} &= \mathbf{a}\left(\cos^2\frac{\varphi}{2} - \sin^2\frac{\varphi}{2}\right) - (\mathbf{a}\times\mathbf{n})\sin\varphi + 2(\mathbf{a}\cdot\mathbf{n})\mathbf{n}\sin^2\frac{\varphi}{2} \\
&= \mathbf{a}\cos\varphi - (\mathbf{a}\times\mathbf{n})\sin\varphi + (\mathbf{a}\cdot\mathbf{n})\mathbf{n}(1-\cos\varphi) \\
&= (\mathbf{a}\cdot\mathbf{n})\mathbf{n} + [\mathbf{a}-(\mathbf{a}\cdot\mathbf{n})\mathbf{n}]\cos\varphi - (\mathbf{a}\times\mathbf{n})\sin\varphi
\end{aligned}$$

If we define the parts of $\mathbf{a}$ parallel and perpendicular to $\mathbf{n}$ as

$$\begin{aligned}
\mathbf{a} &= \mathbf{a}_{\parallel} + \mathbf{a}_{\perp} \\
\mathbf{a}_{\parallel} &= (\mathbf{a}\cdot\mathbf{n})\mathbf{n} \\
\mathbf{a}_{\perp} &= [\mathbf{a}-(\mathbf{a}\cdot\mathbf{n})\mathbf{n}]
\end{aligned}$$

this becomes

$$\mathbf{b} = \mathbf{a}_{\parallel} + \mathbf{a}_{\perp}\cos\varphi - (\mathbf{a}_{\perp}\times\mathbf{n})\sin\varphi$$

showing that the rotation has left the parallel part of $\mathbf{a}$ unchanged, while the part of $\mathbf{a}$ perpendicular to $\mathbf{n}$ is rotated through an angle φ in the plane perpendicular to $\mathbf{n}$.

4 All representations for rotations

We now may find all finite-dimensional representations for the three operators $\hat{J}_i$. These will allow us to rotate arbitrary finite-dimensional states, including the 2-dim spin states, 3-dimensional real vectors, and in general, n-dim states. All results follow from the fundamental commutation relation for hermitian rotational generators,

$$\left[\hat{J}_i, \hat{J}_j\right] = i\hbar\varepsilon_{ijk}\hat{J}_k$$

where i, j, k each take values $1, 2, 3$ and we sum on k.

4.1 A maximal set of commuting observables

To begin, we ask how many mutually commuting operators we can build from $\hat{J}_i$. We can diagonalize any one of $\hat{J}_1, \hat{J}_2, \hat{J}_3$, but since none commute with either or the others, we cannot diagonalize more than one. We choose $\hat{J}_3$ diagonal. There is one further commuting combination – since rotations preserve lengths, the length of $\hat{J}_i$ itself is preserved by rotations,

$$\begin{aligned}
\left[\hat{J}_i, \hat{\mathbf{J}}^2\right] &= \left[\hat{J}_i, \hat{J}_k\hat{J}_k\right] \\
&= \hat{J}_k\left[\hat{J}_i, \hat{J}_k\right] + \left[\hat{J}_i, \hat{J}_k\right]\hat{J}_k \\
&= \hat{J}_k i\hbar\varepsilon_{ikm}\hat{J}_m + i\hbar\varepsilon_{ikm}\hat{J}_m\hat{J}_k \\
&= i\hbar\varepsilon_{ikm}\left(\hat{J}_k\hat{J}_m + \hat{J}_m\hat{J}_k\right) \\
&= 0
\end{aligned}$$

where the last step follows because $\hat{J}_k\hat{J}_m + \hat{J}_m\hat{J}_k$ is symmetric in mk while ε_{ikm} is antisymmetric. In particular, we have

$$\left[\hat{J}_3, \hat{\mathbf{J}}^2\right] = 0$$

so these may be simultaneously diagonalized.

Since we now have a maximal set of commuting observables, we may use their eigenvalues to label states. Let

$$\begin{aligned} \hat{\mathbf{J}}^2 \left|\alpha,\beta\right\rangle &= \alpha^2\hbar^2\left|\alpha,\beta\right\rangle \\ \hat{J}_3\left|\alpha,\beta\right\rangle &= \beta\hbar\left|\alpha,\beta\right\rangle \end{aligned}$$

We seek all allowed values of the real eigenvalues, α, β.

4.2 Raising and lowering operators

Next, define

$$\hat{J}_\pm \equiv \hat{J}_1 \pm i\hat{J}_2$$

where we note that $\hat{J}_+^\dagger = \hat{J}_-$. These satisfy:

$$\begin{aligned} \left[\hat{J}_+, \hat{J}_-\right] &= \left[\hat{J}_1 + i\hat{J}_2, \hat{J}_1 - i\hat{J}_2\right] \\ &= -i\left[\hat{J}_1, \hat{J}_2\right] + i\left[\hat{J}_2, \hat{J}_1\right] \\ &= 2\hbar\hat{J}_3 \end{aligned}$$

and

$$\begin{aligned} \left[\hat{J}_\pm, \hat{J}_3\right] &= \left[\hat{J}_3, \hat{J}_1 \pm i\hat{J}_2\right] \\ &= \left[\hat{J}_3, \hat{J}_1\right] \pm i\left[\hat{J}_3, \hat{J}_2\right] \\ &= i\hbar\hat{J}_2 \pm i\left(-i\hbar\hat{J}_1\right) \\ &= \pm\hbar\hat{J}_\pm \end{aligned}$$

as well as commuting with the length,

$$\left[\hat{J}_\pm, \hat{\mathbf{J}}^2\right] = 0$$

Consider the action of $\hat{\mathbf{J}}^2$ and $\hat{J}_3$ on the state $\hat{J}_+\left|\alpha,\beta\right\rangle$,

$$\begin{aligned} \hat{\mathbf{J}}^2\hat{J}_+\left|\alpha,\beta\right\rangle &= \hat{J}_+\hat{\mathbf{J}}^2\left|\alpha,\beta\right\rangle \\ &= \alpha^2\hbar^2\hat{J}_+\left|\alpha,\beta\right\rangle \end{aligned}$$

so this state is also an eigenstate of $\hat{\mathbf{J}}^2$ with the eigenvalue α^2, while

$$\begin{aligned} \hat{J}_3\hat{J}_+\left|\alpha,\beta\right\rangle &= \left(\left[\hat{J}_3, \hat{J}_+\right] + \hat{J}_+\hat{J}_3\right)\left|\alpha,\beta\right\rangle \\ &= \hbar\hat{J}_+\left|\alpha,\beta\right\rangle + \hat{J}_+\hat{J}_3\left|\alpha,\beta\right\rangle \\ &= (\beta+1)\,\hbar\hat{J}_+\left|\alpha,\beta\right\rangle \end{aligned}$$

We once again have an eigenstate, but the eigenvalue β has increased by $\hbar$. Similarly, $\hat{J}_-$ lowers the eigenvalue by $\hbar$:

$$\begin{aligned} \hat{J}_3\hat{J}_-\left|\alpha,\beta\right\rangle &= \left(\left[\hat{J}_3, \hat{J}_-\right] + \hat{J}_-\hat{J}_3\right)\left|\alpha,\beta\right\rangle \\ &= -\hbar\hat{J}_-\left|\alpha,\beta\right\rangle + \hat{J}_-\hat{J}_3\left|\alpha,\beta\right\rangle \\ &= (\beta-1)\,\hbar\hat{J}_-\left|\alpha,\beta\right\rangle \end{aligned}$$

4.3 Limits on eigenvalues

Now consider the products

$$\begin{aligned}
\hat{J}_+\hat{J}_- &= \left(\hat{J}_1 + i\hat{J}_2\right)\left(\hat{J}_1 - i\hat{J}_2\right) \\
&= \hat{J}_1^2 + \hat{J}_2^2 - i\hat{J}_1\hat{J}_2 + i\hat{J}_2\hat{J}_1 \\
&= \hat{J}_1^2 + \hat{J}_2^2 - i\left[\hat{J}_1, \hat{J}_2\right] \\
&= \hat{J}_1^2 + \hat{J}_2^2 + \hbar\hat{J}_3 \\
&= \hat{\mathbf{J}}^2 - \hat{J}_3^2 + \hbar\hat{J}_3
\end{aligned}$$

and

$$\begin{aligned}
\hat{J}_-\hat{J}_+ &= \left(\hat{J}_1 - i\hat{J}_2\right)\left(\hat{J}_1 + i\hat{J}_2\right) \\
&= \hat{J}_1^2 + \hat{J}_2^2 + i\hat{J}_1\hat{J}_2 - i\hat{J}_2\hat{J}_1 \\
&= \hat{J}_1^2 + \hat{J}_2^2 + i\left[\hat{J}_1, \hat{J}_2\right] \\
&= \hat{J}_1^2 + \hat{J}_2^2 - \hbar\hat{J}_3 \\
&= \hat{\mathbf{J}}^2 - \hat{J}_3^2 - \hbar\hat{J}_3
\end{aligned}$$

The product may be expressed in term of our diagonal operators. Furthermore, since $\hat{J}_+^\dagger = \hat{J}_-$ and $\hat{J}_-^\dagger = \hat{J}_+$ we have

$$\begin{aligned}
\langle\alpha,\beta|\,\hat{J}_-\hat{J}_+\,|\alpha,\beta\rangle &= \left[\langle\alpha,\beta|\,\hat{J}_+^\dagger\right]\left[\hat{J}_+\,|\alpha,\beta\rangle\right] \\
&\geq 0
\end{aligned}$$

and

$$\begin{aligned}
\langle\alpha,\beta|\,\hat{J}_+\hat{J}_-\,|\alpha,\beta\rangle &= \left[\langle\alpha,\beta|\,\hat{J}_-^\dagger\right]\left[\hat{J}_+\,|\alpha,\beta\rangle\right] \\
&\geq 0
\end{aligned}$$

since these expressions give the norms of the kets $\left[\hat{J}_+\,|\alpha,\beta\rangle\right]$ and $\left[\hat{J}_-\,|\alpha,\beta\rangle\right]$, respectively. Substituting for the diagonal expression in each of these, we get two inequalities:

$$\begin{aligned}
0 &\leq \langle\alpha,\beta|\,\hat{J}_-\hat{J}_+\,|\alpha,\beta\rangle \\
&= \langle\alpha,\beta|\left(\hat{\mathbf{J}}^2 - \hat{J}_3^2 - \hbar\hat{J}_3\right)|\alpha,\beta\rangle \\
&= \left(\alpha^2 - \beta^2 - \beta\right)\hbar^2\,\langle\alpha,\beta\,|\alpha,\beta\rangle \\
&= \left(\alpha^2 - \beta^2 - \beta\right)\hbar^2
\end{aligned}$$

and

$$\begin{aligned}
0 &\leq \langle\alpha,\beta|\,\hat{J}_+\hat{J}_-\,|\alpha,\beta\rangle \\
&= \langle\alpha,\beta|\left(\hat{\mathbf{J}}^2 - \hat{J}_3^2 + \hbar\hat{J}_3\right)|\alpha,\beta\rangle \\
&= \left(\alpha^2 - \beta^2 + \beta\right)\hbar^2\,\langle\alpha,\beta\,|\alpha,\beta\rangle \\
&= \left(\alpha^2 - \beta^2 + \beta\right)\hbar^2
\end{aligned}$$

and therefore, both

$$\begin{aligned}
\beta^2 + \beta &\leq \alpha^2 \\
\beta^2 - \beta &\leq \alpha^2
\end{aligned}$$

Now, just as we did for the simple harmonic oscillator, we start with any eigenstate and lower the eigenvalue k times,

$$\hat{J}_3 \left(\hat{J}_-\right)^k |\alpha,\beta\rangle = (\beta - k)\,\hbar \left(\hat{J}_-\right)^k |\alpha,\beta\rangle$$

We may set

$$\left(\hat{J}_-\right)^k |\alpha,\beta\rangle = \lambda_{\beta-k}\,|\alpha,\beta-k\rangle$$

for some normalization constant, $\lambda_{\beta-k}$. However, this series must terminate, since we require

$$k^2 - 2\beta k - k + \beta^2 + \beta \leq \alpha^2$$

For any fixed β, there is some value of k which is sufficiently large to violate this inequality. Therefore, there must exist some β_{min} such that

$$\hat{J}_- |\alpha,\beta_{min}\rangle = 0$$

where, recognizing that $\beta_{min} < 0$, we have

$$\beta_{min}^2 - \beta_{min} \leq \alpha^2$$

From this state, we apply $\hat{J}_+$ to produce eigenkets of the form

$$\hat{J}_+^k |\alpha,\beta_{min}\rangle = \lambda_{\beta_{min}+k}\,|\alpha,\beta_{min}+k\rangle$$

Once again we hit a limit, so there exists some maximum β_{max}, satisfying

$$\beta_{max}^2 + \beta_{max} \leq \alpha^2$$

Notice that if $\beta_{min} = -\beta_{max} = -m$ then both inequalities give

$$m\,(m+1) \leq \alpha^2$$

Now acting on the highest state, $|\alpha,\beta_{max}\rangle$ with $\hat{J}_+$, or acting on the lowest state, $|\alpha,\beta_{min}\rangle$, with $\hat{J}_-$ must give zero

$$\begin{aligned} \hat{J}_+ |\alpha,\beta_{max}\rangle &= 0 \\ \hat{J}_- |\alpha,\beta_{min}\rangle &= 0 \end{aligned}$$

and therefore, acting on the first with $\hat{J}_-$ and the second with $\hat{J}_+$

$$\begin{aligned} 0 &= \hat{J}_-\hat{J}_+ |\alpha,\beta_{max}\rangle \\ &= \left(\hat{\mathbf{J}}^2 - \hat{J}_3^2 - \hbar\hat{J}_3\right) |\alpha,\beta_{max}\rangle \\ &= \left(\alpha^2 - \beta_{max}^2 - \beta_{max}\right)\hbar^2 |\alpha,\beta_{max}\rangle \end{aligned}$$

and

$$\begin{aligned} 0 &= \hat{J}_+\hat{J}_- |\alpha,\beta_{min}\rangle \\ &= \left(\hat{\mathbf{J}}^2 - \hat{J}_3^2 + \hbar\hat{J}_3\right) |\alpha,\beta_{min}\rangle \\ &= \left(\alpha^2 - \beta_{min}^2 + \beta_{min}\right)\hbar^2 |\alpha,\beta_{min}\rangle \end{aligned}$$

so that

$$\begin{aligned} \alpha^2 &= \beta_{max}^2 + \beta_{max} \\ &= \beta_{min}^2 - \beta_{min} \end{aligned}$$

We also know that $\beta_{max} - \beta_{min} = k$ for some non-negative integer, k. Thus

$$\begin{aligned}
\beta_{min}^2 - \beta_{min} &= \beta_{max}^2 + \beta_{max} \\
&= (\beta_{min} + k)^2 + (\beta_{min} + k) \\
\beta_{min}^2 - \beta_{min} &= \beta_{min}^2 + 2\beta_{min} k + k^2 + \beta_{min} + k \\
0 &= (k+1)\, 2\beta_{min} + k\,(k+1) \\
0 &= 2\beta_{min} + k \\
\beta_{min} &= -\frac{k}{2}
\end{aligned}$$

so that β_{min} is some negative integer or half-integer,

$$\beta_{min} = j \in \left\{ -\frac{1}{2}, -1, -\frac{3}{2}, -2, \ldots \right\}$$

and $\beta_{max} = \beta_{min} + k = +j$, and

$$\begin{aligned}
\alpha^2 &= \frac{k}{2}\left(\frac{k}{2} + 1\right) \\
&= j\,(j+1)
\end{aligned}$$

The labeling of our states is complete. Letting $\beta = m$, the complete set of possible states for any fixed half-integer j is given by the $2j+1$ states,

$$|\alpha, \beta\rangle = \{ |j, m\rangle \mid m = -j, -j+1, \ldots, j+1, j \}$$

and we have one such set for every choice of $j = 0, \frac{1}{2}, 1, \frac{3}{2}, 2, \frac{5}{2}, \ldots$. The eigenvalues of these states are given by

$$\begin{aligned}
\hat{\mathbf{J}}^2 |j, m\rangle &= j\,(j+1)\,\hbar^2\, |j, m\rangle \\
\hat{J}_3 |j, m\rangle &= m\hbar\, |j, m\rangle
\end{aligned}$$

These states will be referred to as "spin-j" representations.

4.4 Normalization

We define these eigenstates to be normalized, and since they are nondegenerate, they are orthonormal,

$$\langle j_1, m_1 \,|j_2, m_2\rangle = \delta_{j_1 j_2} \delta_{m_1 m_2}$$

However, we need to know how to the effect of the raising and lowering operators. We already know that

$$\hat{J}_\pm |j, m\rangle = \lambda_{m\pm1} |j, m \pm 1\rangle$$

To find $\lambda_{m\pm1}$, look again at the norm

$$\begin{aligned}
\langle j, m| \hat{J}_- \hat{J}_+ |j, m\rangle &= \langle j, m| \left(\hat{\mathbf{J}}^2 - \hat{J}_3^2 - \hbar \hat{J}_3 \right) |j, m\rangle \\
|\lambda_{m+1}|^2 &= (j\,(j+1) - m\,(m+1))\,\hbar^2 \\
\lambda_{m+1} &= \sqrt{j\,(j+1) - m\,(m+1)}\,\hbar
\end{aligned}$$

where we choose the phase so that λ_{m+1} is real. For $\hat{J}_-$ we have

$$\begin{aligned}
\langle j, m| \hat{J}_+ \hat{J}_- |j, m\rangle &= \langle j, m| \left(\hat{\mathbf{J}}^2 - \hat{J}_3^2 + \hbar \hat{J}_3 \right) |j, m\rangle \\
|\lambda_{m-1}|^2 &= (j\,(j+1) - m\,(m-1))\,\hbar^2 \\
\lambda_{m-1} &= \sqrt{j\,(j+1) - m\,(m-1)}\,\hbar
\end{aligned}$$

Therefore, the action of the raising and lowering operators is

$$\hat{J}_\pm |j, m\rangle = \sqrt{j\,(j+1) - m\,(m \pm 1)}\,\hbar\, |j, m \pm 1\rangle$$

4.5 Examples of representations

4.5.1 Spin 0

For $j = 0$, we only have the single allowed value $m = 0$ and there is only one state,

$$|j, m\rangle = |0, 0\rangle$$

These are scalars. We may find the expectation value of any component of angular momentum using

$$\begin{aligned} J_1 &= \frac{1}{2}\left(\hat{J}_+ + \hat{J}_-\right) \\ J_2 &= \frac{1}{2i}\left(\hat{J}_+ - \hat{J}_-\right) \end{aligned}$$

Since $m = 0 = \beta_{min} = \beta_{max}$, both $\hat{J}_+$ and $\hat{J}_-$ must give zero:

$$\hat{J}_\pm |0, 0\rangle = 0$$

and we have

$$\begin{aligned} \langle 0, 0| \hat{J}_x |0, 0\rangle &= 0 \\ \langle 0, 0| \hat{J}_y |0, 0\rangle &= 0 \\ \langle 0, 0| \hat{J}_z |0, 0\rangle &= 0 \end{aligned}$$

so every component of angular momentum has zero expectation value.

4.5.2 Spin 1/2

For $j = \frac{1}{2}$ we have our familiar algebra of Pauli matrices, but we now have a more systematic labelling for the states. When we wish to be explicit about the value of j, we will write

$$\left|\frac{1}{2}, \pm\frac{1}{2}\right\rangle$$

instead of $|\pm\rangle$. Notice that in all cases here we are taking $\hat{J}_z$ diagonal. We already know the expectation values of $\hat{J}_i$ in these states. For $\hat{\mathbf{J}}^2$ and $\hat{J}_\pm$ we have

$$\begin{aligned} \hat{\mathbf{J}}^2 \left|\frac{1}{2}, \pm\frac{1}{2}\right\rangle &= \frac{1}{2}\left(\frac{1}{2} + 1\right)\hbar^2 \left|\frac{1}{2}, \pm\frac{1}{2}\right\rangle \\ &= \frac{3}{4}\hbar^2 \left|\frac{1}{2}, \pm\frac{1}{2}\right\rangle \end{aligned}$$

and

$$\begin{aligned} \hat{J}_+ \left|\frac{1}{2}, \frac{1}{2}\right\rangle &= 0 \\ \hat{J}_- \left|\frac{1}{2}, \frac{1}{2}\right\rangle &= \sqrt{j(j+1) - m(m-1)}\hbar \left|\frac{1}{2}, \frac{1}{2} - 1\right\rangle \\ &= \sqrt{\frac{1}{2}\left(\frac{3}{2}\right) - \frac{1}{2}\left(-\frac{1}{2}\right)}\hbar \left|\frac{1}{2}, -\frac{1}{2}\right\rangle \\ &= \hbar \left|\frac{1}{2}, -\frac{1}{2}\right\rangle \end{aligned}$$

$$\begin{aligned}
\hat{J}_- \left|\frac{1}{2}, -\frac{1}{2}\right\rangle &= \sqrt{\frac{1}{2}\left(\frac{3}{2}\right) - \left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)} \left|\frac{1}{2}, -\frac{1}{2} - 1\right\rangle \\
&= 0 \\
\hat{J}_+ \left|\frac{1}{2}, -\frac{1}{2}\right\rangle &= \sqrt{j(j+1) - m(m+1)}\hbar \left|\frac{1}{2}, \frac{1}{2}\right\rangle \\
&= \sqrt{\frac{3}{4} - \left(-\frac{1}{2}\right)\left(\frac{1}{2}\right)}\hbar \left|\frac{1}{2}, \frac{1}{2}\right\rangle \\
&= \hbar \left|\frac{1}{2}, \frac{1}{2}\right\rangle
\end{aligned}$$

Notice that in matrix notation,

$$\begin{aligned}
\hat{J}_+ &= \hat{J}_x + i\hat{J}_y \\
&= \frac{\hbar}{2}(\sigma_x + i\sigma_y) \\
&= \hbar \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \\
\hat{J}_- &= \hbar \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}
\end{aligned}$$

so that

$$\begin{aligned}
\hat{J}_+ \begin{pmatrix} 1 \\ 0 \end{pmatrix} &= 0 \\
\hat{J}_+ \begin{pmatrix} 0 \\ 1 \end{pmatrix} &= \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\
\hat{J}_- \begin{pmatrix} 0 \\ 1 \end{pmatrix} &= 0 \\
\hat{J}_+ \begin{pmatrix} 1 \\ 0 \end{pmatrix} &= \begin{pmatrix} 0 \\ 1 \end{pmatrix}
\end{aligned}$$

Quite generally, the components of the raising and lowering operators are unit off-diagonal matrices.

4.5.3 Spin 1

We now have a total of three $j = 1$ states,

$$|j, m\rangle = |1, 1\rangle, |1, 0\rangle, |1, -1\rangle$$

related by

$$\begin{aligned}
\hat{J}_- |1, 1\rangle &= \sqrt{1(1+1) - 1(1-1)}\hbar |1, 1-1\rangle \\
&= \sqrt{2}\hbar |1, 0\rangle
\end{aligned}$$

and

$$\begin{aligned}
\hat{J}_- |1, 0\rangle &= \sqrt{1(1+1) - 0(0-1)}\hbar |1, 0-1\rangle \\
&= \sqrt{2}\hbar |1, -1\rangle
\end{aligned}$$

with similar relations for the raising operator. The eigenvalue of $\hat{\mathbf{J}}^2$ is $j(j+1)\hbar^2 = 2\hbar^2$.

4.5.4 Spin 3/2

We have $2j+1=4$ states,

$$|j,m\rangle = \left|\frac{3}{2},\frac{3}{2}\right\rangle, \left|\frac{3}{2},\frac{1}{2}\right\rangle, \left|\frac{3}{2},-\frac{1}{2}\right\rangle, \left|\frac{3}{2},-\frac{3}{2}\right\rangle$$

related by

$$\begin{aligned}
\hat{J}_-\left|\frac{3}{2},\frac{3}{2}\right\rangle &= \sqrt{\frac{3}{2}\left(\frac{3}{2}+1\right)-\frac{3}{2}\left(\frac{3}{2}-1\right)}\hbar\left|\frac{3}{2},\frac{1}{2}\right\rangle \\
&= \sqrt{3}\hbar\left|\frac{3}{2},\frac{1}{2}\right\rangle \\
\hat{J}_-\left|\frac{3}{2},\frac{1}{2}\right\rangle &= \sqrt{\frac{3}{2}\left(\frac{3}{2}+1\right)-\frac{1}{2}\left(\frac{1}{2}-1\right)}\hbar\left|\frac{3}{2},-\frac{1}{2}\right\rangle \\
&= 2\hbar\left|\frac{3}{2},-\frac{1}{2}\right\rangle \\
\hat{J}_-\left|\frac{3}{2},-\frac{1}{2}\right\rangle &= \sqrt{\frac{3}{2}\left(\frac{3}{2}+1\right)-\left(-\frac{1}{2}\right)\left(-\frac{1}{2}-1\right)}\hbar\left|\frac{3}{2},-\frac{3}{2}\right\rangle \\
&= \sqrt{3}\hbar\left|\frac{3}{2},-\frac{3}{2}\right\rangle \\
\hat{J}_-\left|\frac{3}{2},-\frac{3}{2}\right\rangle &= 0
\end{aligned}$$

with similar relations for the raising operator. The eigenvalue of $\hat{\mathbf{J}}^2$ is $j(j+1)\hbar^2 = \frac{15}{4}\hbar^2$

4.5.5 Spin j

We summarize here the general results we have shown above.

For spin-j, where $j=\frac{n}{2}$ is any integer or half-integer there are $2j+1=n+1$ orthonormal states labelled $|j,m\rangle$, where m ranges over all $2j+1$ values from $-j$ to $+j$.

$$|j,m\rangle$$

The actions of $\hat{\mathbf{J}}^2, \hat{J}_z, \hat{J}_\pm$ on these are given by

$$\begin{aligned}
\hat{\mathbf{J}}^2|j,m\rangle &= j(j+1)\hbar^2|j,m\rangle \\
\hat{J}_z|j,m\rangle &= m\hbar|j,m\rangle \\
\hat{J}_+|j,m\rangle &= \sqrt{j(j+1)-m(m+1)}\hbar|j,m+1\rangle \\
\hat{J}_-|j,m\rangle &= \sqrt{j(j+1)-m(m-1)}\hbar|j,m-1\rangle
\end{aligned}$$

while the actions of $\hat{J}_x, \hat{J}_y$ may be found using

$$\begin{aligned}
\hat{J}_x &= \frac{1}{2}\left(\hat{J}_+ + \hat{J}_-\right) \\
\hat{J}_y &= \frac{1}{2i}\left(\hat{J}_+ - \hat{J}_-\right)
\end{aligned}$$

There is a vector space of every positive integer dimension spanned by $|j,m\rangle$ for some j. Taken together, these give all of the irreducible representations of the 3-dimensional rotation group. This means that any tensor, i.e., any object that the 3-dim rotation group acts on multi-linearly and homogeneously, may be decomposed into some combination of the $|j,m\rangle$ vector space.

4.6 Decomposition of tensors

We have observed previously that a matrix can be decomposed into its trace, its antisymmetric part, and its traceless symmetric part:

$$M_{ij} = \frac{1}{2}\delta_{ij} trM + \frac{1}{2}\left(M_{ij} - M_{ji}\right) + \frac{1}{2}\left(M_{ij} + M_{ji} - \frac{2}{3}trM\right)$$

When we rotate M_{ij} with an orthogonal transformation,

$$\begin{aligned} \tilde{M}_{ij} &= O_i^{\ m} O_j^{\ n} M_{mn} \\ &= O_i^{\ m} M_{mn} \left[O^t\right]^m_{\ i} \\ \tilde{M} &= OMO^{-1} \end{aligned}$$

each of these parts is preserved. For example, the antisymmetric part of the new matrix is a linear combination of the components of only the antisymmetric part of the original matrix,

$$\begin{aligned} O\frac{1}{2}\left(M - M^t\right) O^{-1} &= \frac{1}{2}\left(OMO^{-1} - OM^tO^{-1}\right) \\ &= \frac{1}{2}\left(\tilde{M} - \tilde{M}^t\right) \end{aligned}$$

We say that the usual matrix representation is reducible, and from the fact that these three invariant subspace have one degree of freedom for the trace, three for the antisymmetric part, and five degrees of freedom for the traceless symmetric part, we expect that we can write M as a combination of the three vector spaces,

$$|0,0\rangle, |1,m\rangle, |2,m\rangle$$

which are of dimensions $1, 3$ and 5, respectively.

There is notation for this equivalence. Letting the boldface number **3** stand for one index of M, we think of the nine components of M as the outer product of 3-dimensional things,

$$M \to \mathbf{3} \otimes \mathbf{3}$$

and we write this as the sum, in the new notation, of three irreducible vector spaces:

$$\mathbf{3} \otimes \mathbf{3} = \mathbf{1} \oplus \mathbf{3} \oplus \mathbf{5}$$

There are more general objects that rotations can act on. By taking outer products of vectors, we construct "tensors" with arbitrarily many indices,

$$T_{ij\ldots k} = u_i v_j \ldots w_k$$

Since we can rotate each vector, we know how $T_{ij\ldots k}$ changes under rotations. We may take abritrary linear combinations of objects of this form to construct n-index objects with 3^n degrees of freedom. For example, a general tensor with three indices, T_{ijk}, has $3^3 = 27$ independent components.

A systematic analysis along these same lines shows that a rank three tensor, that is, an object with three indices like the Levi-Civita tensor, T_{ijk}, may be decomposed into four irreducible parts,

$$\mathbf{3} \otimes \mathbf{3} \otimes \mathbf{3} = \mathbf{1} \oplus \mathbf{8} \oplus \mathbf{8} \oplus \mathbf{10}$$

Notice that the degrees of freedom always match, $3^3 = 27 = 1 + 8 + 8 + 10$, so we have accounted for all 27 independent components of T_{ijk}. There are general techniques for finding this decomposition for any tensor.

One familiar example of this sort of decomposition is given by the spherical harmonics. If we have any bounded, piecewise continuous function on a sphere, $f(\theta, \varphi)$, it may be expanded in spherical harmonics,

$$f(\theta,\varphi) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} A_{lm} Y^l_m(\theta,\varphi)$$

But such functions form an infinite dimensional vector space, since sums of such functions give other functions on the sphere. The collection of spherical harmonics for any fixed l, $\left\{Y^l_m(\theta,\varphi) \,|m = -l, -l+1, \ldots, l\right\}$ also form a vector space, since we may take linear combinations of any two linear combinations of these, to form another linear combinations of the same set. Moreover, these sets are rotationally invariant: any rotation of the sphere $(\theta,\varphi) \to (\theta+\alpha, \varphi+\beta)$ mixes m but leaves l fixed. Since the dimension of these invariant subspaces is $2l+1$, while the dimension of the function space is infinite, the sum above gives us an infinite decomposition,

$$\boldsymbol{\infty} = \mathbf{1} \oplus \mathbf{3} \oplus \mathbf{5} \oplus \cdots \oplus (\mathbf{2l+1}) \oplus \cdots$$

We show in the next set of notes that these odd-dimensional vector spaces are, in fact, the spherical harmonics.

The importance of such decompositions becomes evident when we look at atoms, nuclei, mesons or baryons, all of which are composite. Atoms are described by electrons orbiting nuclei, while the others are comprised of quarks and gluons. In each of these multi-particle systems, the constituents may have both orbital angular momentum and spin, and we need to know how these various contributions to the total angular momentum combine to give a total number of states for the system. Therefore, we will later develop rules for the addition of angular momentum states.

4.7 Rotations

We conclude with the form of the matrix elements of the finite rotation operators,

$$\hat{\mathscr{D}}(\mathbf{n},\varphi) = e^{-\frac{i\varphi}{\hbar}\mathbf{n}\cdot\hat{\mathbf{J}}}$$

Since the generators

$$\begin{aligned} \hat{\mathbf{J}} &= \left(\hat{J}_1, \hat{J}_2, \hat{J}_3\right) \\ &= \left(\frac{1}{2}\left(\hat{J}_+ + \hat{J}_-\right), \frac{1}{2i}\left(\hat{J}_+ - \hat{J}_-\right), \hat{J}_3\right) \end{aligned}$$

change m but never change the value of j, we have

$$\langle j_1, m_1 \; \hat{\mathscr{D}}(\mathbf{n},\varphi) \,|j_2, m_2\rangle = \langle j_1, m_1 \; e^{-\frac{i\varphi}{\hbar}\mathbf{n}\cdot\hat{\mathbf{J}}} \,|j_2, m_2\rangle$$

vanishes unless $j_1 = j_2$. We therefore need consider only matrix elements of a single value of j. Define the matrix element of rotations of $(2j+1)$-dimensional states as the $(2j+1)\times(2j+1)$ matrix with elements

$$\hat{\mathscr{D}}^j_{m',m}(\mathbf{n},\varphi) \equiv \langle j, m'|\, e^{-\frac{i\varphi}{\hbar}\mathbf{n}\cdot\hat{\mathbf{J}}} \,|j, m\rangle$$

In general, if we start with a given state, $|j,m\rangle$, and rotate it, the result is given by acting with these matrices. Concretely, by multiplying by the identity matrix,

$$\hat{\mathbf{1}} = \sum_j \sum_{m=-j}^{j} |j,m\rangle \langle j,m|$$

we have

$$\begin{aligned}
\hat{\mathscr{D}}\left(\mathbf{n},\varphi\right)\left|j,m\right\rangle &= \left(\sum_{j'}\sum_{m'=-j'}^{j'}\left|j',m'\right\rangle\left\langle j',m'\right|\right)\hat{\mathscr{D}}\left(\mathbf{n},\varphi\right)\left|j,m\right\rangle \\
&= \sum_{j'}\sum_{m'=-j'}^{j'}\left|j',m'\right\rangle\delta_{j'j}\left\langle j,m'\right|\hat{\mathscr{D}}\left(\mathbf{n},\varphi\right)\left|j,m\right\rangle \\
&= \sum_{m'=-j}^{j}\left|j',m'\right\rangle\hat{\mathscr{D}}^{j}_{m'm}\left(\mathbf{n},\varphi\right)
\end{aligned}$$