Addition of angular momentum

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Often we need to combine different sources of angular momentum to characterize the total angular momentum of a system, or to divide the total angular momentum into parts to evaluate the effect of a potential. We now develop general techniques for this.

1 States

Suppose we have a state which has several parts. For example, we might describe a particle by the product of a spinor and a spatial wave function,

$$\psi(\mathbf{x},t)\,\chi(\theta,\varphi)$$

where χ is a two component spinor. Or, we might have a composite particle such as a proton, made up of three quarks,

$$p = uud$$

This latter involves both the spatial wave function and the spinor for each quark,

$$p = U_1(\mathbf{x},t)\,\chi_{u1}(\theta,\varphi)\,U_2(\mathbf{x},t)\,\chi_{u2}(\theta,\varphi)\,D(\mathbf{x},t)\,\chi_d(\theta,\varphi)$$

Each of the wave functions may have orbital angular momentum described by $|l,m_l\rangle$ states, while each spinor is a $\left|\frac{1}{2},m_s\right\rangle$, so the total angular momentum is built from the 6-fold product,

$$|l_1,m_{1l}\rangle\left|\frac{1}{2},m_{s1}\right\rangle|l_2,m_{2l}\rangle\left|\frac{1}{2},m_{s2}\right\rangle|l_d,m_{dl}\rangle\left|\frac{1}{2},m_{s3}\right\rangle$$

Even if all three quarks are in the $l=0$ ground state, there are $2\times2\times2=8$ possible combinations of the spins. We need techniques to find all possible total angular momentum states, $|j,m\rangle$, for such systems.

When we have products like this, we think of them as general outer products and write the combined spin state with a generic product, $\otimes$. For example, a system comprised of an electron (spin-$\frac{1}{2}$) and pion (spin-1), we may write e_m for $m=-\frac{1}{2},\frac{1}{2}$ and π_M for $M=-1,0,1$ we see that the outer produce will have a combined spin state

$$e_m\pi_M = \left|\frac{1}{2},m\right\rangle_e\otimes|1,M\rangle_\pi$$

This is just like an outer product of vectors, u^iv^j, forming a matrix, except these two vectors live in spaces of different dimension so that $e_m\pi_M$ is a matrix with two rows and three columns. The electron and pion states behave independently of one another. If our electron-pion system has the electron in a spin up state in the plus direction along the x-axis, and the pion has its angular momentum in the positive z direction,

$$\begin{aligned}\chi_e &= \frac{1}{\sqrt{2}}\left(\left|\frac{1}{2},\frac{1}{2}\right\rangle+\left|\frac{1}{2},-\frac{1}{2}\right\rangle\right)\\ \psi_\pi &= |1,1\rangle\end{aligned}$$

then the total state is

$$
\begin{aligned}
\chi_e\psi_\pi &= \frac{1}{\sqrt{2}}\left(\left|\frac{1}{2},\frac{1}{2}\right\rangle + \left|\frac{1}{2},-\frac{1}{2}\right\rangle\right)\otimes|1,1\rangle \\
&= \frac{1}{\sqrt{2}}\left(\left|\frac{1}{2},\frac{1}{2}\right\rangle\otimes|1,1\rangle + \left|\frac{1}{2},-\frac{1}{2}\right\rangle\otimes|1,1\rangle\right) \qquad (1)
\end{aligned}
$$

For more states, we just continue the product. For example, eight possible spin states for the ground state of the positron will have the form

$$
\left|\frac{1}{2},m_1\right\rangle_{u_1}\otimes\left|\frac{1}{2},m_2\right\rangle_{u_2}\otimes\left|\frac{1}{2},m_3\right\rangle_d
$$

We would like to re-express such compound states in terms of states, $|j,m\rangle$ of *total* angular momentum j and z-component, m.

2 Operators

Angular momentum is additive, so the operators representing dynamical variable of angular momentum, $\hat{\mathbf{J}}$, will add when we have multiple particles. Thus, for the electron-pion system, measuring the total z-component of spin amounts to measuring the z-component of spin of each particle and adding them,

$$
\hat{J}_3 = \hat{J}_3^e + \hat{J}_3^\pi
$$

where each of the operators on the right only acts on its corresponding spinor. In a matrix representation, this seems like an odd combination, since the matrices are of different sizes:

$$
\begin{aligned}
\left[\hat{J}_3^e\right]_{mm'} &= \hbar\begin{pmatrix}\frac{1}{2} & \\ & -\frac{1}{2}\end{pmatrix} \\
\left[\hat{J}_3^\pi\right]_{MM'} &= \hbar\begin{pmatrix}1 & & \\ & 0 & \\ & & -1\end{pmatrix}
\end{aligned}
$$

To write this with formal precision acting on the product state we write each as a pair,

$$
\begin{aligned}
\left[\hat{J}_3\right]_{AB} &= \left[\hat{J}_3^e\right]_{mm'}[\mathbf{1}]_{MM'} + [\mathbf{1}]_{mm'}\left[\hat{J}_3^\pi\right]_{MM'} \\
&= \hat{J}_3^e\otimes\hat{\mathbf{1}} + \hat{\mathbf{1}}\otimes\hat{J}_3^\pi
\end{aligned}
$$

where A,B take six values over the pairs (m,M).

The action of any product operator, $\hat{A}\otimes\hat{B}$ on a product state is given by

$$
\left(\hat{A}\otimes\hat{B}\right)(|\psi\rangle\otimes|\chi\rangle) \equiv \left(\hat{A}|\psi\rangle\right)\otimes\left(\hat{B}|\chi\rangle\right)
$$

Notice that if we use this ordered pair notation, we don't need the superscripts. Either notation is clear with the understanding that $\hat{J}_3^{(e^-)}$ sees only the electron state and $\hat{J}_3^{(e^+)}$ only the positron.

For example, for the state described above, the action of $\hat{J}_3$ is

$$
\begin{aligned}
\hat{J}_3(\chi_e\psi_\pi) &= \left(\hat{J}_3^e + \hat{J}_3^\pi\right)\frac{1}{\sqrt{2}}\left(\left|\frac{1}{2},\frac{1}{2}\right\rangle\otimes|1,1\rangle + \left|\frac{1}{2},-\frac{1}{2}\right\rangle\otimes|1,1\rangle\right) \\
&= \frac{1}{\sqrt{2}}\left(\hat{J}_3^e\left|\frac{1}{2},\frac{1}{2}\right\rangle + \hat{J}_3^e\left|\frac{1}{2},-\frac{1}{2}\right\rangle\right)\otimes|1,1\rangle + \frac{1}{\sqrt{2}}\left(\left|\frac{1}{2},\frac{1}{2}\right\rangle + \left|\frac{1}{2},-\frac{1}{2}\right\rangle\right)\otimes\hat{J}_3^\pi|1,1\rangle
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{\sqrt{2}}\hbar\left(\left(\frac{1}{2}\left|\frac{1}{2},\frac{1}{2}\right\rangle - \frac{1}{2}\left|\frac{1}{2},-\frac{1}{2}\right\rangle\right)\otimes|1,1\rangle + \left(\left|\frac{1}{2},\frac{1}{2}\right\rangle + \left|\frac{1}{2},-\frac{1}{2}\right\rangle\right)\otimes|1,1\rangle\right) \\
&= \frac{1}{\sqrt{2}}\hbar\left(\frac{1}{2}\left|\frac{1}{2},\frac{1}{2}\right\rangle - \frac{1}{2}\left|\frac{1}{2},-\frac{1}{2}\right\rangle + \left|\frac{1}{2},\frac{1}{2}\right\rangle + \left|\frac{1}{2},-\frac{1}{2}\right\rangle\right)\otimes|1,1\rangle \\
&= \frac{1}{\sqrt{2}}\hbar\left(\frac{3}{2}\left|\frac{1}{2},\frac{1}{2}\right\rangle + \frac{1}{2}\left|\frac{1}{2},-\frac{1}{2}\right\rangle\right)\otimes|1,1\rangle
\end{aligned}
$$

Since the electron is not in an eigenstate of $J_3^{(e^+)}$, the combined system is not in an eigenstate of $\hat{J}_3$. However, if both states are eigenstates, so is the combined state. If we let $\chi_e\psi_\pi = \left|\frac{1}{2},\frac{1}{2}\right\rangle\otimes|1,-1\rangle$, then

$$
\begin{aligned}
\hat{J}_3\left(\chi_e\psi_\pi\right) &= \left(\hat{J}_3^e + \hat{J}_3^\pi\right)\left|\frac{1}{2},\frac{1}{2}\right\rangle\otimes|1,-1\rangle \\
&= \hat{J}_3^e\left|\frac{1}{2},\frac{1}{2}\right\rangle\otimes|1,-1\rangle + \left|\frac{1}{2},\frac{1}{2}\right\rangle\otimes\hat{J}_3^\pi|1,-1\rangle \\
&= -\frac{\hbar}{2}\left|\frac{1}{2},\frac{1}{2}\right\rangle\otimes|1,-1\rangle
\end{aligned}
$$

Similar considerations apply to any other spin operators, $\hat{J}_1, \hat{J}_2, \hat{J}_\pm$. and so on. Notice that the angular momentum operators for components of angular momentum for different particles commute,

$$
\begin{aligned}
\left[\hat{J}_i^e, \hat{J}_j^\pi\right] &= 0 \\
\left[\hat{J}_i^e, \hat{J}_j^e\right] &= i\hbar\varepsilon_{ijk}\hat{J}_k^\pi \\
\left[\hat{J}_i^\pi, \hat{J}_j^\pi\right] &= i\hbar\varepsilon_{ijk}\hat{J}_k^\pi
\end{aligned}
$$

Significantly, the spin vector for the total angular momentum,

$$
\hat{J}_i = \hat{J}_i^{(1)} + \hat{J}_i^{(2)}
$$

satisfies the fundamental commutation relations:

$$
\begin{aligned}
\left[\hat{J}_i, \hat{J}_j\right] &= \hat{J}_i\hat{J}_j - \hat{J}_j\hat{J}_i \\
&= \left(\hat{J}_i^e + \hat{J}_i^\pi\right)\left(\hat{J}_j^e + \hat{J}_j^\pi\right) - \left(\hat{J}_j^e + \hat{J}_j^\pi\right)\left(\hat{J}_i^e + \hat{J}_i^\pi\right) \\
&= \left(\hat{J}_i^e\hat{J}_j^e + \hat{J}_i^\pi\hat{J}_j^e + \hat{J}_i^e\hat{J}_j^\pi + \hat{J}_i^\pi\hat{J}_j^\pi\right) - \left(\hat{J}_j^e\hat{J}_i^e + \hat{J}_j^\pi\hat{J}_i^e + \hat{J}_j^e\hat{J}_i^\pi + \hat{J}_j^\pi\hat{J}_i^\pi\right) \\
&= \left(\hat{J}_i^e\hat{J}_j^e + \hat{J}_i^\pi\hat{J}_j^e + \hat{J}_i^e\hat{J}_j^\pi + \hat{J}_i^\pi\hat{J}_j^\pi\right) - \left(\hat{J}_j^e\hat{J}_i^e + \hat{J}_j^\pi\hat{J}_i^e + \hat{J}_j^e\hat{J}_i^\pi + \hat{J}_j^\pi\hat{J}_i^\pi\right) \\
&= \left[\hat{J}_i^e, \hat{J}_j^e\right] + \left[\hat{J}_i^\pi, \hat{J}_j^e\right] + \left[\hat{J}_i^e, \hat{J}_j^\pi\right] + \left[\hat{J}_i^\pi, \hat{J}_j^\pi\right] \\
&= i\hbar\varepsilon_{ijk}\hat{J}_k^e + 0 + 0 + i\hbar\varepsilon_{ijk}\hat{J}_k^\pi \\
&= i\hbar\varepsilon_{ijk}\hat{J}_k
\end{aligned}
$$

This means that the *total vector will also be described by* $|j,m\rangle$ *states.* This means that we are able to set up a 1-1, onto equality between states $|j,m\rangle$ of total angular momentum, and linear combinations of products states of constituent particles.

3 Equivalence of simple and composite states

Because the same algebra holds for the composite state as for the individual particle states, eigenstates of the two are the same. This lets us establish unique relationships between the individual particle angular momentum and the total angular momentum of the system. We establish a systematic way of deriving this correspondence.

If we have an equivalence between some $|j,m\rangle$ state and any product of states, of the form

$$|j,m\rangle = |j_1,m_1\rangle \otimes |j_2,m_2\rangle$$

we will always have

$$\begin{aligned}
\hat{J}_3\,|j,m\rangle &= \left(\hat{J}_3^{(1)} + \hat{J}_3^{(2)}\right)|j_1,m_1\rangle \otimes |j_2,m_2\rangle \\
m\hbar\,|j,m\rangle &= \hat{J}_3^{(1)}\,|j_1,m_1\rangle \otimes |j_2,m_2\rangle + |j_1,m_1\rangle \otimes \hat{J}_3^{(2)}\,|j_2,m_2\rangle \\
m\hbar\,|j,m\rangle &= m_1\hbar\,|j_1,m_1\rangle \otimes |j_2,m_2\rangle + m_2\hbar\,|j_1,m_1\rangle \otimes |j_2,m_2\rangle \\
m\hbar\,|j,m\rangle &= (m_1+m_2)\,\hbar\,|j_1,m_1\rangle \otimes |j_2,m_2\rangle
\end{aligned}$$

so that

$$m = m_1 + m_2$$

that is:

- *The total z-component of spin is always the sum of the individual z-components.*

For the total spin, we must compute

$$\begin{aligned}
\hat{\mathbf{J}}^2 &= \left(\hat{\mathbf{J}}^{(1)} + \hat{\mathbf{J}}^{(2)}\right)^2 \\
&= \left(\hat{\mathbf{J}}^{(1)}\right)^2 + \left(\hat{\mathbf{J}}^{(2)}\right)^2 + 2\hat{\mathbf{J}}^{(1)} \cdot \hat{\mathbf{J}}^{(2)} \qquad (2)
\end{aligned}$$

Using $\hat{J}_\pm \equiv \hat{J}_1 \pm i\hat{J}_2$ for each particle, we notice that

$$\begin{aligned}
\hat{J}_+^{(1)}\hat{J}_-^{(2)} &= \left(\hat{J}_1^{(1)} + i\hat{J}_2^{(1)}\right)\left(\hat{J}_1^{(2)} - i\hat{J}_2^{(2)}\right) \\
&= \hat{J}_1^{(1)}\hat{J}_1^{(2)} + i\hat{J}_2^{(1)}\hat{J}_1^{(2)} - i\hat{J}_1^{(1)}\hat{J}_2^{(2)} + \hat{J}_2^{(1)}\hat{J}_2^{(2)} \\
\hat{J}_-^{(1)}\hat{J}_+^{(2)} &= \hat{J}_1^{(1)}\hat{J}_1^{(2)} - i\hat{J}_2^{(1)}\hat{J}_1^{(2)} + i\hat{J}_1^{(1)}\hat{J}_2^{(2)} + \hat{J}_2^{(1)}\hat{J}_2^{(2)}
\end{aligned}$$

so that adding these together, $\frac{1}{2}\left(\hat{J}_+^{(1)}\hat{J}_-^{(2)} + \hat{J}_-^{(1)}\hat{J}_+^{(2)}\right) = \hat{J}_1^{(1)}\hat{J}_1^{(2)} + \hat{J}_2^{(1)}\hat{J}_2^{(2)}$. Therefore, we may write the dot product as

$$\begin{aligned}
\hat{\mathbf{J}}^{(1)} \cdot \hat{\mathbf{J}}^{(2)} &= \hat{J}_1^{(1)}\hat{J}_1^{(2)} + \hat{J}_2^{(1)}\hat{J}_2^{(2)} + \hat{J}_3^{(1)}\hat{J}_3^{(2)} \\
&= \frac{1}{2}\hat{J}_+^{(1)}\hat{J}_-^{(2)} + \frac{1}{2}\hat{J}_-^{(1)}\hat{J}_+^{(2)} + \hat{J}_3^{(1)}\hat{J}_3^{(2)}
\end{aligned}$$

allowing us to write $\hat{\mathbf{J}}^2$ as

$$\hat{\mathbf{J}}^2 = \hat{\mathbf{J}}_{(1)}^2 + \hat{\mathbf{J}}_{(2)}^2 + 2\hat{J}_3^{(1)}\hat{J}_3^{(2)} + \hat{J}_+^{(1)}\hat{J}_-^{(2)} + \hat{J}_-^{(1)}\hat{J}_+^{(2)}$$

for the product states.

For the general case, apply eq.(2) to the *highest* value of m for each particle,

$$\begin{aligned}
\hat{\mathbf{J}}^2\,|j_1,j_1\rangle \otimes |j_2,j_2\rangle &= \hat{\mathbf{J}}_{(1)}^2\,|j_1,j_1\rangle \otimes |j_2,j_2\rangle + \hat{\mathbf{J}}_{(2)}^2\,|j_1,j_1\rangle \otimes |j_2,j_2\rangle \\
&\quad +2\left(\hat{J}_3^{(1)}\hat{J}_3^{(2)} + \frac{1}{2}\hat{J}_+^{(1)}\hat{J}_-^{(2)} + \frac{1}{2}\hat{J}_-^{(1)}\hat{J}_+^{(2)}\right)|j_1,j_1\rangle \otimes |j_2,j_2\rangle \\
&= \left(j_1(j_1+1) + j_2(j_2+1) + 2j_1j_2\right)\hbar^2\,|j_1,j_1\rangle \otimes |j_2,j_2\rangle \\
&= (j_1+j_2)(j_1+j_2+1)\,\hbar^2\,|j_1,j_1\rangle \otimes |j_2,j_2\rangle
\end{aligned}$$

so that this state is an eigenstate of $\hat{\mathbf{J}}^2$ with eigenvalue $j_1 + j_2$, and we write

$$|j, j\rangle = |j_1, +j_2, j_1, +j_2\rangle = |j_1, j_1\rangle \otimes |j_2, j_2\rangle$$

We can find $2(j_1 + j_2) + 1$ of states $|j_1, +j_2, j_1, +j_2\rangle$ by acting with the lowering operator.

However, we have $(2j_1 + 1)(2j_2 + 1)$ possible product states, but only $2j_1 + 2j_2 + 1$ states of the form $|j_1, +j_2, m\rangle$. To find more of the possible combinations, consider the $|j_1, +j_2, j_1, +j_2 - 1\rangle$ state. Recalling that

$$\hat{J}_{\pm} |j, m\rangle = \sqrt{j(j+1) - m(m \pm 1)}\hbar |j, m \pm 1\rangle \tag{3}$$

the effect of the lowering operator is

$$\begin{aligned}
\hat{J}_- |j_1, +j_2, j_1, +j_2\rangle &= \left(\hat{J}_-^{(1)} + \hat{J}_-^{(2)}\right) |j_1, j_1\rangle \otimes |j_2, j_2\rangle \\
\sqrt{(j_1, +j_2)(j_1, +j_2 + 1) - (j_1, +j_2)(j_1, +j_2 - 1)}\hbar |j_1, +j_2, j_1, +j_2 - 1\rangle &= \hat{J}_-^{(1)} |j_1, j_1\rangle \otimes |j_2, j_2\rangle \\
&\quad + |j_1, j_1\rangle \otimes \hat{J}_-^{(2)} |j_2, j_2\rangle \\
\sqrt{2(j_1, +j_2)}\hbar |j_1, +j_2, j_1, +j_2 - 1\rangle &= \sqrt{j_1(j_1+1) - j_1(j_1-1)}\hbar |j_1, j_1 - 1\rangle \otimes |j_2, j_2\rangle \\
&\quad + \sqrt{j_2(j_2+1) - j_2(j_2-1)}\hbar |j_1, j_1\rangle \otimes |j_2, j_2 - 1\rangle \\
&\quad \sqrt{2j_1}\hbar |j_1, j_1 - 1\rangle \otimes |j_2, j_2\rangle + \sqrt{2j_2}\hbar |j_1, j_1\rangle \otimes |j_2
\end{aligned}$$

While this state has total $j = j_1 + j_2$, there is a second combination of $|j_1, j_1 - 1\rangle \otimes |j_2, j_2\rangle$ and $|j_1, j_1\rangle \otimes |j_2, j_2 - 1\rangle$ which is orthogonal to this one, found by interchanging the coefficients on the right and changing the relative sign,

$$|\alpha\rangle = \sqrt{2j_2} |j_1, j_1 - 1\rangle \otimes |j_2, j_2\rangle - \sqrt{2j_1} |j_1, j_1\rangle \otimes |j_2, j_2 - 1\rangle$$

Acting with $\hat{\mathbf{J}}^2$ on this state will tell us what representation it belongs to. Act first on each part.

Applying $\hat{\mathbf{J}}^2$ the first term gives

$$\begin{aligned}
\hat{\mathbf{J}}^2 |j_1, j_1 - 1\rangle \otimes |j_2, j_2\rangle &= \hat{\mathbf{J}}_{(1)}^2 |j_1, j_1 - 1\rangle \otimes |j_2, j_2\rangle + \hat{\mathbf{J}}_{(2)}^2 |j_1, j_1 - 1\rangle \otimes |j_2, j_2\rangle \\
&\quad + 2\left(\hat{J}_3^{(1)}\hat{J}_3^{(2)} + \frac{1}{2}\hat{J}_+^{(1)}\hat{J}_-^{(2)} + \frac{1}{2}\hat{J}_-^{(1)}\hat{J}_+^{(2)}\right) |j_1, j_1 - 1\rangle \otimes |j_2, j_2\rangle \\
&= ((j_1(j_1+1) + j_2(j_2+1)) + 2(j_1 - 1)j_2)\hbar^2 |j_1, j_1 - 1\rangle \otimes |j_2, j_2\rangle \\
&\quad + \hbar^2 \sqrt{j_1(j_1+1) - (j_1 - 1)j_1}\sqrt{j_2(j_2+1) - j_2(j_2-1)} |j_1, j_1\rangle \otimes |j_2, j_2 - 1\rangle \\
&= \left(j_1^2 + j_2^2 + j_1 + j_2 + 2j_1j_2 - 2j_2\right)\hbar^2 |j_1, j_1 - 1\rangle \otimes |j_2, j_2\rangle \\
&\quad + 2\sqrt{j_1j_2}\hbar^2 |j_1, j_1\rangle \otimes |j_2, j_2 - 1\rangle
\end{aligned}$$

while the second term gives

$$\begin{aligned}
\hat{\mathbf{J}}^2 |j_1, j_1\rangle \otimes |j_2, j_2 - 1\rangle &= \hat{\mathbf{J}}_{(1)}^2 |j_1, j_1\rangle \otimes |j_2, j_2 - 1\rangle + \hat{\mathbf{J}}_{(2)}^2 |j_1, j_1\rangle \otimes |j_2, j_2 - 1\rangle \\
&\quad + 2\left(\hat{J}_3^{(1)}\hat{J}_3^{(2)} + \frac{1}{2}\hat{J}_+^{(1)}\hat{J}_-^{(2)} + \frac{1}{2}\hat{J}_-^{(1)}\hat{J}_+^{(2)}\right) |j_1, j_1\rangle \otimes |j_2, j_2 - 1\rangle \\
&= (j_1(j_1+1) + j_2(j_2+1) + 2j_1(j_2 - 1))\hbar^2 |j_1, j_1\rangle \otimes |j_2, j_2 - 1\rangle \\
&\quad + \hbar^2 \sqrt{j_1(j_1+1) - j_1(j_1-1)}\sqrt{j_2(j_2+1) - (j_2 - 1)j_2} |j_1, j_1 - 1\rangle \otimes |j_2, j_2\rangle \\
&= \left(j_1^2 + 2j_1j_2 + j_2^2 - j_1 + j_2\right)\hbar^2 |j_1, j_1\rangle \otimes |j_2, j_2 - 1\rangle \\
&\quad + 2\sqrt{j_1j_2}\hbar^2 |j_1, j_1 - 1\rangle \otimes |j_2, j_2\rangle
\end{aligned}$$

Putting this all together, we find the action of $\hat{\mathbf{J}}^2$ on $|\alpha\rangle$

$$\hat{\mathbf{J}}^2 |\alpha\rangle = \hat{\mathbf{J}}^2 \left(\sqrt{2j_2} |j_1, j_1 - 1\rangle \otimes |j_2, j_2\rangle - \sqrt{2j_1} |j_1, j_1\rangle \otimes |j_2, j_2 - 1\rangle\right)$$

$$\begin{aligned}
&= \sqrt{2j_2}\left(\left(j_1^2+j_2^2+j_1+j_2+2j_1j_2-2j_2\right)\hbar^2\left|j_1,j_1-1\right\rangle\otimes\left|j_2,j_2\right\rangle\right)\\
&\quad +2\sqrt{2j_2}\sqrt{j_1j_2}\hbar^2\left|j_1,j_1\right\rangle\otimes\left|j_2,j_2-1\right\rangle\\
&\quad -\sqrt{2j_1}\left(j_1^2+2j_1j_2+j_2^2-j_1+j_2\right)\hbar^2\left|j_1,j_1\right\rangle\otimes\left|j_2,j_2-1\right\rangle\\
&\quad -2\sqrt{2j_1}\sqrt{j_1j_2}\hbar^2\left|j_1,j_1-1\right\rangle\otimes\left|j_2,j_2\right\rangle\\
&= \sqrt{2j_2}\left(j_1^2+2j_1j_2+j_2^2-j_1-j_2\right)\hbar^2\left|j_1,j_1-1\right\rangle\otimes\left|j_2,j_2\right\rangle\\
&\quad -\sqrt{2j_1}\left(j_1^2+2j_1j_2+j_2^2-j_1-j_2\right)\hbar^2\left|j_1,j_1\right\rangle\otimes\left|j_2,j_2-1\right\rangle\\
&= \left(\left(j_1+j_2\right)^2-\left(j_1+j_2\right)\right)\hbar^2\left|\alpha\right\rangle\\
&= \left(j_1+j_2-1\right)\left(j_1+j_2\right)\hbar^2\left|\alpha\right\rangle
\end{aligned}$$

so we have $j=j_1+j_2-1$.

This may or may not exhaust all possibilities. If not, we can lower twice from the top state to get linear combinations of the three $m=j-2$ states,

$$\left|j_1,j_1-2\right\rangle\otimes\left|j_2,j_2\right\rangle,\left|j_1,j_1-2\right\rangle\otimes\left|j_2,j_2-1\right\rangle,\left|j_1,j_1\right\rangle\otimes\left|j_2,j_2-2\right\rangle$$

The two sets of states we have found, $\left|j_1+j_2,m\right\rangle$ and $\left|j_1+j_2-1,m\right\rangle$ account for two linearly independent combinations of these, so if there remain more degrees of freedom we can find a third combination orthogonal to these. It will have j lower by 1, giving a $\left|j_1+j_2-2,m\right\rangle$ representation. We continue in this manner until we have $\left(2j_1+1\right)\left(2j_2+1\right)$ states. Since the $\left|j,m\right\rangle$ account for $2j+1$ of the degrees of freedom, this occurs when

$$\begin{aligned}
\left(2j_1+1\right)\left(2j_2+1\right) &= \sum_{k=0}^{K}\left(2\left(j_1+j_2-k\right)+1\right)\\
&= \sum_{k=0}^{K}\left(2\left(j_1+j_2\right)+1\right)-2\sum_{k=0}^{K}k\\
&= 2\left(K+1\right)\left(j_1+j_2\right)+\left(K+1\right)-2\frac{K\left(K+1\right)}{2}\\
&= 2\left(K+1\right)\left(j_1+j_2\right)+1-K^2
\end{aligned}$$

Solving for K,

$$\begin{aligned}
0 &= K^2-2\left(K+1\right)\left(j_1+j_2\right)-1+\left(2j_1+1\right)\left(2j_2+1\right)\\
&= K^2-2K\left(j_1+j_2\right)+4j_1j_2\\
K &= \frac{2\left(j_1+j_2\right)\pm\sqrt{4\left(j_1+j_2\right)^2-16j_1j_2}}{2}\\
&= j_1+j_2\pm\sqrt{\left(j_1-j_2\right)^2}\\
&= 2j_1,2j_2
\end{aligned}$$

Therefore, the representations include all j from j_1+j_2 to the first of $j=j_1+j_2-2j_2$ or $j=j_1+j_2-2j_2$ to occur. If $j_1>j_2$, then $j=j_1-j_2>0$ will be the first to occur; if $j_2>j_1$ then $j_2-j_1>0$ will occur first, so in either case, the lowest value is $\left|j_1-j_2\right|$ and we will have representations

$$\begin{gathered}
\left|j_1+j_2,m\right\rangle\\
\left|j_1+j_2-1,m\right\rangle\\
\vdots\\
\left|\left|j_1-j_2\right|,m\right\rangle
\end{gathered}$$

and will have exactly accounted for all states of the system.

4 Example 1: Two spin 1/2 particles

The simplest nontrivial addition comes when we combine two spin-$\frac{1}{2}$ particles to get four states of the form

$$\left|\frac{1}{2},m_1\right\rangle_1\otimes\left|\frac{1}{2},m_2\right\rangle_2$$

With the highest$j=j_1+j_2=1$, and stepping down to $j_1-j_2=0$, we expect $j=1, j=0$ states. Begin with the highest state,

$$|1,1\rangle=\left|\frac{1}{2},\frac{1}{2}\right\rangle_1\otimes\left|\frac{1}{2},\frac{1}{2}\right\rangle_2$$

and apply $\hat{J}_-=\hat{J}_-^{(1)}+\hat{J}_-^{(2)}$,

$$\begin{aligned}
\hat{J}_-|1,1\rangle &= \left(\hat{J}_-^{(1)}+\hat{J}_-^{(2)}\right)\left|\frac{1}{2},\frac{1}{2}\right\rangle_1\otimes\left|\frac{1}{2},\frac{1}{2}\right\rangle_2 \\
\sqrt{1\cdot(1+1)-1\,(1-1)}\hbar\,|1,0\rangle &= \hat{J}_-^{(1)}\left|\frac{1}{2},\frac{1}{2}\right\rangle\otimes\left|\frac{1}{2},\frac{1}{2}\right\rangle_2+\left|\frac{1}{2},\frac{1}{2}\right\rangle_1\otimes\hat{J}_-^{(2)}\left|\frac{1}{2},\frac{1}{2}\right\rangle_2 \\
\sqrt{2}\hbar\,|1,0\rangle &= \sqrt{\frac{1}{2}\left(\frac{3}{2}\right)-\frac{1}{2}\left(\frac{1}{2}-1\right)}\hbar\left|\frac{1}{2},-\frac{1}{2}\right\rangle\otimes\left|\frac{1}{2},\frac{1}{2}\right\rangle_2+\left|\frac{1}{2},\frac{1}{2}\right\rangle_1\otimes\sqrt{\frac{3}{4}+\frac{1}{4}}\hbar\left|\frac{1}{2},-\frac{1}{2}\right\rangle_2 \\
|1,0\rangle &= \frac{1}{\sqrt{2}}\left(\left|\frac{1}{2},-\frac{1}{2}\right\rangle\otimes\left|\frac{1}{2},\frac{1}{2}\right\rangle_2+\left|\frac{1}{2},\frac{1}{2}\right\rangle_1\otimes\left|\frac{1}{2},-\frac{1}{2}\right\rangle_2\right)
\end{aligned}$$

Lowering again, we complete the $j=1$ triplet,

$$\begin{aligned}
\hat{J}_-|1,0\rangle &= \frac{1}{\sqrt{2}}\left(\hat{J}_-^{(1)}+\hat{J}_-^{(2)}\right)\left(\left|\frac{1}{2},-\frac{1}{2}\right\rangle\otimes\left|\frac{1}{2},\frac{1}{2}\right\rangle_2+\left|\frac{1}{2},\frac{1}{2}\right\rangle_1\otimes\left|\frac{1}{2},-\frac{1}{2}\right\rangle_2\right) \\
\sqrt{2}\hbar\,|1,-1\rangle &= \frac{1}{\sqrt{2}}\left(\left|\frac{1}{2},-\frac{1}{2}\right\rangle\otimes\hat{J}_-^{(2)}\left|\frac{1}{2},\frac{1}{2}\right\rangle_2+\hat{J}_-^{(1)}\left|\frac{1}{2},\frac{1}{2}\right\rangle_1\otimes\left|\frac{1}{2},-\frac{1}{2}\right\rangle_2\right)
\end{aligned}$$

where we use the fact that $\hat{J}_-\left|\frac{1}{2},-\frac{1}{2}\right\rangle=0$. Then

$$\begin{aligned}
\sqrt{2}\,|1,-1\rangle &= \frac{1}{\sqrt{2}}\left(\left|\frac{1}{2},-\frac{1}{2}\right\rangle\otimes\left|\frac{1}{2},-\frac{1}{2}\right\rangle_2+\left|\frac{1}{2},-\frac{1}{2}\right\rangle_1\otimes\left|\frac{1}{2},-\frac{1}{2}\right\rangle_2\right) \\
|1,-1\rangle &= \left|\frac{1}{2},-\frac{1}{2}\right\rangle\otimes\left|\frac{1}{2},-\frac{1}{2}\right\rangle_2
\end{aligned}$$

so the full triplet is

$$\begin{aligned}
|1,1\rangle &= \left|\frac{1}{2},\frac{1}{2}\right\rangle_1\otimes\left|\frac{1}{2},\frac{1}{2}\right\rangle_2 \\
|1,0\rangle &= \frac{1}{\sqrt{2}}\left(\left|\frac{1}{2},-\frac{1}{2}\right\rangle\otimes\left|\frac{1}{2},\frac{1}{2}\right\rangle_2+\left|\frac{1}{2},\frac{1}{2}\right\rangle_1\otimes\left|\frac{1}{2},-\frac{1}{2}\right\rangle_2\right) \\
|1,-1\rangle &= \left|\frac{1}{2},-\frac{1}{2}\right\rangle\otimes\left|\frac{1}{2},-\frac{1}{2}\right\rangle_2
\end{aligned}$$

There is one remaining state, and it must be the single $j=0$ state. Since we must also have $m=0$, it must be constructed from the $m_1+m_2=0$ combinations, $\left|\frac{1}{2},-\frac{1}{2}\right\rangle\otimes\left|\frac{1}{2},\frac{1}{2}\right\rangle_2$ and $\left|\frac{1}{2},\frac{1}{2}\right\rangle_1\otimes\left|\frac{1}{2},-\frac{1}{2}\right\rangle_2$. Also, it must be orthogonal to the other three states. This is immediate for the $\left|\frac{1}{2},\frac{1}{2}\right\rangle_1\otimes\left|\frac{1}{2},\frac{1}{2}\right\rangle_2$ and $\left|\frac{1}{2},-\frac{1}{2}\right\rangle\otimes\left|\frac{1}{2},-\frac{1}{2}\right\rangle_2$, while for the $|1,0\rangle$ state orthogonality forces us to write

$$|0,0\rangle=\frac{1}{\sqrt{2}}\left(\left|\frac{1}{2},-\frac{1}{2}\right\rangle\otimes\left|\frac{1}{2},\frac{1}{2}\right\rangle_2-\left|\frac{1}{2},\frac{1}{2}\right\rangle_1\otimes\left|\frac{1}{2},-\frac{1}{2}\right\rangle_2\right)$$

This completes the identification of states.

5 Example 2: Real 3x3 matrix: Add two j = 1 states

We have noted before that a real, 3×3 matrix can be decomposed into a 1-dim trace term, a 3-dim space of antisymmetric matrices, and a 5-dim space of traceless, symmetric matrices. We consider this decomposition in terms of irreducible representations. Since the $|1,m\rangle$ states form a 3-dim vector space, we can think of a matrix as an outer product of two of them,

$$|1,m_1\rangle\,|1,m_2\rangle$$

We have $j_1 = j_2 = 1$, so the range of total j should be from $j = j_1 + j_2 = 2$ to $j = |j_1 - j_2| = 0$. Thus, we have three irreducible representations,

$$\begin{array}{c}|2,m\rangle\\ |1,m\rangle\\ |0,0\rangle\end{array}$$

of dimensions $2j+1$, that is, $5, 3$ and 1 as expected.

To compute the states in detail, we start with the $j = 2$ states,

$$\begin{aligned}
|1,1\rangle\,|1,1\rangle &= |2,2\rangle\\
\frac{1}{\sqrt{2}}\left(|1,0\rangle\,|1,1\rangle + |1,1\rangle\,|1,0\rangle\right) &= |2,1\rangle\\
\frac{1}{\sqrt{6}}\,|1,-1\rangle\,|1,1\rangle + \frac{2}{\sqrt{6}}\,|1,0\rangle\,|1,0\rangle + \frac{1}{\sqrt{6}}\,|1,1\rangle\,|1,-1\rangle &= |2,0\rangle\\
\frac{1}{\sqrt{2}}\,|1,-1\rangle\,|1,0\rangle + \frac{1}{\sqrt{2}}\,|1,0\rangle\,|1,-1\rangle &= |2,-1\rangle\\
|1,-1\rangle\,|1,-1\rangle &= |2,-2\rangle
\end{aligned}$$

The $j = 1$ states start with the unique state orthogonal to $|2,1\rangle$, namely $\frac{1}{\sqrt{2}}\left(|1,0\rangle\,|1,1\rangle - |1,1\rangle\,|1,0\rangle\right)$. Therefore,

$$\begin{aligned}
\frac{1}{\sqrt{2}}\left(|1,0\rangle\,|1,1\rangle - |1,1\rangle\,|1,0\rangle\right) &= |1,1\rangle\\
\frac{1}{\sqrt{2}}\left(|1,-1\rangle\,|1,1\rangle - |1,1\rangle\,|1,-1\rangle\right) &= |1,0\rangle\\
\frac{1}{\sqrt{2}}\left(|1,-1\rangle\,|1,0\rangle - |1,0\rangle\,|1,-1\rangle\right) &= |1,-1\rangle
\end{aligned}$$

The final state is the unique normalized state orthogonal to both:

$$\begin{aligned}
|2,0\rangle &= \frac{1}{\sqrt{6}}\,|1,-1\rangle\,|1,1\rangle + \frac{2}{\sqrt{6}}\,|1,0\rangle\,|1,0\rangle + \frac{1}{\sqrt{6}}\,|1,1\rangle\,|1,-1\rangle\\
|1,0\rangle &= \frac{1}{\sqrt{2}}\left(|1,-1\rangle\,|1,1\rangle - |1,1\rangle\,|1,-1\rangle\right)
\end{aligned}$$

and built as a linear combination

$$|0,0\rangle = \alpha\,|1,-1\rangle\,|1,1\rangle + \beta\,|1,0\rangle\,|1,0\rangle + \gamma\,|1,1\rangle\,|1,-1\rangle$$

since this is the most general combination with $m = 0$. Orthogonality between $|0,0\rangle$ and $|1,0\rangle$ shows that $\alpha = \gamma$, then orthogonality between $|0,0\rangle$ and $|2,0\rangle$ gives

$$\alpha\frac{1}{\sqrt{6}} + \beta\frac{2}{\sqrt{6}} + \alpha\frac{1}{\sqrt{6}} = 0$$

so that $\beta = -\alpha$. This means that $|0,0\rangle = \alpha\left(|1,-1\rangle|1,1\rangle - |1,0\rangle|1,0\rangle + |1,1\rangle|1,-1\rangle\right)$ and we choose $\alpha = \frac{1}{\sqrt{3}}$ to normalize, giving the final one of the nine states,

$$|0,0\rangle = \frac{1}{\sqrt{3}}\left(|1,-1\rangle|1,1\rangle - |1,0\rangle|1,0\rangle + |1,1\rangle|1,-1\rangle\right)$$

6 Example 3: Add j = 1 and j = 3/2 states

Suppose we want to add $|1,m\rangle$ and $\left|\frac{3}{2},m\right\rangle$ angular momenta. Then there are $(2j_1+1)(2j_2+1) = (2\cdot 1+1)\left(2\cdot\frac{3}{2}+1\right) = 12$ states of the form

$$\left|\frac{3}{2},m_2\right\rangle|1,m_1\rangle$$

and we expect the total angular momentum to run from $j = 1+\frac{3}{2} = \frac{5}{2}$ down to $j = \frac{3}{2}-1 = \frac{1}{2}$, that is,

$$\begin{array}{c} 6\ states\ \left|\frac{5}{2},m\right\rangle \\ 4\ states\ \left|\frac{3}{2},m\right\rangle \\ 2\ states\ \left|\frac{1}{2},m\right\rangle \end{array}$$

to get the requisite 12 states.

6.1 The j = 5/2 states

We start with the highest state,

$$\left|\frac{5}{2},\frac{5}{2}\right\rangle = \left|\frac{3}{2},\frac{3}{2}\right\rangle|1,1\rangle$$

and apply the lowering operator:

$$\begin{aligned}
\hat{J}_-\left|\frac{5}{2},\frac{5}{2}\right\rangle &= \left(\hat{J}_-^{(1)} + \hat{J}_-^{(2)}\right)\left|\frac{3}{2},\frac{3}{2}\right\rangle|1,1\rangle \\
\sqrt{\frac{5}{2}\cdot\frac{7}{2} - \frac{5}{2}\cdot\frac{3}{2}}\left|\frac{5}{2},\frac{5}{2}-1\right\rangle &= \sqrt{\frac{3}{2}\cdot\frac{5}{2} - \frac{3}{2}\cdot\frac{1}{2}}\left|\frac{3}{2},\frac{1}{2}\right\rangle|1,1\rangle + \sqrt{1\cdot 2 - 1\cdot 0}\left|\frac{3}{2},\frac{3}{2}\right\rangle|1,0\rangle \\
\sqrt{5}\left|\frac{5}{2},\frac{3}{2}\right\rangle &= \sqrt{3}\left|\frac{3}{2},\frac{1}{2}\right\rangle|1,1\rangle + \sqrt{2}\left|\frac{3}{2},\frac{3}{2}\right\rangle|1,0\rangle
\end{aligned}$$

This gives the second state,

$$\left|\frac{5}{2},\frac{3}{2}\right\rangle = \sqrt{\frac{3}{5}}\left|\frac{3}{2},\frac{1}{2}\right\rangle|1,1\rangle + \sqrt{\frac{2}{5}}\left|\frac{3}{2},\frac{3}{2}\right\rangle|1,0\rangle$$

Notice that the state is normalized and that each term has a total $m = \frac{3}{2}$, i.e., for the first term on the right, $\frac{1}{2}+1 = \frac{3}{2}$ and for the second term, $\frac{3}{2}+0 = \frac{3}{2}$. Continue, lowering four more times:

$$\begin{aligned}
\hat{J}_-\left|\frac{5}{2},\frac{3}{2}\right\rangle &= \left(\hat{J}_-^{(1)} + \hat{J}_-^{(2)}\right)\left(\sqrt{\frac{3}{5}}\left|\frac{3}{2},\frac{1}{2}\right\rangle|1,1\rangle + \sqrt{\frac{2}{5}}\left|\frac{3}{2},\frac{3}{2}\right\rangle|1,0\rangle\right) \\
\sqrt{\frac{5}{2}\cdot\frac{7}{2} - \frac{3}{2}\cdot\frac{1}{2}}\left|\frac{5}{2},\frac{1}{2}\right\rangle &= \sqrt{\frac{3}{5}}\hat{J}_-^{(1)}\left|\frac{3}{2},\frac{1}{2}\right\rangle|1,1\rangle + \sqrt{\frac{3}{5}}\hat{J}_-^{(2)}\left|\frac{3}{2},\frac{1}{2}\right\rangle|1,1\rangle \\
&\quad + \sqrt{\frac{2}{5}}\hat{J}_-^{(1)}\left|\frac{3}{2},\frac{3}{2}\right\rangle|1,0\rangle + \sqrt{\frac{2}{5}}\hat{J}_-^{(2)}\left|\frac{3}{2},\frac{3}{2}\right\rangle|1,0\rangle
\end{aligned}$$

$$
\begin{aligned}
2\sqrt{2}\left|\frac{5}{2},\frac{1}{2}\right\rangle &= \sqrt{\frac{3}{5}}\sqrt{\frac{3}{2}\cdot\frac{5}{2}-\frac{1}{2}\cdot\left(-\frac{1}{2}\right)}\left|\frac{3}{2},-\frac{1}{2}\right\rangle|1,1\rangle+\sqrt{\frac{3}{5}}\sqrt{2}\left|\frac{3}{2},\frac{1}{2}\right\rangle|1,0\rangle \\
&\quad+\sqrt{\frac{2}{5}}\sqrt{\frac{3}{2}\cdot\frac{5}{2}-\frac{3}{2}\cdot\frac{1}{2}}\left|\frac{3}{2},\frac{1}{2}\right\rangle|1,0\rangle+\sqrt{\frac{2}{5}}\sqrt{1\cdot 2-0\cdot(-1)}\left|\frac{3}{2},\frac{3}{2}\right\rangle|1,-1\rangle \\
2\sqrt{2}\left|\frac{5}{2},\frac{1}{2}\right\rangle &= 2\sqrt{\frac{3}{5}}\left|\frac{3}{2},-\frac{1}{2}\right\rangle|1,1\rangle+\sqrt{\frac{6}{5}}\left|\frac{3}{2},\frac{1}{2}\right\rangle|1,0\rangle+\sqrt{\frac{6}{5}}\left|\frac{3}{2},\frac{1}{2}\right\rangle|1,0\rangle+\frac{2}{\sqrt{5}}\left|\frac{3}{2},\frac{3}{2}\right\rangle|1,-1\rangle
\end{aligned}
$$

so we have

$$
\left|\frac{5}{2},\frac{1}{2}\right\rangle=\sqrt{\frac{3}{10}}\left|\frac{3}{2},-\frac{1}{2}\right\rangle|1,1\rangle+\sqrt{\frac{6}{10}}\left|\frac{3}{2},\frac{1}{2}\right\rangle|1,0\rangle+\frac{1}{\sqrt{10}}\left|\frac{3}{2},\frac{3}{2}\right\rangle|1,-1\rangle
$$

Once again, the state is normalized and $m_1+m_2=\frac{1}{2}$ in each term.

Continue

$$
\begin{aligned}
\hat{J}_-\left|\frac{5}{2},\frac{1}{2}\right\rangle &= \sqrt{\frac{3}{10}}\left(\hat{J}_-^{(1)}+\hat{J}_-^{(2)}\right)\left|\frac{3}{2},-\frac{1}{2}\right\rangle|1,1\rangle \\
&\quad+\sqrt{\frac{6}{10}}\left(\hat{J}_-^{(1)}+\hat{J}_-^{(2)}\right)\left|\frac{3}{2},\frac{1}{2}\right\rangle|1,0\rangle+\frac{1}{\sqrt{10}}\left(\hat{J}_-^{(1)}+\hat{J}_-^{(2)}\right)\left|\frac{3}{2},\frac{3}{2}\right\rangle|1,-1\rangle \\
3\left|\frac{5}{2},-\frac{1}{2}\right\rangle &= \sqrt{\frac{3}{10}}\sqrt{3}\left|\frac{3}{2},-\frac{3}{2}\right\rangle|1,1\rangle+\sqrt{\frac{3}{10}}\sqrt{2}\left|\frac{3}{2},-\frac{1}{2}\right\rangle|1,0\rangle+\sqrt{\frac{6}{10}}2\left|\frac{3}{2},-\frac{1}{2}\right\rangle|1,0\rangle \\
&\quad+\sqrt{\frac{6}{10}}\sqrt{2}\left|\frac{3}{2},\frac{1}{2}\right\rangle|1,-1\rangle+\frac{1}{\sqrt{10}}\sqrt{3}\left|\frac{3}{2},\frac{1}{2}\right\rangle|1,-1\rangle+0
\end{aligned}
$$

and therefore,

$$
\left|\frac{5}{2},-\frac{1}{2}\right\rangle = \sqrt{\frac{1}{10}}\left|\frac{3}{2},-\frac{3}{2}\right\rangle|1,1\rangle+\sqrt{\frac{6}{10}}\left|\frac{3}{2},-\frac{1}{2}\right\rangle|1,0\rangle+\sqrt{\frac{3}{10}}\left|\frac{3}{2},\frac{1}{2}\right\rangle|1,-1\rangle
$$

Notice the symmetry in the coefficients between this state and the previous one. The symmetry continues, with the next lowering leading us to

$$
\left|\frac{5}{2},-\frac{3}{2}\right\rangle=\sqrt{\frac{3}{5}}\left|\frac{3}{2},-\frac{1}{2}\right\rangle|1,-1\rangle+\sqrt{\frac{2}{5}}\left|\frac{3}{2},-\frac{3}{2}\right\rangle|1,0\rangle
$$

and finally, the lowest state is unique just like the highest state. Just to check the consistency, we work it out:

$$
\begin{aligned}
\hat{J}_-\left|\frac{5}{2},-\frac{3}{2}\right\rangle &= \sqrt{\frac{3}{5}}\left(\hat{J}_-^{(1)}+\hat{J}_-^{(2)}\right)\left|\frac{3}{2},-\frac{1}{2}\right\rangle|1,-1\rangle+\sqrt{\frac{2}{5}}\left(\hat{J}_-^{(1)}+\hat{J}_-^{(2)}\right)\left|\frac{3}{2},-\frac{3}{2}\right\rangle|1,0\rangle \\
\sqrt{5}\left|\frac{5}{2},-\frac{5}{2}\right\rangle &= \sqrt{\frac{3}{5}}\hat{J}_-^{(1)}\left|\frac{3}{2},-\frac{1}{2}\right\rangle|1,-1\rangle+\sqrt{\frac{3}{5}}\hat{J}_-^{(2)}\left|\frac{3}{2},-\frac{1}{2}\right\rangle|1,-1\rangle \\
&\quad+\sqrt{\frac{2}{5}}\hat{J}_-^{(1)}\left|\frac{3}{2},-\frac{3}{2}\right\rangle|1,0\rangle+\sqrt{\frac{2}{5}}\hat{J}_-^{(2)}\left|\frac{3}{2},-\frac{3}{2}\right\rangle|1,0\rangle \\
&= \sqrt{\frac{3}{5}}\sqrt{3}\left|\frac{3}{2},-\frac{3}{2}\right\rangle|1,-1\rangle+0+0+\sqrt{\frac{2}{5}}\sqrt{2}\left|\frac{3}{2},-\frac{3}{2}\right\rangle|1,0\rangle
\end{aligned}
$$

so that, combining the terms,

$$
\left|\frac{5}{2},-\frac{5}{2}\right\rangle = \left|\frac{3}{2},-\frac{3}{2}\right\rangle|1,-1\rangle
$$

as expected.

Collecting the full multiplet,

$$\begin{aligned}
\left|\frac{5}{2},\frac{5}{2}\right\rangle &= \left|\frac{3}{2},\frac{3}{2}\right\rangle|1,1\rangle \\
\left|\frac{5}{2},\frac{3}{2}\right\rangle &= \sqrt{\frac{3}{5}}\left|\frac{3}{2},\frac{1}{2}\right\rangle|1,1\rangle+\sqrt{\frac{2}{5}}\left|\frac{3}{2},\frac{3}{2}\right\rangle|1,0\rangle \\
\left|\frac{5}{2},\frac{1}{2}\right\rangle &= \sqrt{\frac{3}{10}}\left|\frac{3}{2},-\frac{1}{2}\right\rangle|1,1\rangle+\sqrt{\frac{6}{10}}\left|\frac{3}{2},\frac{1}{2}\right\rangle|1,0\rangle+\frac{1}{\sqrt{10}}\left|\frac{3}{2},\frac{3}{2}\right\rangle|1,-1\rangle \\
\left|\frac{5}{2},-\frac{1}{2}\right\rangle &= \sqrt{\frac{1}{10}}\left|\frac{3}{2},-\frac{3}{2}\right\rangle|1,1\rangle+\sqrt{\frac{6}{10}}\left|\frac{3}{2},-\frac{1}{2}\right\rangle|1,0\rangle+\sqrt{\frac{3}{10}}\left|\frac{3}{2},\frac{1}{2}\right\rangle|1,-1\rangle \\
\left|\frac{5}{2},-\frac{3}{2}\right\rangle &= \sqrt{\frac{3}{5}}\left|\frac{3}{2},-\frac{1}{2}\right\rangle|1,-1\rangle+\sqrt{\frac{2}{5}}\left|\frac{3}{2},-\frac{3}{2}\right\rangle|1,0\rangle \\
\left|\frac{5}{2},-\frac{5}{2}\right\rangle &= \left|\frac{3}{2},-\frac{3}{2}\right\rangle|1,-1\rangle
\end{aligned}$$

6.2 The j = 3/2 states

The second state,

$$\left|\frac{5}{2},\frac{3}{2}\right\rangle=\sqrt{\frac{3}{5}}\left|\frac{3}{2},\frac{1}{2}\right\rangle|1,1\rangle+\sqrt{\frac{2}{5}}\left|\frac{3}{2},\frac{3}{2}\right\rangle|1,0\rangle$$

involved a linear combination of two product states, $\left|\frac{3}{2},\frac{1}{2}\right\rangle|1,1\rangle$ and $\left|\frac{3}{2},\frac{3}{2}\right\rangle|1,0\rangle$, so there is a unique second state built from these,

$$|j,m\rangle=\sqrt{\frac{2}{5}}\left|\frac{3}{2},\frac{1}{2}\right\rangle|1,1\rangle-\sqrt{\frac{3}{5}}\left|\frac{3}{2},\frac{3}{2}\right\rangle|1,0\rangle$$

As shown above, this state will have a total angular momentum of $j=\frac{3}{2}$. It is the highest state, as can be seen by noting that $m_1+m_2=\frac{3}{2}$ in each term. Therefore,

$$\left|\frac{3}{2},\frac{3}{2}\right\rangle=\sqrt{\frac{2}{5}}\left|\frac{3}{2},\frac{1}{2}\right\rangle|1,1\rangle-\sqrt{\frac{3}{5}}\left|\frac{3}{2},\frac{3}{2}\right\rangle|1,0\rangle$$

and we may lower to find the remaining states,

$$\sqrt{3}\left|\frac{3}{2},\frac{1}{2}\right\rangle=2\sqrt{\frac{2}{5}}\left|\frac{3}{2},-\frac{1}{2}\right\rangle|1,1\rangle+\sqrt{2}\sqrt{\frac{2}{5}}\left|\frac{3}{2},\frac{1}{2}\right\rangle|1,0\rangle-\sqrt{3}\sqrt{\frac{3}{5}}\left|\frac{3}{2},\frac{1}{2}\right\rangle|1,0\rangle-\sqrt{2}\sqrt{\frac{3}{5}}\left|\frac{3}{2},\frac{3}{2}\right\rangle|1,-1\rangle$$

so

$$\left|\frac{3}{2},\frac{1}{2}\right\rangle=\sqrt{\frac{8}{15}}\left|\frac{3}{2},-\frac{1}{2}\right\rangle|1,1\rangle-\sqrt{\frac{1}{15}}\left|\frac{3}{2},\frac{1}{2}\right\rangle|1,0\rangle-\sqrt{\frac{6}{15}}\left|\frac{3}{2},\frac{3}{2}\right\rangle|1,-1\rangle$$

Lowering again,

$$\begin{aligned}
2\left|\frac{3}{2},-\frac{1}{2}\right\rangle &= \sqrt{3}\sqrt{\frac{8}{15}}\left|\frac{3}{2},-\frac{3}{2}\right\rangle|1,1\rangle+\sqrt{\frac{8}{15}}\sqrt{2}\left|\frac{3}{2},-\frac{1}{2}\right\rangle|1,0\rangle-\frac{2}{\sqrt{15}}\left|\frac{3}{2},-\frac{1}{2}\right\rangle|1,0\rangle \\
&\quad -\frac{1}{\sqrt{15}}\sqrt{2}\left|\frac{3}{2},\frac{1}{2}\right\rangle|1,-1\rangle-\sqrt{3}\sqrt{\frac{6}{15}}\left|\frac{3}{2},\frac{1}{2}\right\rangle|1,-1\rangle \\
\left|\frac{3}{2},-\frac{1}{2}\right\rangle &= \sqrt{\frac{6}{15}}\left|\frac{3}{2},-\frac{3}{2}\right\rangle|1,1\rangle+\frac{1}{\sqrt{15}}\left|\frac{3}{2},-\frac{1}{2}\right\rangle|1,0\rangle-\sqrt{\frac{8}{15}}\left|\frac{3}{2},\frac{1}{2}\right\rangle|1,-1\rangle
\end{aligned}$$

and finally,

$$\begin{aligned}
\sqrt{3}\left|\frac{3}{2},-\frac{3}{2}\right\rangle &= \sqrt{\frac{6}{15}}\sqrt{2}\left|\frac{3}{2},-\frac{3}{2}\right\rangle|1,0\rangle + \frac{1}{\sqrt{15}}\sqrt{3}\left|\frac{3}{2},-\frac{3}{2}\right\rangle|1,0\rangle \\
&\quad + \frac{1}{\sqrt{15}}\sqrt{2}\left|\frac{3}{2},-\frac{1}{2}\right\rangle|1,-1\rangle - 2\sqrt{\frac{8}{15}}\left|\frac{3}{2},-\frac{1}{2}\right\rangle|1,-1\rangle \\
\left|\frac{3}{2},-\frac{3}{2}\right\rangle &= \sqrt{\frac{3}{5}}\left|\frac{3}{2},-\frac{3}{2}\right\rangle|1,0\rangle - \sqrt{\frac{2}{5}}\left|\frac{3}{2},-\frac{1}{2}\right\rangle|1,-1\rangle
\end{aligned}$$

which is what we expect for the lowest state.

Just as a check, try to lower this last state again:

$$\begin{aligned}
0 &= \sqrt{2}\sqrt{\frac{3}{5}}\left|\frac{3}{2},-\frac{3}{2}\right\rangle|1,-1\rangle - \sqrt{3}\sqrt{\frac{2}{5}}\left|\frac{3}{2},-\frac{3}{2}\right\rangle|1,-1\rangle \\
&= 0
\end{aligned}$$

We therefore have all of the $\left|\frac{3}{2},m\right\rangle$ states,

$$\begin{aligned}
\left|\frac{3}{2},\frac{3}{2}\right\rangle &= \sqrt{\frac{2}{5}}\left|\frac{3}{2},\frac{1}{2}\right\rangle|1,1\rangle - \sqrt{\frac{3}{5}}\left|\frac{3}{2},\frac{3}{2}\right\rangle|1,0\rangle \\
\left|\frac{3}{2},\frac{1}{2}\right\rangle &= \sqrt{\frac{8}{15}}\left|\frac{3}{2},-\frac{1}{2}\right\rangle|1,1\rangle - \sqrt{\frac{1}{15}}\left|\frac{3}{2},\frac{1}{2}\right\rangle|1,0\rangle - \sqrt{\frac{6}{15}}\left|\frac{3}{2},\frac{3}{2}\right\rangle|1,-1\rangle \\
\left|\frac{3}{2},-\frac{1}{2}\right\rangle &= \sqrt{\frac{6}{15}}\left|\frac{3}{2},-\frac{3}{2}\right\rangle|1,1\rangle + \frac{1}{\sqrt{15}}\left|\frac{3}{2},-\frac{1}{2}\right\rangle|1,0\rangle - \sqrt{\frac{8}{15}}\left|\frac{3}{2},\frac{1}{2}\right\rangle|1,-1\rangle \\
\left|\frac{3}{2},-\frac{3}{2}\right\rangle &= \sqrt{\frac{3}{5}}\left|\frac{3}{2},-\frac{3}{2}\right\rangle|1,0\rangle - \sqrt{\frac{2}{5}}\left|\frac{3}{2},-\frac{1}{2}\right\rangle|1,-1\rangle
\end{aligned}$$

6.3 The j = 1/2 states

We have exhausted all states with $m_1+m_2=\frac{5}{2}$ and $m_1+m_2=\frac{3}{2}$, but there remain two linear combinations unaccounted for. We have two states built from

$$\left|\frac{3}{2},-\frac{1}{2}\right\rangle|1,1\rangle, \left|\frac{3}{2},\frac{1}{2}\right\rangle|1,0\rangle, \left|\frac{3}{2},\frac{3}{2}\right\rangle|1,-1\rangle$$

and two states built out of

$$\left|\frac{3}{2},-\frac{3}{2}\right\rangle|1,1\rangle, \left|\frac{3}{2},-\frac{1}{2}\right\rangle|1,0\rangle, \left|\frac{3}{2},\frac{1}{2}\right\rangle|1,-1\rangle$$

Consider the two states built from the first triple, with $m_1+m_2=\frac{1}{2}$,

$$\begin{aligned}
\left|\frac{5}{2},\frac{1}{2}\right\rangle &= \sqrt{\frac{3}{10}}\left|\frac{3}{2},-\frac{1}{2}\right\rangle|1,1\rangle + \sqrt{\frac{6}{10}}\left|\frac{3}{2},\frac{1}{2}\right\rangle|1,0\rangle + \frac{1}{\sqrt{10}}\left|\frac{3}{2},\frac{3}{2}\right\rangle|1,-1\rangle \\
\left|\frac{3}{2},\frac{1}{2}\right\rangle &= \sqrt{\frac{8}{15}}\left|\frac{3}{2},-\frac{1}{2}\right\rangle|1,1\rangle - \sqrt{\frac{1}{15}}\left|\frac{3}{2},\frac{1}{2}\right\rangle|1,0\rangle - \sqrt{\frac{6}{15}}\left|\frac{3}{2},\frac{3}{2}\right\rangle|1,-1\rangle
\end{aligned}$$

These are orthogonal to one another, since

$$\begin{aligned}
\left\langle\frac{3}{2},\frac{1}{2}\middle|\frac{5}{2},\frac{1}{2}\right\rangle &= \sqrt{\frac{8}{15}}\sqrt{\frac{3}{10}} - \sqrt{\frac{1}{15}}\sqrt{\frac{6}{10}} - \sqrt{\frac{6}{15}}\frac{1}{\sqrt{10}} \\
&= \frac{2}{5} - \frac{1}{5} - \frac{1}{5} \\
&= 0
\end{aligned}$$

and there is exactly one more normalized state orthogonal to both of these. This will be the $|j,m\rangle = \left|\frac{1}{2},\frac{1}{2}\right\rangle$ state. Start with an arbitrary linear combination,

$$\left|\frac{1}{2},\frac{1}{2}\right\rangle = \alpha\left|\frac{3}{2},-\frac{1}{2}\right\rangle|1,1\rangle + \beta\left|\frac{3}{2},\frac{1}{2}\right\rangle|1,0\rangle + \gamma\left|\frac{3}{2},\frac{3}{2}\right\rangle|1,-1\rangle$$

and demand the vanishing of the inner products

$$\begin{aligned} 0 = \left\langle\frac{1}{2},\frac{1}{2}\right|\left.\frac{5}{2},\frac{1}{2}\right\rangle &= \sqrt{\frac{3}{10}}\alpha + \sqrt{\frac{6}{10}}\beta + \sqrt{\frac{1}{10}}\gamma \\ 0 = \left\langle\frac{1}{2},\frac{1}{2}\right|\left.\frac{3}{2},\frac{1}{2}\right\rangle &= \sqrt{\frac{8}{15}}\alpha - \sqrt{\frac{1}{15}}\beta - \sqrt{\frac{6}{15}}\gamma \end{aligned}$$

Cancelling the denominators,

$$\begin{aligned} 0 &= \sqrt{3}\alpha + \sqrt{6}\beta + \gamma \\ 0 &= \sqrt{8}\alpha - \beta - \sqrt{6}\gamma \end{aligned}$$

so that $\beta = \sqrt{8}\alpha - \sqrt{6}\gamma$ from the second, leaving

$$\begin{aligned} 0 &= \sqrt{3}\alpha + \sqrt{6}\left(\sqrt{8}\alpha - \sqrt{6}\gamma\right) + \gamma \\ &= \sqrt{3}\alpha + 4\sqrt{3}\alpha - 6\gamma + \gamma \\ &= 5\sqrt{3}\alpha - 5\gamma \end{aligned}$$

Therefore,

$$\begin{aligned} \gamma &= \sqrt{3}\alpha \\ \beta &= \sqrt{8}\alpha - \sqrt{6}\gamma \\ &= \sqrt{8}\alpha - \sqrt{18}\sqrt{3}\alpha \\ &= -\sqrt{2}\alpha \end{aligned}$$

so our state is

$$\left|\frac{1}{2},\frac{1}{2}\right\rangle = \alpha\left|\frac{3}{2},-\frac{1}{2}\right\rangle|1,1\rangle - \sqrt{2}\alpha\left|\frac{3}{2},\frac{1}{2}\right\rangle|1,0\rangle + \sqrt{3}\alpha\left|\frac{3}{2},\frac{3}{2}\right\rangle|1,-1\rangle$$

and normalization requires

$$\begin{aligned} 1 &= \left\langle\frac{1}{2},\frac{1}{2}\right|\left.\frac{1}{2},\frac{1}{2}\right\rangle \\ &= \alpha^2(1+2+3) \\ \alpha &= \frac{1}{\sqrt{6}} \end{aligned}$$

Finally,

$$\left|\frac{1}{2},\frac{1}{2}\right\rangle = \sqrt{\frac{1}{6}}\left|\frac{3}{2},-\frac{1}{2}\right\rangle|1,1\rangle - \sqrt{\frac{2}{6}}\left|\frac{3}{2},\frac{1}{2}\right\rangle|1,0\rangle + \sqrt{\frac{3}{6}}\left|\frac{3}{2},\frac{3}{2}\right\rangle|1,-1\rangle$$

Now lower to find the final state,

$$\begin{aligned} \left|\frac{1}{2},-\frac{1}{2}\right\rangle &= \sqrt{\frac{1}{6}}\sqrt{3}\left|\frac{3}{2},-\frac{3}{2}\right\rangle|1,1\rangle + \sqrt{\frac{1}{6}}\sqrt{2}\left|\frac{3}{2},-\frac{1}{2}\right\rangle|1,0\rangle - \sqrt{\frac{2}{6}}2\left|\frac{3}{2},-\frac{1}{2}\right\rangle|1,0\rangle \\ &\quad -\sqrt{\frac{2}{6}}\sqrt{2}\left|\frac{3}{2},\frac{1}{2}\right\rangle|1,-1\rangle + \sqrt{\frac{3}{6}}\sqrt{3}\left|\frac{3}{2},\frac{1}{2}\right\rangle|1,-1\rangle \\ &= \sqrt{\frac{3}{6}}\left|\frac{3}{2},-\frac{3}{2}\right\rangle|1,1\rangle - \sqrt{\frac{3}{6}}\left|\frac{3}{2},-\frac{1}{2}\right\rangle|1,0\rangle + \sqrt{\frac{1}{6}}\left|\frac{3}{2},\frac{1}{2}\right\rangle|1,-1\rangle \end{aligned}$$

and we have the complete doublet:

$$\left|\frac{1}{2},\frac{1}{2}\right\rangle = \sqrt{\frac{1}{6}}\left|\frac{3}{2},-\frac{1}{2}\right\rangle|1,1\rangle - \sqrt{\frac{2}{6}}\left|\frac{3}{2},\frac{1}{2}\right\rangle|1,0\rangle + \sqrt{\frac{3}{6}}\left|\frac{3}{2},\frac{3}{2}\right\rangle|1,-1\rangle$$

$$\left|\frac{1}{2},-\frac{1}{2}\right\rangle = \sqrt{\frac{3}{6}}\left|\frac{3}{2},-\frac{3}{2}\right\rangle|1,1\rangle - \sqrt{\frac{3}{6}}\left|\frac{3}{2},-\frac{1}{2}\right\rangle|1,0\rangle + \sqrt{\frac{1}{6}}\left|\frac{3}{2},\frac{1}{2}\right\rangle|1,-1\rangle$$

Exercise: Compute all eight total angular momentum states of the combination of three spin-$\frac{1}{2}$ particles,

$$\left|\frac{1}{2},m_1\right\rangle_1 \otimes \left|\frac{1}{2},m_2\right\rangle_2 \otimes \left|\frac{1}{2},m_3\right\rangle_3$$

by first deriving the total angular momentum of the combination of the first two $\left|\frac{1}{2},m_1\right\rangle_1 \otimes \left|\frac{1}{2},m_2\right\rangle_2$ as a triplet $|1,m\rangle$ and a singlet $|0,0\rangle$, then combining each of these with the remaining spin-$\frac{1}{2}$ state:

$$|1,m\rangle \otimes \left|\frac{1}{2},m_3\right\rangle_3$$

$$|0,0\rangle \otimes \left|\frac{1}{2},m_3\right\rangle_3$$

Check that you have all eight states.