Orbital angular momentum and the spherical harmonics

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1 Orbital angular momentum

We compare our result on representations of rotations with our previous experience of angular momentum, defined for a point particle as

$$\mathbf{L} = \mathbf{x} \times \mathbf{p}$$

or, for a quantum system as the operator relationship

$$\hat{\mathbf{L}} = \hat{\mathbf{x}} \times \hat{\mathbf{p}}$$

Notice that since

$$\hat{L}_i = \varepsilon_{ijk} \hat{x}_j \hat{p}_k$$

there is no ordering ambiguity: $\hat{x}_j$ and $\hat{p}_k$ commute as long as $j \neq k$, and the cross product insures this.

Computing commutators of the components of $\mathbf{L}$, we have

$$\begin{aligned}
\left[\hat{L}_i, \hat{L}_m\right] &= \left[\varepsilon_{ijk} \hat{x}_j \hat{p}_k, \varepsilon_{mns} \hat{x}_n \hat{p}_s\right] \\
&= \varepsilon_{ijk} \varepsilon_{mns} \left[\hat{x}_j \hat{p}_k, \hat{x}_n \hat{p}_s\right] \\
&= \varepsilon_{ijk} \varepsilon_{mns} \left(\hat{x}_j \left[\hat{p}_k, \hat{x}_n \hat{p}_s\right] + \left[\hat{x}_j, \hat{x}_n \hat{p}_s\right] \hat{p}_k\right) \\
&= \varepsilon_{ijk} \varepsilon_{mns} \left(\hat{x}_j \left[\hat{p}_k, \hat{x}_n\right] \hat{p}_s + \hat{x}_n \left[\hat{x}_j, \hat{p}_s\right] \hat{p}_k\right) \\
&= \varepsilon_{ijk} \varepsilon_{mns} \left(-i\hbar \delta_{kn} \hat{x}_j \hat{p}_s + i\hbar \delta_{js} \hat{x}_n \hat{p}_k\right) \\
&= i\hbar \left(-\varepsilon_{ijk} \varepsilon_{mks} \hat{x}_j \hat{p}_s + \varepsilon_{ijk} \varepsilon_{mnj} \hat{x}_n \hat{p}_k\right) \\
&= i\hbar \left(\varepsilon_{ijk} \varepsilon_{msk} + \varepsilon_{iks} \varepsilon_{mjk}\right) \hat{x}_j \hat{p}_s
\end{aligned}$$

Using the Jacobi identity (eq.(2) in *Angular Momentum Notes*) $\varepsilon_{jkm} \varepsilon_{inm} + \varepsilon_{kim} \varepsilon_{jnm} + \varepsilon_{ijm} \varepsilon_{knm} = 0$, we rewrite

$$\begin{aligned}
\varepsilon_{ijk} \varepsilon_{msk} + \varepsilon_{iks} \varepsilon_{mjk} &= \varepsilon_{ijk} \varepsilon_{msk} + \varepsilon_{jmk} \varepsilon_{isk} \\
&= -\varepsilon_{mik} \varepsilon_{jsk}
\end{aligned}$$

and the commutator becomes

$$\begin{aligned}
\left[\hat{L}_i, \hat{L}_m\right] &= i\hbar \left(\varepsilon_{ijk} \varepsilon_{msk} + \varepsilon_{iks} \varepsilon_{mjk}\right) \hat{x}_j \hat{p}_s \\
&= -i\hbar \varepsilon_{mik} \varepsilon_{jsk} \hat{x}_j \hat{p}_s \\
&= i\hbar \varepsilon_{imk} \hat{L}_k
\end{aligned}$$

We see that $\hat{L}_m$ satisfies the fundamental angular momentum commutation relations and must therefore admit $|l, m\rangle$ representations satisfying

$$\begin{aligned}
\hat{L}_z |l, m\rangle &= m\hbar |l, m\rangle \\
\hat{\mathbf{L}}^2 |l, m\rangle &= l(l+1) \hbar^2 |l, m\rangle
\end{aligned}$$

along with raising and lowering operators, $\hat{L}_{\pm}$. However, in the case of orbital angular momentum, we have an explicit coordinate representation for the operators. For the z-component,

$$\begin{aligned} \langle\mathbf{x}| \hat{L}_3 |\alpha\rangle &= \langle\mathbf{x}| (\hat{x}_1\hat{p}_2 - \hat{x}_2\hat{p}_1) |\alpha\rangle \\ &= -i\hbar \left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x} \right) \langle\mathbf{x} \mid \alpha\rangle \end{aligned}$$

The eigenvalues of $\hat{L}_3$ are given by solving

$$-i\hbar \left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x} \right) \langle\mathbf{x} \mid \alpha\rangle = m\hbar \langle\mathbf{x} \mid \alpha\rangle$$

but this takes a simpler form in terms of an azimuthal coordinate. Let $x = \rho\cos\varphi$ and $y = \rho\sin\varphi$. Then

$$\begin{aligned} \frac{\partial}{\partial\varphi} &= \frac{\partial x}{\partial\varphi}\frac{\partial}{\partial x} + \frac{\partial y}{\partial\varphi}\frac{\partial}{\partial y} \\ &= -\rho\sin\varphi\frac{\partial}{\partial x} + \rho\cos\varphi\frac{\partial}{\partial y} \\ &= -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y} \end{aligned}$$

and we rewrite the eigenvalue equation as

$$-i\hbar\frac{\partial}{\partial\varphi} \langle\mathbf{x} \mid \alpha\rangle = m\hbar \langle\mathbf{x} \mid \alpha\rangle$$

with the immediate solutions

$$\langle\mathbf{x} \mid \alpha\rangle = e^{im\varphi}$$

Single valuedness of the wave function means that we must have

$$e^{2\pi mi} = 1$$

and therefore only *integer* values for m are allowed.

This exposes an essential asymmetry between spinors and vectors. We have seen that 3-vectors may be represented as matrices in a complex, 2-dim spinor representation, but there does not exist a similar representation of spinors using 3-dim coordinates. Having shown that j and m may take both integer and half-integer values, we now see that classical angular momentun is not the whole story. While the physical existence of *intrinsic angular momentum* or *spin* was only discovered after the advent of quantum mechanics, its existence is a consequence of the group-theoretic nature of rotations, and could have existed classically.

We continue with our examination of *integer j representations*, and the states $|l,m\rangle$ of orbital angular momentum.

2 Changing to spherical coordinates

It is not surprising that orbital angular momentum is most transparently studied in terms of spherical coordinates. Here we rewrite $\hat{L}_z, \hat{L}_{\pm}$ and $\hat{\mathbf{L}}^2$ in spherical coordinates. The coordinate transformation and its inverse are given by

$$\begin{aligned} r &= \sqrt{x^2+y^2+z^2} \\ \theta &= \tan^{-1}\left(\frac{x^2+y^2}{r^2}\right) \\ \varphi &= \tan^{-1}\left(\frac{y}{x}\right) \end{aligned}$$

and

$$\begin{aligned}
x &= r\sin\theta\cos\varphi \\
y &= r\sin\theta\sin\varphi \\
z &= r\cos\theta
\end{aligned}$$

We also need the derivative operators, $\frac{\partial}{\partial x^i}$. Using the chain rule, we have

$$\begin{aligned}
\frac{\partial}{\partial x} &= \frac{\partial r}{\partial x}\frac{\partial}{\partial r} + \frac{\partial \theta}{\partial x}\frac{\partial}{\partial \theta} + \frac{\partial \varphi}{\partial x}\frac{\partial}{\partial \varphi} \\
\frac{\partial}{\partial y} &= \frac{\partial r}{\partial y}\frac{\partial}{\partial r} + \frac{\partial \theta}{\partial y}\frac{\partial}{\partial \theta} + \frac{\partial \varphi}{\partial y}\frac{\partial}{\partial \varphi} \\
\frac{\partial}{\partial z} &= \frac{\partial r}{\partial z}\frac{\partial}{\partial r} + \frac{\partial \theta}{\partial z}\frac{\partial}{\partial \theta} + \frac{\partial \varphi}{\partial z}\frac{\partial}{\partial \varphi} \qquad (1)
\end{aligned}$$

We would like to write the right hand sides of these equations in spherical coordinates.

We may find the partials by writing the total differentials of r, θ and φ. Starting with the differential of r,

$$dr = \frac{x}{r}dx + \frac{y}{r}dy + \frac{z}{r}dz$$

we read off the partial derivatives,

$$\begin{aligned}
\frac{\partial r}{\partial x} &= \frac{x}{r} = \sin\theta\cos\varphi \\
\frac{\partial r}{\partial y} &= \frac{y}{r} = \sin\theta\sin\varphi \\
\frac{\partial r}{\partial z} &= \frac{z}{r} = \cos\theta
\end{aligned}$$

Next, for θ, we take the differential of $\tan\theta$,

$$\begin{aligned}
\tan\theta &= \frac{\sqrt{x^2+y^2}}{z} \\
\frac{1}{\cos^2\theta}d\theta &= \frac{1}{\sqrt{x^2+y^2}}\frac{x}{z}dx + \frac{1}{\sqrt{x^2+y^2}}\frac{y}{z}dy - \frac{\sqrt{x^2+y^2}}{z^2}dz
\end{aligned}$$

Then, with

$$\begin{aligned}
\frac{1}{\sqrt{x^2+y^2}} &= \frac{1}{r\sin\theta} \\
\frac{x}{z} &= \frac{r\sin\theta\cos\varphi}{r\cos\theta} \\
\frac{y}{z} &= \frac{r\sin\theta\sin\varphi}{r\cos\theta}
\end{aligned}$$

the differential of θ becomes

$$\begin{aligned}
d\theta &= \cos^2\theta\left(\frac{1}{r\sin\theta}\frac{\sin\theta\cos\varphi}{\cos\theta}dx + \frac{1}{r\sin\theta}\frac{\sin\theta\sin\varphi}{\cos\theta}dy - \frac{r\sin\theta}{r^2\cos^2\theta}dz\right) \\
&= \frac{1}{r}\cos\theta\cos\varphi dx + \frac{1}{r}\cos\theta\sin\varphi dy - \frac{\sin\theta}{r}dz
\end{aligned}$$

and read off the partials

$$\begin{aligned}\frac{\partial\theta}{\partial x} &= \frac{1}{r}\cos\theta\cos\varphi\\ \frac{\partial\theta}{\partial y} &= \frac{1}{r}\cos\theta\sin\varphi\\ \frac{\partial\theta}{\partial z} &= -\frac{\sin\theta}{r}\end{aligned}$$

Finally, we compute the differential of $\tan\varphi = \frac{y}{x}$, and use $\cos^2\varphi = \frac{x^2}{x^2+y^2}$

$$\begin{aligned}\frac{1}{\cos^2\varphi}d\varphi &= -\frac{y}{x^2}dx + \frac{1}{x}dy\\ d\varphi &= -\cos^2\varphi\frac{r\sin\theta\sin\varphi}{r^2\sin^2\theta\cos^2\varphi}dx + \cos^2\varphi\frac{1}{r\sin\theta\cos\varphi}dy\\ &= -\frac{\sin\varphi}{r\sin\theta}dx + \frac{\cos\varphi}{r\sin\theta}dy\end{aligned}$$

and once again read off the partials

$$\begin{aligned}\frac{\partial\varphi}{\partial x} &= -\frac{\sin\varphi}{r\sin\theta}\\ \frac{\partial\varphi}{\partial y} &= \frac{\cos\varphi}{r\sin\theta}\\ \frac{\partial\varphi}{\partial z} &= 0\end{aligned}$$

Now, returning to the chain rule expansions, eqs.(1), we substitute to find

$$\begin{aligned}\frac{\partial}{\partial x} &= \frac{x}{r}\frac{\partial}{\partial r} + \frac{1}{\sqrt{x^2+y^2}}\frac{xz}{r^2}\frac{\partial}{\partial\theta} - \frac{y}{x^2+y^2}\frac{\partial}{\partial\varphi}\\ &= \sin\theta\cos\varphi\frac{\partial}{\partial r} + \frac{1}{r}\cos\theta\cos\varphi\frac{\partial}{\partial\theta} - \frac{1}{r}\frac{\sin\varphi}{\sin\theta}\frac{\partial}{\partial\varphi}\\ \frac{\partial}{\partial y} &= \frac{y}{r}\frac{\partial}{\partial r} + \frac{1}{\sqrt{x^2+y^2}}\frac{yz}{r^2}\frac{\partial}{\partial\theta} + \frac{x}{x^2+y^2}\frac{\partial}{\partial\varphi}\\ &= \sin\theta\sin\varphi\frac{\partial}{\partial r} + \frac{1}{r}\cos\theta\sin\varphi\frac{\partial}{\partial\theta} + \frac{1}{r}\frac{\cos\varphi}{\sin\theta}\frac{\partial}{\partial\varphi}\\ \frac{\partial}{\partial z} &= \frac{z}{r}\frac{\partial}{\partial r} - \frac{\sqrt{x^2+y^2}}{r^2}\frac{\partial}{\partial\theta}\\ &= \cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial\theta} \qquad (2)\end{aligned}$$

In the next section, we substitute to find the orbital angular momentum operators in angular coordinates.

Finally, it is easy to find the Laplacian in spherical coordinates using the techniques of differential geometry. Using the metric in spherical coordinates

$$g_{ij} = \begin{pmatrix} 1 & & \\ & r^2 & \\ & & r^2\sin^2\theta \end{pmatrix}$$

and the divergence theorem, the result is immediate:

$$\nabla^2 = D_i D^i$$

$$
\begin{aligned}
&= \frac{1}{\sqrt{g}}\frac{\partial}{\partial x^i}\left(\sqrt{g}g^{ij}\frac{\partial}{\partial x^j}\right)\\
&= \frac{1}{r^2\sin\theta}\left[\frac{\partial}{\partial r}\left(r^2\sin\theta\frac{\partial}{\partial r}\right)+\frac{\partial}{\partial\theta}\left(r^2\sin\theta\frac{1}{r^2}\frac{\partial}{\partial\theta}\right)+\frac{\partial}{\partial\varphi}\left(r^2\sin\theta\frac{1}{r^2\sin^2\theta}\frac{\partial}{\partial\varphi}\right)\right]\\
&= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\varphi^2} \qquad (3)
\end{aligned}
$$

3 Orbital angular momentum operators in spherical coordiates

Carrying out the coordinate substitutions, for $\hat{L}_3$ we have

$$
\begin{aligned}
-i\hbar\left(x\frac{\partial}{\partial y}-y\frac{\partial}{\partial x}\right) &= -i\hbar r\sin\theta\cos\varphi\left(\sin\theta\sin\varphi\frac{\partial}{\partial r}+\frac{1}{r}\cos\theta\sin\varphi\frac{\partial}{\partial\theta}+\frac{1}{r}\frac{\cos\varphi}{\sin\theta}\frac{\partial}{\partial\varphi}\right)\\
&\quad +i\hbar r\sin\theta\sin\varphi\left(\sin\theta\cos\varphi\frac{\partial}{\partial r}+\frac{1}{r}\cos\theta\cos\varphi\frac{\partial}{\partial\theta}-\frac{1}{r}\frac{\sin\varphi}{\sin\theta}\frac{\partial}{\partial\varphi}\right)\\
&= -i\hbar\frac{\partial}{\partial\varphi}
\end{aligned}
$$

as found above. For the raising operator, we have

$$
\begin{aligned}
\frac{1}{\hbar}\hat{L}_+ &= z\frac{\partial}{\partial x}+iz\frac{\partial}{\partial y}-(x+iy)\frac{\partial}{\partial z}\\
&= r\cos\theta\left(\sin\theta\cos\varphi\frac{\partial}{\partial r}+\frac{1}{r}\cos\theta\cos\varphi\frac{\partial}{\partial\theta}-\frac{1}{r}\frac{\sin\varphi}{\sin\theta}\frac{\partial}{\partial\varphi}\right)\\
&\quad +r\cos\theta\left(i\sin\theta\sin\varphi\frac{\partial}{\partial r}+\frac{i}{r}\cos\theta\sin\varphi\frac{\partial}{\partial\theta}+\frac{i}{r}\frac{\cos\varphi}{\sin\theta}\frac{\partial}{\partial\varphi}\right)\\
&\quad -r\sin\theta\left(\cos\varphi+i\sin\varphi\right)\left(\cos\theta\frac{\partial}{\partial r}-\frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\right)\\
&= \left(\cos\varphi+i\sin\varphi-e^{i\varphi}\right)r\cos\theta\sin\theta\frac{\partial}{\partial r}+\cos^2\theta e^{i\varphi}\frac{\partial}{\partial\theta}+e^{i\varphi}\sin^2\theta\frac{\partial}{\partial\theta}\\
&\quad +i\left(\cos\varphi+i\sin\varphi\right)\frac{\cos\theta}{\sin\theta}\frac{\partial}{\partial\varphi}\\
&= e^{i\varphi}\left(\frac{\partial}{\partial\theta}+i\frac{\cos\theta}{\sin\theta}\frac{\partial}{\partial\varphi}\right)
\end{aligned}
$$

while the lowering operator is

$$
\begin{aligned}
\frac{1}{\hbar}\hat{L}_- &= -z\frac{\partial}{\partial x}+iz\frac{\partial}{\partial y}+(x-iy)\frac{\partial}{\partial z}\\
&= -r\cos\theta\left(\sin\theta\cos\varphi\frac{\partial}{\partial r}+\frac{1}{r}\cos\theta\cos\varphi\frac{\partial}{\partial\theta}-\frac{1}{r}\frac{\sin\varphi}{\sin\theta}\frac{\partial}{\partial\varphi}\right)\\
&\quad +r\cos\theta\left(i\sin\theta\sin\varphi\frac{\partial}{\partial r}+\frac{i}{r}\cos\theta\sin\varphi\frac{\partial}{\partial\theta}+\frac{i}{r}\frac{\cos\varphi}{\sin\theta}\frac{\partial}{\partial\varphi}\right)\\
&\quad +r\sin\theta\left(\cos\varphi-i\sin\varphi\right)\left(\cos\theta\frac{\partial}{\partial r}-\frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\right)\\
&= re^{-i\varphi}\sin\theta\cos\theta\frac{\partial}{\partial r}-re^{-i\varphi}\cos\theta\sin\theta\frac{\partial}{\partial r}\\
&\quad -e^{-i\varphi}\cos^2\theta\frac{\partial}{\partial\theta}-e^{-i\varphi}\sin^2\theta\frac{\partial}{\partial\theta}+ie^{-i\varphi}\frac{\cos\theta}{\sin\theta}\frac{\partial}{\partial\varphi}
\end{aligned}
$$

$$= -e^{-i\varphi}\left(\frac{\partial}{\partial\theta} - i\frac{\cos\theta}{\sin\theta}\frac{\partial}{\partial\varphi}\right)$$

Collecting the results so far, we have

$$\hat{L}_3 = -i\hbar\frac{\partial}{\partial\varphi} \tag{4}$$

$$\hat{L}_+ = \hbar e^{i\varphi}\left(\frac{\partial}{\partial\theta} + i\frac{\cos\theta}{\sin\theta}\frac{\partial}{\partial\varphi}\right) \tag{5}$$

$$\hat{L}_- = \hbar e^{-i\varphi}\left(-\frac{\partial}{\partial\theta} + i\frac{\cos\theta}{\sin\theta}\frac{\partial}{\partial\varphi}\right) \tag{6}$$

Exercise: Find the form of $\hat{L}_x$ and $\hat{L}_y$ from eqs.(5) and (6), together with the definitions $\hat{J}_\pm \equiv \hat{J}_1 \pm i\hat{J}_2$.

Exercise: Confirm the form of the Laplacian operator by direct substitution into

$$\hat{\mathbf{L}}^2 = \hat{L}_x^2 + \hat{L}_y^2 + \hat{L}_z^2$$

Now, since

$$\hat{L}_+\hat{L}_- = \hat{\mathbf{L}}^2 - \hat{L}_3^2 + \hbar\hat{L}_3$$

the magnitude squared of the total angular momentum is

$$\begin{aligned}
\mathbf{L}^2 &= \hat{L}_+\hat{L}_- + L_3^2 - \hbar L_3 \\
&= \left(\hbar e^{i\varphi}\left(\frac{\partial}{\partial\theta} + i\frac{\cos\theta}{\sin\theta}\frac{\partial}{\partial\varphi}\right)\right)\left(\hbar e^{-i\varphi}\left(-\frac{\partial}{\partial\theta} + i\frac{\cos\theta}{\sin\theta}\frac{\partial}{\partial\varphi}\right)\right) - \hbar^2\frac{\partial^2}{\partial\varphi^2} + i\hbar^2\frac{\partial}{\partial\varphi} \\
&= \hbar^2\left(-\frac{\cos\theta}{\sin\theta}\frac{\partial}{\partial\theta} + i\frac{\cos^2\theta}{\sin^2\theta}\frac{\partial}{\partial\varphi} + \left(\frac{\partial}{\partial\theta} + i\frac{\cos\theta}{\sin\theta}\frac{\partial}{\partial\varphi}\right)\left(-\frac{\partial}{\partial\theta} + i\frac{\cos\theta}{\sin\theta}\frac{\partial}{\partial\varphi}\right)\right) - \hbar^2\frac{\partial^2}{\partial\varphi^2} + i\hbar^2\frac{\partial}{\partial\varphi} \\
&= \hbar^2\left(-\frac{\cos\theta}{\sin\theta}\frac{\partial}{\partial\theta} - \frac{\partial^2}{\partial\theta^2} - i\frac{\partial}{\partial\varphi} - \frac{\cos^2\theta}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\right) - \hbar^2\frac{\partial^2}{\partial\varphi^2} + i\hbar^2\frac{\partial}{\partial\varphi} \\
&= \hbar^2\left(-\frac{\cos\theta}{\sin\theta}\frac{\partial}{\partial\theta} - \frac{\partial^2}{\partial\theta^2} - \frac{\partial^2}{\partial\varphi^2} - \frac{\cos^2\theta}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\right) \\
&= -\hbar^2\left(\frac{1}{\sin\theta}\left(\cos\theta\frac{\partial}{\partial\theta} + \sin\theta\frac{\partial^2}{\partial\theta^2}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\right) \\
&= -\hbar^2\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\right)
\end{aligned}$$

This last equation establishes the relationship between the spherical harmonics and the angular momentum states, because the Laplace equation in spherical coordinates is

$$\begin{aligned}
\nabla^2 &= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\varphi^2} \\
&= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) - \frac{1}{\hbar^2 r^2}\hat{\mathbf{L}}^2
\end{aligned}$$

and we know that separation of variables leads to general solution of the Laplace equation, $f(r,\theta,\varphi)$ with the angular solution given in terms of spherical harmonics,

$$f(r,\theta,\varphi) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} A_l(r) Y_m^l(\theta,\varphi)$$

The spherical harmonics satisfy the separated angular eigenvalue equation,

$$\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\right) Y_m^l(\theta,\varphi) = -l(l+1) Y_m^l(\theta,\varphi)$$

for integer l and $m = -l, -l+1, \ldots + l$. Expressing this in terms of $\hat{\mathbf{L}}^2$,

$$\hat{\mathbf{L}}^2 |\psi\rangle = l(l+1)\hbar^2 |\psi\rangle$$

we see that $|\psi\rangle = |l,m\rangle$ and therefore identify the spherical harmonics as the integer spin eigenstates of angular momentum in a coordinate basis,

$$Y_m^l(\theta,\varphi) = \langle\theta,\varphi \mid l,m\rangle$$

These describe only integer j states.

4 Spherical harmonics

We can now use the quantum formalism to find the spherical harmonics, $Y_m^l(\theta,\varphi) = \langle\theta,\varphi\,|l,m\rangle$. For any state $|\alpha\rangle$, we know the effect of $\hat{L}_z$ is given by eq.(4), so

$$\langle\theta,\varphi|\,\hat{L}_z\,|\alpha\rangle = -i\hbar\frac{\partial}{\partial\varphi}\langle\theta,\varphi \mid \alpha\rangle$$

Since the eigenstates satisfy $\hat{L}_z|l,m\rangle = m\hbar|l,m\rangle$ in general, placing this equation in a coordinate basis it becomes

$$-i\hbar\frac{\partial}{\partial\varphi}\langle\theta,\varphi \mid l,m\rangle = m\hbar\langle\theta,\varphi \mid l,m\rangle$$

This is trivially integrated to give

$$\langle\theta,\varphi \mid l,m\rangle = e^{im\varphi}\langle\theta,\varphi \mid l\rangle$$

Furthermore, we know that the raising operator will anihilate the state with the highest value of m,

$$\hat{L}_+|l,m=l\rangle = 0$$

Again choosing a coordinate basis, $\hat{L}_+$ is given by eq.(5) so this translates to a differential equation,

$$\begin{aligned}
0 &= \langle\theta,\varphi|\,\hat{L}_+\,|l,l\rangle \\
&= \hbar e^{i\varphi}\left(\frac{\partial}{\partial\theta} + i\frac{\cos\theta}{\sin\theta}\frac{\partial}{\partial\varphi}\right)\langle\theta,\varphi \mid l,m=l\rangle \\
&= \hbar e^{i\varphi}\left(\frac{\partial}{\partial\theta} + i\frac{\cos\theta}{\sin\theta}\frac{\partial}{\partial\varphi}\right)\left(e^{il\varphi}\langle\theta,\varphi \mid l\rangle\right) \\
&= \hbar e^{i(l+1)\varphi}\left(\frac{\partial}{\partial\theta}\langle\theta,\varphi \mid l\rangle - l\frac{\cos\theta}{\sin\theta}\langle\theta,\varphi \mid l\rangle\right)
\end{aligned}$$

Setting $\langle\theta,\varphi \mid l\rangle = f_l(\theta)$, we rewrite this as

$$0 = \sin\theta\frac{\partial f_l}{\partial\theta} - l\cos\theta f_l$$

This is solved by $f_l = \sin^l \theta$, so we have, for $m = l$

$$Y^l{}_l(\theta, \varphi) = A_{ll} e^{il\varphi} \sin^l \theta$$

Now we can find all other $Y^l{}_m(\theta, \varphi)$ by acting with the lowering operator,

$$\langle \theta, \varphi | \hat{L}_- | l, m \rangle = \sqrt{l(l+1) - m(m-1)} \hbar \langle \theta, \varphi \mid l, m-1 \rangle$$

Inserting the coordinate expression, eq.(6), for $\langle \theta, \varphi | \hat{L}_- | l, m \rangle$ and solving for the next lower state, we have

$$\begin{aligned}
\langle \theta, \varphi \mid l, m-1 \rangle &= \frac{e^{-i\varphi}}{\sqrt{l(l+1) - m(m-1)}} \left(-\frac{\partial}{\partial \theta} + i \frac{\cos \theta}{\sin \theta} \frac{\partial}{\partial \varphi} \right) \langle \theta, \varphi \mid l, m \rangle \\
&= -\frac{e^{-i\varphi} e^{im\varphi}}{\sqrt{l(l+1) - m(m-1)}} \left(\frac{\partial}{\partial \theta} + m \frac{\cos \theta}{\sin \theta} \right) \langle \theta \mid l \rangle
\end{aligned}$$

thereby defining all $Y^l{}_m(\theta, \varphi)$ recursively.

Exercise: Find the $Y^1{}_m(\theta, \varphi)$ for all allowed m.