All representations for rotations

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We now may find a basis for states of angular momentum, that is, all finite-dimensional representations for the three operators $\hat{J}_i$. All results follow from the fundamental commutation relation for hermitian rotational generators,

$$\left[\hat{J}_i, \hat{J}_j\right] = i\hbar\varepsilon_{ijk}\hat{J}_k$$

where i, j, k each take values $1, 2, 3$ and we sum on k.

1 A maximal set of commuting observables

To begin, we ask how many mutually commuting operators we can build from $\hat{J}_i$. We can diagonalize any one of $\hat{J}_1, \hat{J}_2, \hat{J}_3$, but since none commute with either or the others, we cannot diagonalize more than one. We choose $\hat{J}_3$ diagonal. There is one further commuting combination – since rotations preserve lengths, the length of $\hat{J}_i$ itself is preserved by rotations,

$$\begin{aligned}
\left[\hat{J}_i, \hat{\mathbf{J}}^2\right] &= \left[\hat{J}_i, \hat{J}_k\hat{J}_k\right] \\
&= \hat{J}_k\left[\hat{J}_i, \hat{J}_k\right] + \left[\hat{J}_i, \hat{J}_k\right]\hat{J}_k \\
&= \hat{J}_k i\hbar\varepsilon_{ikm}\hat{J}_m + i\hbar\varepsilon_{ikm}\hat{J}_m\hat{J}_k \\
&= i\hbar\varepsilon_{ikm}\left(\hat{J}_k\hat{J}_m + \hat{J}_m\hat{J}_k\right) \\
&= 0
\end{aligned}$$

where the last step follows because $\hat{J}_k\hat{J}_m + \hat{J}_m\hat{J}_k$ is symmetric in mk while ε_{ikm} is antisymmetric. In particular, we have

$$\left[\hat{J}_3, \hat{\mathbf{J}}^2\right] = 0$$

so these may be simultaneously diagonalized. Since we already know that the Pauli matrices give a 2-dimensional example for the generators, there cannot be more than two indepedent diagonal combinations.

Having found a maximal set of commuting observables, we may use their eigenvalues to label their simultaneous eigenkets. Let

$$\begin{aligned}
\hat{\mathbf{J}}^2\left|\alpha, \beta\right\rangle &= \alpha^2\hbar^2\left|\alpha, \beta\right\rangle \\
\hat{J}_3\left|\alpha, \beta\right\rangle &= \beta\hbar\left|\alpha, \beta\right\rangle
\end{aligned}$$

We take these kets to be orthonormal and seek all allowed values of the real eigenvalues, α, β.

2 Raising and lowering operators

We combine the remaing two generators in the useful combinations,

$$\hat{J}_\pm \equiv \hat{J}_1 \pm i\hat{J}_2$$

where we note that $\hat{J}_+^\dagger = \hat{J}_-$. These satisfy:

$$\begin{aligned}
\left[\hat{J}_+, \hat{J}_-\right] &= \left[\hat{J}_1 + i\hat{J}_2, \hat{J}_1 - i\hat{J}_2\right] \\
&= -i\left[\hat{J}_1, \hat{J}_2\right] + i\left[\hat{J}_2, \hat{J}_1\right] \\
&= 2\hbar\hat{J}_3
\end{aligned}$$

and

$$\begin{aligned}
\left[\hat{J}_3, \hat{J}_\pm\right] &= \left[\hat{J}_3, \hat{J}_1 \pm i\hat{J}_2\right] \\
&= \left[\hat{J}_3, \hat{J}_1\right] \pm i\left[\hat{J}_3, \hat{J}_2\right] \\
&= i\hbar\hat{J}_2 \pm i\left(-i\hbar\hat{J}_1\right) \\
&= \pm\hbar\hat{J}_\pm
\end{aligned}$$

as well as commuting with the length,

$$\left[\hat{J}_\pm, \hat{\mathbf{J}}^2\right] = 0$$

Consider the actions of $\hat{\mathbf{J}}^2$ and $\hat{J}_3$ on the state $\hat{J}_+ |\alpha, \beta\rangle$,

$$\begin{aligned}
\hat{\mathbf{J}}^2\hat{J}_+ |\alpha, \beta\rangle &= \hat{J}_+\hat{\mathbf{J}}^2 |\alpha, \beta\rangle \\
&= \alpha^2\hbar^2\hat{J}_+ |\alpha, \beta\rangle
\end{aligned}$$

so this state is also an eigenstate of $\hat{\mathbf{J}}^2$ with the eigenvalue α, while

$$\begin{aligned}
\hat{J}_3\hat{J}_+ |\alpha, \beta\rangle &= \left(\left[\hat{J}_3, \hat{J}_+\right] + \hat{J}_+\hat{J}_3\right) |\alpha, \beta\rangle \\
&= \hbar\hat{J}_+ |\alpha, \beta\rangle + \hat{J}_+\hat{J}_3 |\alpha, \beta\rangle \\
&= (\beta + 1)\,\hbar\hat{J}_+ |\alpha, \beta\rangle
\end{aligned}$$

We once again have an eigenstate, but the eigenvalue β has increased by $\hbar$. Up to an overall constant λ we have

$$\hat{J}_+ |\alpha, \beta\rangle = \lambda |\alpha, \beta + 1\rangle$$

Exercise: Show that $\hat{J}_+ |\alpha, \beta\rangle = \lambda |\alpha, \beta - 1\rangle$ for some constant λ.

3 Limits on eigenvalues

3.1 Inequalities on the eigenvalues

Products of $\hat{J}_+$ and $\hat{J}_-$ may be expressed in term of our diagonal operators. For the product $\hat{J}_+\hat{J}_-$:

$$\begin{aligned}
\hat{J}_+\hat{J}_- &= \left(\hat{J}_1 + i\hat{J}_2\right)\left(\hat{J}_1 - i\hat{J}_2\right) \\
&= \hat{J}_1^2 + \hat{J}_2^2 - i\hat{J}_1\hat{J}_2 + i\hat{J}_2\hat{J}_1 \\
&= \hat{\mathbf{J}}^2 - \hat{J}_3^2 - i\left[\hat{J}_1, \hat{J}_2\right] \\
&= \hat{\mathbf{J}}^2 - \hat{J}_3^2 + \hbar\hat{J}_3
\end{aligned}$$

Exercise: Show that $\hat{J}_-\hat{J}_+ = \hat{\mathbf{J}}^2 - \hat{J}_3^2 - \hbar\hat{J}_3$

Since $\hat{J}_+^\dagger = \hat{J}_-$ and $\hat{J}_-^\dagger = \hat{J}_+$ we have inequalities from the norms of $\hat{J}_+ |\alpha,\beta\rangle$ and $\hat{J}_- |\alpha,\beta\rangle$:

$$\begin{aligned}
\left[\langle\alpha,\beta|\,\hat{J}_+^\dagger\right]\left[\hat{J}_+|\alpha,\beta\rangle\right] = \langle\alpha,\beta|\,\hat{J}_-\hat{J}_+|\alpha,\beta\rangle &\geq 0 \\
\left[\langle\alpha,\beta|\,\hat{J}_-^\dagger\right]\left[\hat{J}_-|\alpha,\beta\rangle\right] = \langle\alpha,\beta|\,\hat{J}_=\hat{J}_-|\alpha,\beta\rangle &\geq 0
\end{aligned}$$

These give, respectively,

$$\begin{aligned}
0 &\leq \langle\alpha,\beta|\,\hat{J}_-\hat{J}_+|\alpha,\beta\rangle \\
&= \langle\alpha,\beta|\left(\hat{\mathbf{J}}^2 - \hat{J}_3^2 - \hbar\hat{J}_3\right)|\alpha,\beta\rangle \\
&= \left(\alpha^2 - \beta^2 - \beta\right)\hbar^2\,\langle\alpha,\beta\,|\alpha,\beta\rangle \\
&= \left(\alpha^2 - \beta^2 - \beta\right)\hbar^2
\end{aligned}$$

and

$$\begin{aligned}
0 &\leq \langle\alpha,\beta|\,\hat{J}_+\hat{J}_-|\alpha,\beta\rangle \\
&= \langle\alpha,\beta|\left(\hat{\mathbf{J}}^2 - \hat{J}_3^2 + \hbar\hat{J}_3\right)|\alpha,\beta\rangle \\
&= \left(\alpha^2 - \beta^2 + \beta\right)\hbar^2
\end{aligned}$$

so two distinct inequalities must hold:

$$\beta^2 + \beta \leq \alpha^2 \tag{1}$$
$$\beta^2 - \beta \leq \alpha^2 \tag{2}$$

3.2 The eigenvalues

Now, just as we did for the simple harmonic oscillator, we start with any eigenstate and lower the eigenvalue k times,

$$\hat{J}_3\left(\hat{J}_-\right)^k|\alpha,\beta\rangle = \lambda_{\beta-k}\,(\beta-k)\,\hbar\,|\alpha,\beta-k\rangle$$

for some normalization constant, $\lambda_{\beta-k}$. However, this series must terminate, since eq.(1) for the state $|\alpha,\beta-k\rangle$ leads to

$$\begin{aligned}
(\beta-k)^2 + (\beta-k) &\leq \alpha^2 \\
k^2 - 2\beta k - k + \beta^2 + \beta &\leq \alpha^2
\end{aligned}$$

Regardless of the value of α and β, there is some value of k which is sufficiently large to violate this inequality. Therefore, there must exist some β_{min} such that

$$\hat{J}_-|\alpha,\beta_{min}\rangle = 0$$

Since $\beta = 0$ satisfies both inequalities we must have $\beta_{min} < 0$, and therefore

$$\beta_{min}^2 - \beta_{min} \leq \alpha^2$$

gives the strongest constraint on β_{min}.

Now we apply $\hat{J}_+$ to $|\alpha,\beta_{min}\rangle$ to produce eigenkets of larger and larger β,

$$\hat{J}_+^k|\alpha,\beta_{min}\rangle = \lambda_{\beta_{min}+k}|\alpha,\beta_{min}+k\rangle$$

Once again we eventually reach a value of k which violates one of the inequalities, so there exists some positive, maximum β_{max}, satisfying both inequalities. The strongest constraint is

$$\beta_{max}^2 + \beta_{max} \leq \alpha^2$$

Notice that if $\beta_{min} = -\beta_{max} = -m$ then both inequalities give

$$m(m+1) \leq \alpha^2$$

Now acting on the highest state, $|\alpha, \beta_{max}\rangle$ with $\hat{J}_+$, or acting on the lowest state, $|\alpha, \beta_{min}\rangle$, with $\hat{J}_-$ must give zero

$$\begin{aligned} \hat{J}_+ |\alpha, \beta_{max}\rangle &= 0 \\ \hat{J}_- |\alpha, \beta_{min}\rangle &= 0 \end{aligned}$$

and therefore, acting on the first with $\hat{J}_-$ and the second with $\hat{J}_+$

$$\begin{aligned} 0 &= \hat{J}_-\hat{J}_+ |\alpha, \beta_{max}\rangle \\ &= \left(\hat{\mathbf{J}}^2 - \hat{J}_3^2 - \hbar\hat{J}_3\right) |\alpha, \beta_{max}\rangle \\ &= \left(\alpha^2 - \beta_{max}^2 - \beta_{max}\right) \hbar^2 |\alpha, \beta_{max}\rangle \end{aligned}$$

and

$$\begin{aligned} 0 &= \hat{J}_+\hat{J}_- |\alpha, \beta_{min}\rangle \\ &= \left(\hat{\mathbf{J}}^2 - \hat{J}_3^2 + \hbar\hat{J}_3\right) |\alpha, \beta_{min}\rangle \\ &= \left(\alpha^2 - \beta_{min}^2 + \beta_{min}\right) \hbar^2 |\alpha, \beta_{min}\rangle \end{aligned}$$

giving us two equalities for the maximum and minimum values:

$$\begin{aligned} \alpha^2 &= \beta_{max}^2 + \beta_{max} \\ \alpha^2 &= \beta_{min}^2 - \beta_{min} \end{aligned}$$

We also know that $\beta_{max} - \beta_{min} = k$ for some non-negative integer, k. Setting $\beta_{max} = \beta_{min} + k$ and equating the two expressions,

$$\begin{aligned} \beta_{min}^2 - \beta_{min} &= \beta_{max}^2 + \beta_{max} \\ &= (\beta_{min} + k)^2 + (\beta_{min} + k) \\ \beta_{min}^2 - \beta_{min} &= \beta_{min}^2 + 2\beta_{min}k + k^2 + \beta_{min} + k \\ 0 &= (k+1)\, 2\beta_{min} + k(k+1) \\ 0 &= 2\beta_{min} + k \\ \beta_{min} &= -\frac{k}{2} \end{aligned}$$

so that β_{min} is some negative integer or half-integer we will call $-j$:

$$\beta_{min} = -j \in \left\{0, -\frac{1}{2}, -1, -\frac{3}{2}, -2, \ldots\right\}$$

The maximum value $\beta_{max} = \beta_{min} + k = +\frac{k}{2} = +j$, and the remaining eigenvalue is

$$\begin{aligned} \alpha^2 &= \frac{k}{2}\left(\frac{k}{2} + 1\right) \\ &= j(j+1) \end{aligned}$$

The labeling of our states is complete. Letting $\beta = m$, the complete set of possible states for any fixed half-integer j is given by the $2j+1$ states,

$$|\alpha, \beta\rangle = \{|j, m\rangle \mid m = -j, -j+1, \ldots, j+1, j\}$$

and we have one such set for every choice of $j = 0, \frac{1}{2}, 1, \frac{3}{2}, 2, \frac{5}{2}, \ldots$. The eigenvalues of these states are given by

$$\begin{aligned} \hat{\mathbf{J}}^2 |j, m\rangle &= j(j+1)\hbar^2 |j, m\rangle & (3) \\ \hat{J}_3 |j, m\rangle &= m\hbar |j, m\rangle & (4) \end{aligned}$$

These states will be referred to as "spin-j" representations.

3.3 Normalization of raising and lowering

We define these eigenstates to be normalized, and since they are nondegenerate, they are orthonormal,

$$\langle j_1, m_1 | j_2, m_2\rangle = \delta_{j_1 j_2}\delta_{m_1 m_2}$$

However, we need to know the effect of the raising and lowering operators. We already know that

$$\hat{J}_\pm |j, m\rangle = \lambda_{m\pm 1} |j, m \pm 1\rangle$$

for some constants $\lambda_{m\pm1}$. To find $\lambda_{m\pm1}$, look again at the norm

$$\begin{aligned} \langle j, m| \hat{J}_- \hat{J}_+ |j, m\rangle &= \langle j, m| \left(\hat{\mathbf{J}}^2 - \hat{J}_3^2 - \hbar \hat{J}_3\right) |j, m\rangle \\ |\lambda_{m+1}|^2 &= (j(j+1) - m(m+1))\hbar^2 \\ \lambda_{m+1} &= \sqrt{j(j+1) - m(m+1)}\hbar \end{aligned}$$

where we choose the phase so that λ_{m+1} is real. For $\hat{J}_-$ we have

$$\begin{aligned} \langle j, m| \hat{J}_+ \hat{J}_- |j, m\rangle &= \langle j, m| \left(\hat{\mathbf{J}}^2 - \hat{J}_3^2 + \hbar \hat{J}_3\right) |j, m\rangle \\ |\lambda_{m-1}|^2 &= (j(j+1) - m(m-1))\hbar^2 \\ \lambda_{m-1} &= \sqrt{j(j+1) - m(m-1)}\hbar \end{aligned}$$

Therefore, the action of the raising and lowering operators is

$$\hat{J}_\pm |j, m\rangle = \sqrt{j(j+1) - m(m\pm 1)}\hbar |j, m \pm 1\rangle \quad (5)$$

4 Examples of representations

4.1 Spin 0

For $j = 0$, we only have the single allowed value $m = 0$ and there is only one state,

$$|j, m\rangle = |0, 0\rangle$$

These are scalars. We may find the expectation value of any component of angular momentum using

$$\begin{aligned} J_1 &= \frac{1}{2}\left(\hat{J}_+ + \hat{J}_-\right) \\ J_2 &= \frac{1}{2i}\left(\hat{J}_+ - \hat{J}_-\right) \end{aligned}$$

Since $m = 0 = \beta_{min} = \beta_{max}$, both $\hat{J}_+$ and $\hat{J}_-$ must give zero:

$$\hat{J}_\pm |0,0\rangle = 0$$

and we have

$$\begin{aligned} \hat{J}_1 |0,0\rangle &= 0 \\ \hat{J}_2 |0,0\rangle &= 0 \\ \hat{J}_3 |0,0\rangle &= 0 \end{aligned}$$

so the action of all generators is zero. Furthermore,

$$\begin{aligned} \langle 0,0| \hat{J}_x |0,0\rangle &= 0 \\ \langle 0,0| \hat{J}_y |0,0\rangle &= 0 \\ \langle 0,0| \hat{J}_z |0,0\rangle &= 0 \end{aligned}$$

so every component of angular momentum has zero expectation value.

The effect of a general infinitesimal rotation on a scalar state is given by

$$\begin{aligned} \mathscr{D}(\mathbf{n},\varphi) |0,0\rangle &= \left(\hat{1} - \frac{i\varphi}{\hbar}\mathbf{n}\cdot\hat{\boldsymbol{J}}\right) |0,0\rangle \\ &= |0,0\rangle \end{aligned}$$

so scalars are unaffected by any rotation.

4.2 Spin 1/2

For $j = \frac{1}{2}$ we have our familiar algebra of Pauli matrices, but we now have a more systematic labelling for the states. When we wish to be explicit about the value of j, we will write

$$\left|\frac{1}{2}, \pm\frac{1}{2}\right\rangle$$

instead of $|\pm\rangle$. Notice that in all cases here we are taking $\hat{J}_3$ diagonal. We already know the expectation values of $\hat{J}_i$ in these states. For $\hat{\mathbf{J}}^2$ and $\hat{J}_\pm$ we have

$$\begin{aligned} \hat{\mathbf{J}}^2 \left|\frac{1}{2}, \pm\frac{1}{2}\right\rangle &= \frac{1}{2}\left(\frac{1}{2}+1\right)\hbar^2 \left|\frac{1}{2}, \pm\frac{1}{2}\right\rangle \\ &= \frac{3}{4}\hbar^2 \left|\frac{1}{2}, \pm\frac{1}{2}\right\rangle \end{aligned}$$

and

$$\begin{aligned} \hat{J}_+ \left|\frac{1}{2}, \frac{1}{2}\right\rangle &= 0 \\ \hat{J}_- \left|\frac{1}{2}, \frac{1}{2}\right\rangle &= \sqrt{j(j+1) - m(m-1)}\hbar \left|\frac{1}{2}, \frac{1}{2} - 1\right\rangle \\ &= \sqrt{\frac{1}{2}\left(\frac{3}{2}\right) - \frac{1}{2}\left(-\frac{1}{2}\right)}\hbar \left|\frac{1}{2}, -\frac{1}{2}\right\rangle \\ &= \hbar \left|\frac{1}{2}, -\frac{1}{2}\right\rangle \end{aligned}$$

$$
\begin{aligned}
\hat{J}_- \left|\frac{1}{2},-\frac{1}{2}\right\rangle &= \sqrt{\frac{1}{2}\left(\frac{3}{2}\right)-\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}\left|\frac{1}{2},-\frac{1}{2}-1\right\rangle \\
&= 0 \\
\hat{J}_+ \left|\frac{1}{2},-\frac{1}{2}\right\rangle &= \sqrt{j\left(j+1\right)-m\left(m+1\right)}\hbar\left|\frac{1}{2},\frac{1}{2}\right\rangle \\
&= \sqrt{\frac{3}{4}-\left(-\frac{1}{2}\right)\left(\frac{1}{2}\right)}\hbar\left|\frac{1}{2},\frac{1}{2}\right\rangle \\
&= \hbar\left|\frac{1}{2},\frac{1}{2}\right\rangle
\end{aligned}
$$

The spin-$\frac{1}{2}$ states forem a 2-dimensional representation, so the generators are the Pauli matrices. Writing the raising and lowering operators in matrix notation,

$$
\begin{aligned}
\hat{J}_+ &= \hat{J}_x + i\hat{J}_y \\
&= \frac{\hbar}{2}\left(\sigma_x + i\sigma_y\right) \\
&= \hbar\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \\
\hat{J}_- &= \hbar\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}
\end{aligned}
$$

so that

$$
\begin{aligned}
\hat{J}_+\begin{pmatrix} 1 \\ 0 \end{pmatrix} &= 0 \\
\hat{J}_+\begin{pmatrix} 0 \\ 1 \end{pmatrix} &= \hbar\begin{pmatrix} 1 \\ 0 \end{pmatrix} \\
\hat{J}_-\begin{pmatrix} 0 \\ 1 \end{pmatrix} &= 0 \\
\hat{J}_+\begin{pmatrix} 1 \\ 0 \end{pmatrix} &= \hbar\begin{pmatrix} 0 \\ 1 \end{pmatrix}
\end{aligned}
$$

Quite generally, the components of the raising and lowering operators are unit off-diagonal matrices.

4.3 Spin 1

We have a total of three $j = 1$ states,

$$
\left|j,m\right\rangle = \left|1,1\right\rangle, \left|1,0\right\rangle, \left|1,-1\right\rangle
$$

related by

$$
\begin{aligned}
\hat{J}_-\left|1,1\right\rangle &= \sqrt{1\left(1+1\right)-1\left(1-1\right)}\hbar\left|1,1-1\right\rangle \\
&= \sqrt{2}\hbar\left|1,0\right\rangle
\end{aligned}
$$

and

$$
\begin{aligned}
\hat{J}_-\left|1,0\right\rangle &= \sqrt{1\left(1+1\right)-0\left(0-1\right)}\hbar\left|1,0-1\right\rangle \\
&= \sqrt{2}\hbar\left|1,-1\right\rangle
\end{aligned}
$$

with similar relations for the raising operator. The eigenvalue of $\hat{\mathbf{J}}^2$ is $j\left(j+1\right)\hbar^2 = 2\hbar^2$.

4.4 Spin 3/2

We have $2j+1=4$ states,

$$|j,m\rangle = \left|\frac{3}{2},\frac{3}{2}\right\rangle, \left|\frac{3}{2},\frac{1}{2}\right\rangle, \left|\frac{3}{2},-\frac{1}{2}\right\rangle, \left|\frac{3}{2},-\frac{3}{2}\right\rangle$$

related by

$$\begin{aligned}
\hat{J}_- \left|\frac{3}{2},\frac{3}{2}\right\rangle &= \sqrt{\frac{3}{2}\left(\frac{3}{2}+1\right) - \frac{3}{2}\left(\frac{3}{2}-1\right)}\hbar \left|\frac{3}{2},\frac{1}{2}\right\rangle \\
&= \sqrt{3}\hbar \left|\frac{3}{2},\frac{1}{2}\right\rangle \\
\hat{J}_- \left|\frac{3}{2},\frac{1}{2}\right\rangle &= \sqrt{\frac{3}{2}\left(\frac{3}{2}+1\right) - \frac{1}{2}\left(\frac{1}{2}-1\right)}\hbar \left|\frac{3}{2},-\frac{1}{2}\right\rangle \\
&= 2\hbar \left|\frac{3}{2},-\frac{1}{2}\right\rangle \\
\hat{J}_- \left|\frac{3}{2},-\frac{1}{2}\right\rangle &= \sqrt{\frac{3}{2}\left(\frac{3}{2}+1\right) - \left(-\frac{1}{2}\right)\left(-\frac{1}{2}-1\right)}\hbar \left|\frac{3}{2},-\frac{3}{2}\right\rangle \\
&= \sqrt{3}\hbar \left|\frac{3}{2},-\frac{3}{2}\right\rangle \\
\hat{J}_- \left|\frac{3}{2},-\frac{3}{2}\right\rangle &= 0
\end{aligned}$$

with similar relations for the raising operator. The eigenvalue of $\hat{\mathbf{J}}^2$ is $j(j+1)\hbar^2 = \frac{15}{4}\hbar^2$.

4.5 Spin j

We summarize here the general results we have shown above.

For spin-j, where $j=\frac{n}{2}$ is any integer or half-integer there are $2j+1=n+1$ orthonormal states labeled $|j,m\rangle$, where m ranges over all $2j+1$ values from $-j$ to $+j$. The actions of $\hat{\mathbf{J}}^2, \hat{J}_3, \hat{J}_\pm$ on these are given by

$$\begin{aligned}
\hat{\mathbf{J}}^2 |j,m\rangle &= j(j+1)\hbar^2 |j,m\rangle \\
\hat{J}_3 |j,m\rangle &= m\hbar |j,m\rangle \\
\hat{J}_+ |j,m\rangle &= \sqrt{j(j+1) - m(m+1)}\hbar |j,m+1\rangle \\
\hat{J}_- |j,m\rangle &= \sqrt{j(j+1) - m(m-1)}\hbar |j,m-1\rangle
\end{aligned}$$

while the actions of $\hat{J}_1, \hat{J}_2$ may be found using

$$\begin{aligned}
\hat{J}_1 &= \frac{1}{2}\left(\hat{J}_+ + \hat{J}_-\right) \\
\hat{J}_2 &= \frac{1}{2i}\left(\hat{J}_+ - \hat{J}_-\right)
\end{aligned}$$

There is a vector space of every positive integer dimension spanned by $|j,m\rangle$ for some j. Taken together, these give all of the irreducible representations of the 3-dimensional rotation group. This means that any tensor, i.e., any object that the 3-dim rotation group acts on multi-linearly and homogeneously, may be decomposed into some combination of the $|j,m\rangle$ vector space.

Exercise: Find all spin-2 states by acting repeatedly with $\hat{J}_-$ on the highest state $|2,2\rangle$, including showing that $\hat{J}_- |2,-2\rangle = 0$.

Exercise: Study the effect of infinitesimal rotations on spin-1 states. Consider rotations of each of the three states about the z-axis by arbitrary amounts, and about the x-axis until you can describe what is happening clearly.

5 Decomposition of tensors

We have observed previously that a matrix can be decomposed into its trace, its antisymmetric part, and its traceless symmetric part:

$$\begin{aligned} M_{ij} &= \frac{1}{2}\delta_{ij} trM + \frac{1}{2}(M_{ij} - M_{ji}) + \frac{1}{2}\left(M_{ij} + M_{ji} - \frac{2}{3}trM\right) \\ &= T_{ij} + A_{ij} + S_{ij} \end{aligned}$$

When we rotate M_{ij} with an orthogonal transformation,

$$\begin{aligned} \tilde{M}_{ij} &= O_i^{\ m} O_j^{\ n} M_{mn} \\ &= O_i^{\ m} M_{mn} \left[O^t\right]_{\ i}^{m} \\ \tilde{M} &= OMO^{-1} \end{aligned}$$

each of these parts is preserved. For example, the antisymmetric part of the new matrix is a linear combination of the components of only the antisymmetric part of the original matrix,

$$\begin{aligned} O\frac{1}{2}\left(M - M^t\right)O^{-1} &= \frac{1}{2}\left(OMO^{-1} - OM^tO^{-1}\right) \\ &= \frac{1}{2}\left(\tilde{M} - \tilde{M}^t\right) \end{aligned}$$

We say that the usual matrix representation M_{ij} is reducible, and from the fact that these three invariant subspace have one degree of freedom for the trace, three for the antisymmetric part, and five degrees of freedom for the traceless symmetric part, we might guess that we can write M as a combination of the three vector spaces,

$$|0,0\rangle, |1,m\rangle, |2,m\rangle$$

which are of dimensions $1, 3$ and 5, respectively. What we have accomplished is to find the *irreducible representations* of the rotation group.

There is notation for this equivalence. Letting the boldface number **3** stand for each index of M, we think of the nine components of M as the outer product of 3-dimensional things,

$$M \to \mathbf{3} \otimes \mathbf{3}$$

and we write this as the sum, in the new notation, of three irreducible vector spaces:

$$\mathbf{3} \otimes \mathbf{3} = \mathbf{1} \oplus \mathbf{3} \oplus \mathbf{5}$$

There are more general objects that rotations can act on. By taking outer products of vectors, we construct "tensors" with arbitrarily many indices,

$$T_{ij\ldots k} = u_i v_j \ldots w_k$$

Since we can rotate each vector, we know how $T_{ij\ldots k}$ changes under rotations. We may take abritrary linear combinations of objects of this form to construct n-index objects with 3^n degrees of freedom. For example, a general tensor with three indices, T_{ijk}, has $3^3 = 27$ independent components.

A systematic analysis along these same lines shows that a rank three tensor, that is, an object with three indices like the Levi-Civita tensor, T_{ijk}, may be decomposed into four irreducible parts,

$$\mathbf{3} \otimes \mathbf{3} \otimes \mathbf{3} = \mathbf{1} \oplus \mathbf{8} \oplus \mathbf{8} \oplus \mathbf{10}$$

The 1-dimensional subspace $\mathbf{1}$ is the totally antisymmetric part of T_{ijk}. The two $\mathbf{8}$s are of definite mixed symmetry and the $\mathbf{10}$ is the totally symmetric part. Notice that the degrees of freedom always match, $3^3 = 27 = 1 + 8 + 8 + 10$, so we have accounted for all 27 independent components of T_{ijk}. There are general techniques for finding this decomposition for any tensor.

One familiar example of this sort of decomposition is given by the spherical harmonics. If we have any bounded, piecewise continuous function on a sphere, $f(\theta, \varphi)$, it may be expanded in spherical harmonics,

$$f(\theta, \varphi) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} A_{lm} Y^l_m(\theta, \varphi)$$

But such functions form an infinite dimensional vector space, since sums of such functions give other functions on the sphere. The collection of spherical harmonics for any fixed l, $\left\{Y^l_m(\theta, \varphi) \,|\, m = -l, -l+1, \ldots, l\right\}$ also form a vector space, since we may take linear combinations of any two linear combinations of these, to form another linear combinations of the same set. Moreover, these sets are rotationally invariant: any rotation of the sphere $(\theta, \varphi) \rightarrow (\theta + \alpha, \varphi + \beta)$ mixes m but leaves l fixed. Since the dimension of these invariant subspaces is $2l + 1$, while the dimension of the function space is infinite, the sum above gives us an infinite decomposition,

$$\boldsymbol{\infty} = \mathbf{1} \oplus \mathbf{3} \oplus \mathbf{5} \oplus \cdots \oplus (\mathbf{2l} + \mathbf{1}) \oplus \cdots$$

We show in the next set of notes that these odd-dimensional vector spaces are, in fact, spanned by the spherical harmonics.

The importance of such decompositions becomes evident when we look at atoms, nuclei, mesons or baryons, all of which are composite. Atoms are described by electrons orbiting nuclei, while the others are comprised of quarks and gluons. In each of these multi-particle systems, the constituents may have both orbital angular momentum and spin, and we need to know how these various contributions to the total angular momentum combine to give a total number of states for the system. Therefore, we will develop rules for the addition of angular momentum states.

6 Rotation matrices

We conclude with the form of the matrix elements of the finite rotation operators,

$$\hat{\mathscr{D}}(\mathbf{n}, \varphi) = e^{-\frac{i\varphi}{\hbar}\mathbf{n}\cdot\hat{\mathbf{J}}}$$

Since the generators

$$\begin{aligned} \hat{\mathbf{J}} &= \left(\hat{J}_1, \hat{J}_2, \hat{J}_3\right) \\ &= \left(\frac{1}{2}\left(\hat{J}_+ + \hat{J}_-\right), \frac{1}{2i}\left(\hat{J}_+ - \hat{J}_-\right), \hat{J}_3\right) \end{aligned}$$

change m but never change the value of j, we have

$$\begin{aligned} \langle j_1, m_1 | \hat{\mathscr{D}}(\mathbf{n}, \varphi) | j_2, m_2 \rangle &= \langle j_1, m_1 \sum_m c_m | j_2, m' \rangle \\ &= \sum_m c_m \langle j_1, m_1 | j_2, m' \rangle \end{aligned}$$

and this vanishes unless $j_1 = j_2$. We therefore need consider only matrix elements of a single fixed value of j. The matrix element of rotations of spin-j states is then the $(2j+1)\times(2j+1)$ matrix $\hat{\mathscr{D}}^{j}_{m',m}(\mathbf{n},\varphi)$ with elements

$$\hat{\mathscr{D}}^{j}_{m',m}(\mathbf{n},\varphi) \equiv \langle j,m'|\, e^{-\frac{i\varphi}{\hbar}\mathbf{n}\cdot\hat{\mathbf{J}}}\,|j,m\rangle$$

In general, if we start with a given state, $|j,m\rangle$, and rotate it, the result is given by acting with these matrices. Concretely, by multiplying by the identity matrix,

$$\hat{\mathbf{1}} = \sum_{j}\sum_{m=-j}^{j} |j,m\rangle\langle j,m|$$

we have

$$\begin{aligned}
\hat{\mathscr{D}}(\mathbf{n},\varphi)|j,m\rangle &= \left(\sum_{j'}\sum_{m'=-j'}^{j'} |j',m'\rangle\langle j',m'|\right)\hat{\mathscr{D}}(\mathbf{n},\varphi)|j,m\rangle \\
&= \sum_{j'}\sum_{m'=-j'}^{j'} |j',m'\rangle\,\delta_{j'j}\,\langle j,m'|\,\hat{\mathscr{D}}(\mathbf{n},\varphi)|j,m\rangle \\
&= \sum_{m'=-j}^{j} |j',m'\rangle\,\hat{\mathscr{D}}^{j}_{m'm}(\mathbf{n},\varphi)
\end{aligned}$$

Exercise: Find the matrix elements for the $j=\frac{1}{2}$ rotations $\hat{\mathscr{D}}^{\frac{1}{2}}_{m'm}(\mathbf{n},\varphi)$.