Rotations

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1 Rotations and angular momentum

Rotation about one or more axes is a common and useful symmetry of many physical systems. Central forces, for example, have complete rotational symmetry. Invoking Noether's theorem, we recognize angular momentum as the consequent conserved quantity. Similarly, the rotational symmetry of an isolated system leads to conservation of angular momentum in rigid body motion. In atomic physics, we also deal with central forces, so rotations and angular momentum play an important role. At the quantum level it becomes necessary to represent rotational symmetry using unitary operators. The generators depend on Hermitian operators – observables – which correspond to angular momentum.

In addition to the angular momentum of rigid bodies or the orbital angular momentum of planets or electrons, there is another form of angular momentum called *intrinsic spin*. Intrinsic spin cannot be written in the form $\mathbf{r}\times\mathbf{p}$ as we expect for orbital angular momentum. Instead, it is found by asking what (mathematical) objects can rotate. By finding all objects that we can rotate, we will discover the naturalness of intrinsic spin.

It is an awkward coincidence that intrinsic spin is found in nature only at the quantum level. There is no reason it could not exist macroscopically, but it is not seen.

To define what we mean by spin, we find the algebra of infinitesimal rotations. The situation is similar to what happens with translations and momentum. In that case, we found that translations could be described by a unitary operator. The translation operator, acting on an eigenstate of position, gives a position eigenstate with a different position eigenvalue. Because the translation $\boldsymbol{\mathcal{T}}$ operator is unitary, it has an infinitesimal generator which is anti-Hermitian, $i\hat{\mathbf{P}}$, where $\hat{\mathbf{P}}$ is Hermitian. This Hermitian operator is an observable which we identify with momentum. Thus, we see a direct relationship between the symmetry – translations – and the conserved physical property – momentum.

A similar relationship holds for angular momentum. We will find a set of unitary operators, $\mathscr{D}(\mathbf{n},\varphi)$, to describe rotations, with anti-Hermitian generators, $i\hat{\mathbf{J}}$. The Hermitian operators, $\hat{\mathbf{J}}$ correspond to angular momentum. Here we encounter a difference from our study of translations. The spectrum of translations is continuous and the generators all commute. By contrast, the generators of rotations do not commute, and (because the space of directions is compact) have a discrete spectrum. This leads us to ask for the complete spectrum of angular momentum operators.

The search for a complete spectrum leads us to examine all *linear representations* of the rotation group. Technically, a linear representation is defined as a vector space on which the rotation operators act, but intuitively once we know the vector space we also find a specific matrix representation for the transformations. These representations are not as unfamiliar as they may sound. The spherical harmonics form a vector space: linear combinations of $Y_{lm}(\theta,\varphi)$ allow us to write any piecewise continuous function of angle, $f(\theta,\varphi)$ and therefore form a basis for a vector space. The space of all spherical harmonics partitions into $(2l+1)$-dimensional subspaces of degree l. We shall see these vector spaces emerge uniquely as we develop the full spectrum.

The angular momentum of macroscopic objects which orbit or rotate can be described by $\mathbf{L}=\mathbf{r}\times\mathbf{p}$ is called *orbital angular momentum*. We will also find states of angular momentum which cannot be described this way, but require spinors. Such states describe *intrinsic angular momentum*.

2 Some examples

We begin by developing the two simplest linear representations of rotations. Since there are three independent degrees of freedom to specify any unique rotation (3 Euler angles, or equivalently, two angles to specify the axis of rotation and a third to give the rotation angle around that axis) there will be three infinitesimal generators. No matter how we *represent* the transformations, in any representation, the elements of the rotation group must satisfy the same group multiplication properties. That is, if A, B and C are any three rotations such that $AB = C$, then this must be true no matter what matrices or other mathematical objects we use to represent A, B and C. This observation carries over to the Lie algebra of infintesimal generators as well. It is this that makes it possible to find all representations.

In any Lie algebra, there are commutation relations of the form

$$[G_A, G_B] = c_{AB}^{\ \ C} G_C$$

where the numbers $c_{AB}^{\ \ C}$ are real constants called *structure constants* that characterize the group. Since the group products are independent of representation, these commutation relations are also independent of representation. In this Section, we present two representations of the rotation group, and we will see that the generators of each representation have the same Lie algebra commutators.

2.1 Real, 3-dimensional representation of rotations

2.1.1 Infinitesimal generators

Various classical mechanics books derive the form of 3-dimensional rotations in terms of Euler angles. This is unnecessarily complicated. A more efficient characterization of a rotation is to choose a unit vector, $\mathbf{n} = \mathbf{n}(\theta, \phi)$ as the axis of rotation and specify one further angle φ as the amount of rotation in the right hand sense about $\mathbf{n}$.

The basic property of a rotation is to preserve lengths of real 3-vectors. Choosing the Euclidean metric, we may write all indices down. Then if R_{ij} is a rotation matrix and

$$\tilde{x}_i = R_{ij} x_j$$

we must have

$$\tilde{\mathbf{x}}^2 = \mathbf{x}^2$$

We use the Einstein summation convention, that any repeated index is to be summed, $R_{ij}x_j = \sum_{j=1}^{3} R_{ij}x_j$. The equality of lengths implies

$$\begin{aligned} \tilde{x}_i \tilde{x}_i &= x_i x_i \\ R_{ij} x_j R_{ik} x_k &= x_i x_i \\ R_{ij} R_{ik} x_j x_k &= \delta_{jk} x_j x_k \end{aligned}$$

Since both $R_{ij}R_{ik}$ and δ_{jk} are symmetric, and this holds for all 3-vectors x_i, we may write

$$\begin{aligned} R_{ij} R_{ik} &= \delta_{jk} \\ R^t R &= 1 \end{aligned}$$

This condition is a set of coupled quadratic equations, so we study the linearized system given by the infinitesimal generators instead. Expanding R_{ij} to first order near the identity

$$R = 1 + \varepsilon G$$

Then to first order, preserving lengths implies

$$\begin{aligned}
R^t R &= 1 \\
(1+\varepsilon G)^t (1+\varepsilon G) &= 1 \\
\varepsilon G^t + \varepsilon G &= 0 \\
G^t &= -G
\end{aligned}$$

Therefore, as long as G is antisymmetric, R will be an infinitesimal rotation. The most general 3×3, antisymmetric matrix is a linear combination of

$$\mathrm{J}_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \ \mathrm{J}_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \ \mathrm{J}_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Exercise: Show for each matrix $\mathrm{J}_i, i = 1,2,3$ that its components are given by

$$[\mathrm{J}_i]_{jk} = \varepsilon_{jik}$$

These give one representation of the generators of rotations.

The Lie algebra commutators now follow by working in components,

$$\begin{aligned}
[\mathrm{J}_i, \mathrm{J}_j]_{mn} &= [\mathrm{J}_i]_{mk}[\mathrm{J}_j]_{kn} - [\mathrm{J}_j]_{mk}[\mathrm{J}_i]_{kn} \\
&= \varepsilon_{mik}\varepsilon_{kjn} - \varepsilon_{mjk}\varepsilon_{kin} \\
&= \varepsilon_{mik}\varepsilon_{jnk} - \varepsilon_{mjk}\varepsilon_{ink} \\
&= (\delta_{mj}\delta_{in} - \delta_{mn}\delta_{ij}) - (\delta_{mi}\delta_{jn} - \delta_{mn}\delta_{ji}) \\
&= \delta_{jm}\delta_{in} - \delta_{jn}\delta_{im} \\
&= \varepsilon_{jik}\varepsilon_{mnk} \\
&= (-\varepsilon_{ijk})(-\varepsilon_{mkn}) \\
&= \varepsilon_{ijk}[\mathrm{J}_k]_{mn}
\end{aligned}$$

Now we may drop the component indices and write this as the matrix relation

$$[\mathrm{J}_i, \mathrm{J}_j] = \varepsilon_{ijk}\mathrm{J}_k \tag{1}$$

These are easily checked from the explicit matrices.

The algebra of generators of a Lie group also satisfy the *Jacobi identity,*

$$[G_A, [G_B, G_C]] + [G_B, [G_C, G_A]] + [G_C, [G_A, G_B]] = 0$$

which follows from the associativity of the group elements. For the rotation group, we may accomplish the cycic permutation of indices using the Levi-Civita tensor

$$\begin{aligned}
\varepsilon^{ijk}[J_i, [J_j, J_k]] &= 0 \\
\varepsilon^{ijk}\varepsilon_{jkm}[J_i, J_m] &= 0 \\
\varepsilon^{ijk}\varepsilon_{jkm}\varepsilon_{imn}J_n &= 0
\end{aligned}$$

and this is an identity because

$$\begin{aligned}
\varepsilon^{ijk}\varepsilon_{jkm}\varepsilon_{imn} &= \varepsilon^{ijk}\varepsilon_{mjk}\varepsilon_{imn} \\
&= 2\delta^i_m\varepsilon_{imn} \\
&= 2\varepsilon_{mmn} \\
&= 0
\end{aligned}$$

Therefore, we have the Jacobi identity,

$$\varepsilon_{jkm}\varepsilon_{inm} + \varepsilon_{kim}\varepsilon_{jnm} + \varepsilon_{ijm}\varepsilon_{knm} = 0 \tag{2}$$

Note that eq.(1), and any other relationships that hold for the J_i , must hold for any representation.

2.1.2 Rotation group elements

The full group of rotations formed by 3×3 real matrices is the *3-dimensional orthogonal group*, denoted $O(3)$. These transformations may have determinant $+1$ or -1, with the negitive determinant including an inversion. If we require the handedness to be preserved, so that $\det R = 1$, the group is called the *special orthognonal group, $SO(3)$*. Any transformation in $O(3)$ is either in $SO(3)$ or may be written as an element of $SO(3)$ times the parity transformation, $-\mathbf{1}$. Here we find the general form of an element of $SO(3)$.

Having an explicit form of the generators allows us to find the general form of a rotation. Since an infinitesimal rotation is given by

$$R = 1 + \varepsilon G$$

the n^{th} power of R will give a rotation by approximately $n\varepsilon$. We produce a finite rotation by taking the limit as $n \to \infty$, with $n\varepsilon = \varphi$ staying finite. Taking G to be a linear combination, $\varepsilon G = \mathbf{w}\cdot\mathbf{J} = \varepsilon\mathbf{n}\cdot\mathbf{J}$ for an arbitrary unit vector $\mathbf{n}$

$$\begin{aligned}
R(\mathbf{n},\varphi) &= \lim_{n\to\infty}(1+\varepsilon G)^n \\
&= \lim_{n\to\infty}\sum_{k=0}^{n}\begin{pmatrix} n \\ k \end{pmatrix}\varepsilon^k G^k 1^{n-k} \\
&= \lim_{n\to\infty}\sum_{k=0}^{n}\frac{n!}{k!\,(n-k)!}\frac{1}{n^k}(n\varepsilon)^k G^k \\
&= \lim_{n\to\infty}\sum_{k=0}^{n}\frac{n(n-1)(n-2)\cdots(n-k+1)}{k!}\frac{1}{n^k}(n\varepsilon)^k G^k \\
&= \lim_{n\to\infty}\sum_{k=0}^{n}\frac{1}{k!}\frac{n(n-1)(n-2)\cdots(n-k+1)}{n^k}(n\varepsilon)^k G^k \\
&= \lim_{n\to\infty}\sum_{k=0}^{n}\frac{1}{k!}1\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{k-1}{n}\right)(n\varepsilon)^k G^k
\end{aligned}$$

Now, taking the limit, the finite product $1\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{k-1}{n}\right)$ becomes 1 and $n\varepsilon = \varphi$, so

$$R(\mathbf{n},\varphi) = \sum_{k=0}^{\infty}\frac{1}{k!}\varphi^k G^k = e^{\varphi\mathbf{n}\cdot\mathbf{J}}$$

The exponential is defined by its power series.

This result holds for any Lie algebra. The exponential of a linear combination of the generators reproduces the full group.

To see what this result means geometrically, we must evaluate the power series. Taking powers of $\mathbf{n}\cdot\mathbf{J}$ proves not to be difficult:

$$\begin{aligned} \mathbf{n}\cdot\mathbf{J} &= \mathbf{n}\cdot\mathbf{J} \\ \left[(\mathbf{n}\cdot\mathbf{J})^2\right]_{mn} &= -\left(\delta_{mn}-n_m n_n\right) \\ (\mathbf{n}\cdot\mathbf{J})^2 &\equiv -\mathbf{K} \\ (\mathbf{n}\cdot\mathbf{J})^3 &= -\mathbf{n}\cdot\mathbf{J} \end{aligned}$$

where we define $K_{mn}=\delta_{mn}-n_m n_n$

Exercise: Prove the identity

$$\varepsilon_{ijk}\varepsilon_{mnk}=\delta_{im}\delta_{jn}-\delta_{in}\delta_{jm}$$

then contract this on jn to show that

$$\varepsilon_{ijk}\varepsilon_{mjk}=2\delta_{im}$$

Exercise: Using the identity of the previous exercise, show that

$$\begin{aligned} \mathbf{n}\cdot\mathbf{J} &= \mathbf{n}\cdot\mathbf{J} \\ \left[(\mathbf{n}\cdot\mathbf{J})^2\right]_{mn} &= -\left(\delta_{mn}-n_m n_n\right) \\ (\mathbf{n}\cdot\mathbf{J})^3 &= -\mathbf{n}\cdot\mathbf{J} \end{aligned}$$

Since, except for a sign, we have come back to the original matrix, it is clear that further powers alternate, so that:

$$\begin{aligned} (\mathbf{n}\cdot\mathbf{J})^0 &= \mathbf{1} \\ \left[(\mathbf{n}\cdot\mathbf{J})^{2k}\right] &= (-1)^k\,\mathbf{K} \\ (\mathbf{n}\cdot\mathbf{J})^{2k+1} &= (-1)^k\,\mathbf{n}\cdot\mathbf{J} \end{aligned}$$

for all $k=1,2,3,\ldots$.

Substituting these powers into the power series for the finite rotation,

$$\begin{aligned} R(\mathbf{n},\varphi) &= \sum_{k=0}^{\infty}\frac{1}{k!}(\varphi\mathbf{n}\cdot\mathbf{J})^k \\ &= \mathbf{1}+\sum_{m=1}^{\infty}\frac{1}{(2m)!}(\varphi\mathbf{n}\cdot\mathbf{J})^{2m}+\sum_{m=0}^{\infty}\frac{1}{(2m)!}(\varphi\mathbf{n}\cdot\mathbf{J})^{2m+1} \\ &= \mathbf{1}+\mathbf{K}\sum_{m=1}^{\infty}\frac{(-1)^m}{(2m)!}\varphi^{2m}+\mathbf{n}\cdot\mathbf{J}\sum_{m=0}^{\infty}\frac{(-1)^m}{(2m+1)!}\varphi^{2m+1} \end{aligned}$$

Recognizing the Taylor series for the sine and cosine, we finally have

$$R(\mathbf{n},\varphi) = \mathbf{1}+\mathbf{K}(\cos\varphi-1)+\mathbf{n}\cdot\mathbf{J}\sin\varphi$$

To see the geometric effect of this transformation, we let it act on a generic vector $\mathbf{x}$ to give the new vector $\tilde{\mathbf{x}}$,

$$\tilde{\mathbf{x}}=R(\mathbf{n},\varphi)\,\mathbf{x}$$

Exercise: By writing

$$\tilde{\mathbf{x}} = R(\mathbf{n}, \varphi)\,\mathbf{x}$$

in components, $\tilde{x}_i = R_{ij} x_j$**, show that**

$$\tilde{x}_i = x_i + (x_i - (\mathbf{n}\cdot\mathbf{x})\,n_i)(\cos\varphi - 1) + (\mathbf{n}\times\mathbf{x})_i \sin\varphi$$

or, returning to vector notation,

$$\tilde{\mathbf{x}} = \mathbf{x} + (\mathbf{x} - (\mathbf{n}\cdot\mathbf{x})\,\mathbf{n})(\cos\varphi - 1) + (\mathbf{n}\times\mathbf{x})\sin\varphi$$

The meaning of the expression

$$\tilde{\mathbf{x}} = \mathbf{x} + (\mathbf{x} - (\mathbf{n}\cdot\mathbf{x})\,\mathbf{n})(\cos\varphi - 1) + (\mathbf{n}\times\mathbf{x})\sin\varphi$$

is easy to see if we define two unit vectors orthogonal to $\mathbf{n}$. We may decompose $\mathbf{x}$ into pieces parallel and perpendicular to $\mathbf{n}$, $\mathbf{x} = \mathbf{x}_\parallel + \mathbf{x}_\perp$,

$$\begin{aligned}
\mathbf{x}_\parallel &= (\mathbf{n}\cdot\mathbf{x})\,\mathbf{n} \\
\mathbf{x}_\perp &= (\mathbf{x} - (\mathbf{n}\cdot\mathbf{x})\,\mathbf{n})
\end{aligned}$$

Let unit vectors $\mathbf{k}, \mathbf{m}$ be defined by

$$\mathbf{k} \equiv \frac{\mathbf{x}_\perp}{x_\perp} \tag{3}$$

$$\mathbf{m} \equiv \frac{\mathbf{x}_\perp \times \mathbf{n}}{x_\perp} \tag{4}$$

where we note that $\mathbf{x}\times\mathbf{n} = \mathbf{x}_\perp \times \mathbf{n}$. In terms of these, the transformed $\tilde{\mathbf{x}}$ is

$$\begin{aligned}
\tilde{\mathbf{x}} &= \mathbf{x} + x_\perp \mathbf{k}(\cos\varphi - 1) - x_\perp \mathbf{m}\sin\varphi \\
&= (\mathbf{x} - x_\perp \mathbf{k}) + x_\perp \mathbf{k}\cos\varphi - x_\perp \mathbf{m}\sin\varphi \\
&= \mathbf{x}_\parallel + x_\perp \mathbf{k}\cos\varphi - x_\perp \mathbf{m}\sin\varphi
\end{aligned}$$

and therefore

$$\tilde{\mathbf{x}} = \mathbf{x}_\parallel + x_\perp \mathbf{k}\cos\varphi - x_\perp \mathbf{m}\sin\varphi$$

This shows that the component of $\mathbf{x}$ in the direction of $\mathbf{n}$ is unchanged while the part perpendicular to $\mathbf{n}$ rotates in the $\mathbf{km}$-plane through an angle φ. Therefore, $\mathbf{n}$gives the axis of rotation and φ the angle about that axis.

3 Unitary representations and SU(2)

Rotations preserve lengths, but the length we are concerned with in quantum mechanis is the hermitian norm of a state:

$$\langle \alpha \,|\alpha\rangle$$

Let a rotation of the state $|\alpha\rangle$ be denoted by

$$|\tilde{\alpha}\rangle = \mathscr{D}(\mathbf{n}, \varphi)\,|\alpha\rangle$$

so that $\langle\tilde{\alpha}| = \langle\alpha| \mathscr{D}^\dagger(\mathbf{n},\varphi)$. Then preservation of the norm amounts to

$$\begin{aligned}\langle\alpha|\alpha\rangle &= \langle\tilde{\alpha}|\tilde{\alpha}\rangle \\ &= \langle\alpha| \mathscr{D}^\dagger(\mathbf{n},\varphi)\,\mathscr{D}(\mathbf{n},\varphi)|\alpha\rangle\end{aligned}$$

We conclude that $\mathscr{D}(\mathbf{n},\varphi)$ must be unitary,

$$\mathscr{D}^\dagger(\mathbf{n},\varphi)\,\mathscr{D}(\mathbf{n},\varphi) = \hat{1}$$

This means that $\mathscr{D}^\dagger(\mathbf{n},\varphi) = \mathscr{D}^{-1}(\mathbf{n},\varphi) = \mathscr{D}(\mathbf{n},-\varphi)$.

We therefore consider unitary representations of the rotation group.

3.1 The algebra of infinitesimal generators

We find the infinitesimal generators. Any unitary transformation may be written as $\hat{\mathcal{U}} = e^{-i\hat{H}}$ with $\hat{H}$ Hermitian. Expanding for a small angle φ and writing $\hat{H}$ as a linear combination of the generators, $\hat{H} = \mathbf{w}\cdot\hat{\boldsymbol{J}} = \frac{\varphi}{\hbar}\mathbf{n}\cdot\hat{\boldsymbol{J}}$, we have

$$\mathscr{D}(\mathbf{n},\varphi) = \hat{1} - \frac{i\varphi}{\hbar}\mathbf{n}\cdot\hat{\boldsymbol{J}}$$

where the three operators, $\hat{\boldsymbol{J}} = \left(\hat{J}_1, \hat{J}_2, \hat{J}_3\right)$ are Hermitian with the same units as $\hbar$. Since the commutation relations of the generators are independent of representation, and we have found those commutators of $\hat{\mathrm{J}}_i$ to take the form

$$\left[\hat{\mathrm{J}}_i, \hat{\mathrm{J}}_j\right] = \varepsilon_{ijk}\hat{\mathrm{J}}_k$$

when the $\hat{\mathbf{J}}$ are *anti-Hermitian*. To get Hermitian generators, we set $\hat{\mathbf{J}} = -\frac{i}{\hbar}\hat{\boldsymbol{J}}$ so that $\left[\mathrm{J}_i, \hat{\mathrm{J}}_j\right] = \varepsilon_{ijk}\hat{\mathrm{J}}_k$ becomes

$$\begin{aligned}\left[-\frac{i}{\hbar}\hat{J}_i, -\frac{i}{\hbar}\hat{J}_j\right] &= \varepsilon_{ijk}\left(-\frac{i}{\hbar}\hat{J}_k\right) \\ \left[\hat{J}_i, \hat{J}_j\right] &= i\hbar\varepsilon_{ijk}\hat{J}_k\end{aligned}$$

and we take

$$\left[\hat{J}_i, \hat{J}_j\right] = i\hbar\varepsilon_{ijk}\hat{J}_k \tag{5}$$

as the fundamental commutation relation for unitary representations of generators of the rotation group.

Note that this works as it must for the real, 3-dimensional representation since inserting $(-i)\,i = 1$,

$$\begin{aligned}R(\mathbf{n},\varphi) &= e^{\varphi\mathbf{n}\cdot\mathbf{J}} \\ &= \exp\left(-i\varphi\mathbf{n}\cdot(\mathbf{iJ})\right)\end{aligned}$$

and the antisymmetry of $\mathbf{J}$ means that $(i\mathbf{J})^\dagger = i\mathbf{J}$ so $i\mathbf{J}$ is Hermitian.

3.2 SU(2)

We now develop a second representation of the rotation group.

3.2.1 General elements of SU(2)

We have already seen that the Pauli matrices satisfy

$$[\sigma_i, \sigma_j] = 2i\varepsilon_{ijk}\sigma_k$$

Therefore, if we set

$$\tau_i = \frac{\hbar}{2}\sigma_i$$

then

$$[\tau_i, \tau_j] = i\hbar\varepsilon_{ijk}\tau_k$$
$$\left[\hat{J}_i, \hat{J}_j\right] = i\varepsilon_{ijk}\hat{J}_k$$

in agreement with eq.(5).

A finite rotation is given by

$$\begin{aligned}\mathscr{D}(\mathbf{n}, \varphi) &= \exp\left(-\frac{i\varphi}{\hbar}\mathbf{n}\cdot\boldsymbol{\tau}\right) \\ &= \exp\left(-\frac{i\varphi}{2}\mathbf{n}\cdot\boldsymbol{\sigma}\right)\end{aligned}$$

This 2-dim matrix is easily found from the expansion for the exponential,

$$\exp\left(-\frac{i\varphi}{2}\mathbf{n}\cdot\boldsymbol{\sigma}\right) = \sum_{k=0}^{\infty}\frac{1}{k!}\left(-\frac{i\varphi}{2}\right)^k(\mathbf{n}\cdot\boldsymbol{\sigma})^k$$

Powers of $\mathbf{n}\cdot\boldsymbol{\sigma}$ are given by first computing

$$\begin{aligned}(\mathbf{n}\cdot\boldsymbol{\sigma})^2 &= n_i\sigma_i n_j\sigma_j \\ &= n_i n_j(\sigma_i\sigma_j) \\ &= n_i n_j(\delta_{ij}\mathbf{1} + i\varepsilon_{ijk}\sigma_k) \\ &= (\mathbf{n}\cdot\mathbf{n})\,\mathbf{1} + i(\mathbf{n}\times\mathbf{n})\cdot\boldsymbol{\sigma} \\ &= 1\end{aligned}$$

Then

$$\begin{aligned}(\mathbf{n}\cdot\boldsymbol{\sigma})^3 &= (\mathbf{n}\cdot\boldsymbol{\sigma})^2(\mathbf{n}\cdot\boldsymbol{\sigma}) \\ &= (\mathbf{n}\cdot\boldsymbol{\sigma})\end{aligned}$$

and iterating we have

$$\begin{aligned}(\mathbf{n}\cdot\boldsymbol{\sigma})^{2n} &= \mathbf{1} \\ (\mathbf{n}\cdot\boldsymbol{\sigma})^{2n+1} &= \mathbf{n}\cdot\boldsymbol{\sigma}\end{aligned}$$

The power series for the exponential is therefore

$$\begin{aligned}\exp\left(-\frac{i\varphi}{2}\mathbf{n}\cdot\boldsymbol{\sigma}\right) &= \sum_{k=0}^{\infty}\frac{1}{k!}\left(-\frac{i\varphi}{2}\right)^k(\mathbf{n}\cdot\boldsymbol{\sigma})^k \\ &= \sum_{m=0}^{\infty}\frac{1}{k!}\left(-\frac{i\varphi}{2}\right)^{2m}(\mathbf{n}\cdot\boldsymbol{\sigma})^{2m} + \sum_{m=0}^{\infty}\frac{1}{k!}\left(-\frac{i\varphi}{2}\right)^{2m+1}(\mathbf{n}\cdot\boldsymbol{\sigma})^{2m+1} \\ &= \mathbf{1}\sum_{m=0}^{\infty}\frac{(-1)^m}{k!}\left(\frac{\varphi}{2}\right)^{2m} - i\mathbf{n}\cdot\boldsymbol{\sigma}\sum_{m=0}^{\infty}\frac{(-1)^m}{k!}\left(\frac{\varphi}{2}\right)^{2m+1}\end{aligned}$$

and recognizing the series as the sine and cosine,

$$\exp\left(-\frac{i\varphi}{2}\mathbf{n}\cdot\boldsymbol{\sigma}\right) = \mathbf{1}\cos\frac{\varphi}{2} - i\mathbf{n}\cdot\boldsymbol{\sigma}\sin\frac{\varphi}{2} \tag{6}$$

Either of these forms represents the general form of an element of $SU(2)$.

Notice that we could not extend $SU(2)$ to the full unitary group $U(2)$. This would add an additional infinitesimal 1, which commutes with the Pauli matrices and therefore does not satisfy our fundamental commutation relation for rotations. Such a transformation does not preserve the norm $\langle\alpha\,|\alpha\rangle$.

3.2.2 The action of SU(2) on spinors

To rotate a spinor, $\chi = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$, we act with this $\mathscr{D}(\mathbf{n},\varphi)$,

$$\chi' = \mathscr{D}(\mathbf{n},\varphi)\,\chi$$

Concretely, to rotate the spin up ket $|S_z;+\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ by $\theta = \frac{\pi}{2}$ about the y-axis we use

$$\begin{aligned}
\mathscr{D}\left(\mathbf{j},\frac{\pi}{2},0\right)|S_z;+\rangle &= \left(1\cos\frac{\pi}{4} - i\sigma_y \sin\frac{\pi}{4}\right)\begin{pmatrix} 1 \\ 0 \end{pmatrix} \\
&= \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} \\
&= \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \\
&= \frac{1}{\sqrt{2}}\left(|+\rangle + |-\rangle\right) \\
&= |S_x,+\rangle
\end{aligned}$$

Thus, rotating a state which is spin up with respect to the z-axis by 90° around the y-axis gives a state which is spin up with respect to the x-axis.

We may find a spin up state with respect to any axis $\mathbf{n} = \mathbf{n}(\theta,\varphi)$ by starting with an up ket in the z-direction and rotating first around y by θ and then around z by φ,

$$\begin{aligned}
\chi(\theta,\varphi) &= \mathscr{D}(\mathbf{k},\varphi)\,\mathscr{D}(\mathbf{j},\theta)\begin{pmatrix} 1 \\ 0 \end{pmatrix} \\
&= \mathscr{D}(\mathbf{k},\varphi)\left(1\cos\frac{\theta}{2} - i\sigma_y\sin\frac{\theta}{2}\right)\begin{pmatrix} 1 \\ 0 \end{pmatrix} \\
&= \mathscr{D}(\mathbf{k},\varphi)\begin{pmatrix} \cos\frac{\theta}{2} \\ \sin\frac{\theta}{2} \end{pmatrix} \\
&= \left(1\cos\frac{\varphi}{2} - i\sigma_z\sin\frac{\varphi}{2}\right)\begin{pmatrix} \cos\frac{\theta}{2} \\ \sin\frac{\theta}{2} \end{pmatrix} \\
&= \begin{pmatrix} \cos\frac{\theta}{2} \\ \sin\frac{\theta}{2} \end{pmatrix}\cos\frac{\varphi}{2} - \begin{pmatrix} \cos\frac{\theta}{2} \\ -\sin\frac{\theta}{2} \end{pmatrix} i\sin\frac{\varphi}{2} \\
&= \begin{pmatrix} e^{-\frac{i\varphi}{2}}\cos\frac{\theta}{2} \\ e^{\frac{i\varphi}{2}}\sin\frac{\theta}{2} \end{pmatrix} \\
&= e^{-\frac{i\varphi}{2}}\begin{pmatrix} \cos\frac{\theta}{2} \\ e^{i\varphi}\sin\frac{\theta}{2} \end{pmatrix}
\end{aligned}$$

This a phase times the general $\mathbf{n}\cdot\hat{\mathbf{S}}$ eigenket,

$$\begin{aligned}
\chi(\theta,\varphi) &= e^{-\frac{i\varphi}{2}}\left|\mathbf{n}\cdot\hat{\mathbf{S}},+\right\rangle \\
&= e^{-\frac{i\varphi}{2}}\left[\cos\frac{\theta}{2}|+\rangle + e^{i\varphi}\sin\frac{\theta}{2}|-\rangle\right]
\end{aligned}$$

as expected.

3.2.3 The action of SU(2) on real 3-vectors

Since $SU(2)$ has the same Lie algebra as the rotation group, there must be a way to rotate real 3-vectors using $SU(2)$.

Let $\mathbf{x}$ be any real 3-vector, $\mathbf{x} = (x_1, x_2, x_3)$ and define the 2×2, traceless, Hermitian matrix,

$$X \equiv \mathbf{x} \cdot \boldsymbol{\sigma} = \begin{pmatrix} x_3 & x_1 - ix_2 \\ x_1 + ix_2 & -x_3 \end{pmatrix}$$

This is a 1-1, onto mapping between real three vectors and traceless, Hermitian matrices.

Exercise: Clearly, X is traceless and Hermitian for any real 3-vector x. Prove the converse, that any real, traceless Hermitian matrix defines a unique real 3-vector.

Exercise: Given a real 3-vector, $\boldsymbol{x}$ and its corresponding matrix, $X = \mathbf{x} \cdot \boldsymbol{\sigma}$, show that

$$\det X = -|\boldsymbol{x}|^2$$

Thus, X is a 2-dim complex representation of a real, 3-dim vector, with negative of the vector length given by the determinant.

To transform a matrix (mixed second rank tensor) under $SU(2)$, we require a similarity transformation with two copies of the rotation matrix,

$$\tilde{X} = \mathscr{D}(\mathbf{n}, \varphi) X \mathscr{D}^{-1}(\mathbf{n}, \varphi)$$

Since $\mathscr{D}^{-1}(\mathbf{n}, \varphi) = \mathscr{D}^{\dagger}(\mathbf{n}, \varphi)$ we see that $\tilde{X}$ is Hermitian if and only if X is Hermitian:

$$\begin{aligned} \tilde{X}^{\dagger} &= \left(\mathscr{D}(\mathbf{n}, \varphi) X \mathscr{D}^{-1}(\mathbf{n}, \varphi)\right)^{\dagger} \\ &= \left(\mathscr{D}(\mathbf{n}, \varphi) X \mathscr{D}^{\dagger}(\mathbf{n}, \varphi)\right)^{\dagger} \\ &= \mathscr{D}^{\dagger\dagger}(\mathbf{n}, \varphi) X^{\dagger} \mathscr{D}^{\dagger}(\mathbf{n}, \varphi) \\ &= \mathscr{D}(\mathbf{n}, \varphi) X \mathscr{D}^{\dagger}(\mathbf{n}, \varphi) \\ &= \tilde{X} \end{aligned}$$

Furthermore, the trace $\tilde{X}$ remains zero. By the cyclic property of the trace, $tr(ABC) = tr(CAB)$, we have

$$\begin{aligned} tr\tilde{X} &= tr\left(\mathscr{D}(\mathbf{n}, \varphi) X \mathscr{D}^{-1}(\mathbf{n}, \varphi)\right) \\ &= tr\left(\mathscr{D}^{-1}(\mathbf{n}, \varphi) \mathscr{D}(\mathbf{n}, \varphi) X\right) \\ &= tr(X) = 0 \end{aligned}$$

We conclude that similarity transformation by $\mathscr{D}(\mathbf{n}, \varphi)$ maps real 3-vectors to real 3-vectors, $\left(\tilde{X} = \tilde{\mathbf{x}} \cdot \boldsymbol{\sigma}\right) \leftrightarrow (X = \mathbf{x} \cdot \boldsymbol{\sigma})$.

Finally, the norm is preserved:

$$\begin{aligned} \det \tilde{X} &= \det \mathscr{D}(\mathbf{n}, \varphi) \det X \det \mathscr{D}^{-1}(\mathbf{n}, \varphi) \\ &= \det \mathscr{D}(\mathbf{n}, \varphi) \det X \frac{1}{\det \mathscr{D}(\mathbf{n}, \varphi)} \\ &= \det X \end{aligned}$$

Therefore, the action of $\mathscr{D}(\mathbf{n}, \varphi)$ on X is necessarily a rotation of the 3-vector $\mathbf{x}$ into $\tilde{\mathbf{x}}$.

The actual rotation is accomplished as follows. Let

$$\tilde{X} = \mathscr{D}(\mathbf{n},\varphi) X \mathscr{D}^{\dagger}(\mathbf{n},\varphi)$$

where $\tilde{X} = \tilde{\mathbf{x}} \cdot \boldsymbol{\sigma}$ and $X = \mathbf{x} \cdot \boldsymbol{\sigma}$. Then, expanding,

$$\begin{aligned}
\tilde{\mathbf{x}} \cdot \boldsymbol{\sigma} &= \left(e^{-\frac{i\varphi}{2}\mathbf{n}\cdot\boldsymbol{\sigma}}\right)(\mathbf{x}\cdot\boldsymbol{\sigma})\left(e^{\frac{i\varphi}{2}\mathbf{n}\cdot\boldsymbol{\sigma}}\right) \\
&= \left(\mathbf{1}\cos\frac{\varphi}{2} - i\mathbf{n}\cdot\boldsymbol{\sigma}\sin\frac{\varphi}{2}\right)(\mathbf{x}\cdot\boldsymbol{\sigma})\left(\mathbf{1}\cos\frac{\varphi}{2} + i\mathbf{n}\cdot\boldsymbol{\sigma}\sin\frac{\varphi}{2}\right) \\
&= (\mathbf{x}\cdot\boldsymbol{\sigma})\cos^2\frac{\varphi}{2} + i(\mathbf{x}\cdot\boldsymbol{\sigma})(\mathbf{n}\cdot\boldsymbol{\sigma})\sin\frac{\varphi}{2}\cos\frac{\varphi}{2} - i(\mathbf{n}\cdot\boldsymbol{\sigma})(\mathbf{x}\cdot\boldsymbol{\sigma})\sin\frac{\varphi}{2}\cos\frac{\varphi}{2} \\
&\quad + (\mathbf{n}\cdot\boldsymbol{\sigma})(\mathbf{x}\cdot\boldsymbol{\sigma})(\mathbf{n}\cdot\boldsymbol{\sigma})\sin^2\frac{\varphi}{2} \\
&= (\mathbf{x}\cdot\boldsymbol{\sigma})\cos^2\frac{\varphi}{2} + i\left[(\mathbf{x}\cdot\boldsymbol{\sigma}),(\mathbf{n}\cdot\boldsymbol{\sigma})\right]\sin\frac{\varphi}{2}\cos\frac{\varphi}{2} \\
&\quad + (\mathbf{n}\cdot\boldsymbol{\sigma})(\mathbf{x}\cdot\boldsymbol{\sigma})(\mathbf{n}\cdot\boldsymbol{\sigma})\sin^2\frac{\varphi}{2}
\end{aligned}$$

Working out the commutator

$$\begin{aligned}
\left[(\mathbf{x}\cdot\boldsymbol{\sigma}),(\mathbf{n}\cdot\boldsymbol{\sigma})\right] &= x_i n_j \left[\sigma_i,\sigma_j\right] \\
&= x_i n_j \left(2i\varepsilon_{ijk}\sigma_k\right) \\
&= 2i(\mathbf{x}\times\mathbf{n})\cdot\boldsymbol{\sigma}
\end{aligned}$$

while for the triple product, we have

$$\begin{aligned}
(\mathbf{n}\cdot\boldsymbol{\sigma})(\mathbf{x}\cdot\boldsymbol{\sigma})(\mathbf{n}\cdot\boldsymbol{\sigma}) &= (\mathbf{n}\cdot\boldsymbol{\sigma}) x_i n_j \sigma_i\sigma_j \\
&= (\mathbf{n}\cdot\boldsymbol{\sigma}) x_i n_j \left(\delta_{ij} + i\varepsilon_{ijk}\sigma_k\right) \\
&= (\mathbf{n}\cdot\boldsymbol{\sigma}) x_i n_j \left(\delta_{ij} + i\varepsilon_{ijk}\sigma_k\right) \\
&= (\mathbf{n}\cdot\boldsymbol{\sigma})\left(\mathbf{x}\cdot\mathbf{n} + i x_i n_j \varepsilon_{ijk}\sigma_k\right) \\
&= (\mathbf{x}\cdot\mathbf{n})\,\mathbf{n}\cdot\boldsymbol{\sigma} + i x_i n_j n_m \varepsilon_{ijk}\sigma_m\sigma_k \\
&= (\mathbf{x}\cdot\mathbf{n})\,\mathbf{n}\cdot\boldsymbol{\sigma} + i x_i n_j n_m \varepsilon_{ijk}\left(\delta_{mk} + i\varepsilon_{mkn}\sigma_n\right) \\
&= (\mathbf{x}\cdot\mathbf{n})\,\mathbf{n}\cdot\boldsymbol{\sigma} + i\mathbf{x}\cdot(\mathbf{n}\times\mathbf{n}) - x_i n_j n_m \varepsilon_{ijk}\varepsilon_{mkn}\sigma_n
\end{aligned}$$

Dropping $i\mathbf{x}\cdot(\mathbf{n}\times\mathbf{n}) = 0$ and using $\varepsilon_{ijk}\varepsilon_{mnk} = \delta_{im}\delta_{jn} - \delta_{in}\delta_{jm}$ this becomes

$$\begin{aligned}
(\mathbf{n}\cdot\boldsymbol{\sigma})(\mathbf{x}\cdot\boldsymbol{\sigma})(\mathbf{n}\cdot\boldsymbol{\sigma}) &= (\mathbf{x}\cdot\mathbf{n})\,\mathbf{n}\cdot\boldsymbol{\sigma} + x_i n_j n_m \sigma_n\left(\delta_{im}\delta_{jn} - \delta_{in}\delta_{jm}\right) \\
&= 2(\mathbf{x}\cdot\mathbf{n})\,\mathbf{n}\cdot\boldsymbol{\sigma} - (\mathbf{x}\cdot\boldsymbol{\sigma})
\end{aligned}$$

Substituting into the transformation,

$$\begin{aligned}
\tilde{\mathbf{x}}\cdot\boldsymbol{\sigma} &= (\mathbf{x}\cdot\boldsymbol{\sigma})\cos^2\frac{\varphi}{2} + i\left[(\mathbf{x}\cdot\boldsymbol{\sigma}),(\mathbf{n}\cdot\boldsymbol{\sigma})\right]\sin\frac{\varphi}{2}\cos\frac{\varphi}{2} \\
&\quad + (\mathbf{n}\cdot\boldsymbol{\sigma})(\mathbf{x}\cdot\boldsymbol{\sigma})(\mathbf{n}\cdot\boldsymbol{\sigma})\sin^2\frac{\varphi}{2} \\
&= (\mathbf{x}\cdot\boldsymbol{\sigma})\cos^2\frac{\varphi}{2} - 2(\mathbf{x}\times\mathbf{n})\cdot\boldsymbol{\sigma}\sin\frac{\varphi}{2}\cos\frac{\varphi}{2} \\
&\quad + \left(2(\mathbf{x}\cdot\mathbf{n})\,\mathbf{n}\cdot\boldsymbol{\sigma} - (\mathbf{x}\cdot\boldsymbol{\sigma})\right)\sin^2\frac{\varphi}{2} \\
&= (\mathbf{x}\cdot\boldsymbol{\sigma})\cos\varphi - (\mathbf{x}\times\mathbf{n})\cdot\boldsymbol{\sigma}\sin\varphi + (\mathbf{x}\cdot\mathbf{n})\,\mathbf{n}\cdot\boldsymbol{\sigma}(1-\cos\varphi) \\
&= (\mathbf{x}\cdot\mathbf{n})\,\mathbf{n}\cdot\boldsymbol{\sigma} + \left((\mathbf{x}\cdot\boldsymbol{\sigma}) - (\mathbf{x}\cdot\mathbf{n})\,\mathbf{n}\cdot\boldsymbol{\sigma}\right)\cos\varphi - (\mathbf{x}\times\mathbf{n})\cdot\boldsymbol{\sigma}\sin\varphi
\end{aligned}$$

Notice that as we proved, the right hand side comes out as a real 3-vector dotted into $\boldsymbol{\sigma}$. Equating coefficients on the right and left,

$$\tilde{\mathbf{x}} = (\mathbf{x}\cdot\mathbf{n})\,\mathbf{n} + \left(\mathbf{x} - (\mathbf{x}\cdot\mathbf{n})\,\mathbf{n}\right)\cos\varphi - \mathbf{x}\times\mathbf{n}\sin\varphi$$

Finally, decomposing $\mathbf{x} = \mathbf{x}_{\parallel} + \mathbf{x}_{\perp}$ as before and using the same definitions of unit vectors $\mathbf{k}, \mathbf{m}$, given in eqs.(3) and (4), we recover the same expression

$$\tilde{\mathbf{x}} = \mathbf{x}_{\parallel} + x_{\perp}\mathbf{k}\cos\varphi - x_{\perp}\mathbf{m}\sin\varphi$$

found from $SO(3)$ for a rotation by an angle φ about the direction $\mathbf{n}$.

4 Appendix: Lie algebras

Expanding infinitesimally to second order

$$\begin{aligned}
\mathscr{D}(\mathbf{n},\varphi) &= \left(\hat{1} - \frac{i\varphi}{\hbar}\mathbf{n}\cdot\hat{\mathbf{J}} + \frac{1}{2}\left(-\frac{i\varphi}{\hbar}\mathbf{n}\cdot\hat{\mathbf{J}}\right)^2 + \ldots\right) \\
&= \hat{1} - \frac{i\varphi}{\hbar}\mathbf{n}\cdot\hat{\mathbf{J}} - \frac{\varphi^2}{2\hbar^2}\left(\mathbf{n}\cdot\hat{\mathbf{J}}\right)^2 + \ldots
\end{aligned}$$

These operators must satisfy certain basic properties of rotations. In particular, we know that any two rotations are equivalent to some third rotation,

$$\mathscr{D}(\mathbf{n},\varphi)\,\mathscr{D}(\mathbf{m},\theta) = \mathscr{D}(\mathbf{l},\chi)$$

It follows that a finite sequence of rotations is also equivalent to a single rotation. Consider the combination

$$\mathscr{D}^{\dagger}(\mathbf{n},\varphi)\,\mathscr{D}^{\dagger}(\mathbf{m},\theta)\,\mathscr{D}(\mathbf{n},\varphi)\,\mathscr{D}(\mathbf{m},\theta) = \mathscr{D}(\mathbf{l},\chi)$$

This must hold for some $\mathbf{l}, \chi$. Keeping terms to second order,

$$\begin{aligned}
\hat{1} - \frac{i\chi}{\hbar}\mathbf{l}\cdot\hat{\mathbf{J}} &= \left(\hat{1} + \frac{i\varphi}{\hbar}\mathbf{n}\cdot\hat{\mathbf{J}} - \frac{\varphi^2}{2\hbar^2}\left(\mathbf{n}\cdot\hat{\mathbf{J}}\right)^2\right)\left(\hat{1} + \frac{i\theta}{\hbar}\mathbf{m}\cdot\hat{\mathbf{J}} - \frac{\theta^2}{2\hbar^2}\left(\mathbf{m}\cdot\hat{\mathbf{J}}\right)^2\right) \\
&\quad \times\left(\hat{1} - \frac{i\varphi}{\hbar}\mathbf{n}\cdot\hat{\mathbf{J}} - \frac{\varphi^2}{2\hbar^2}\left(\mathbf{n}\cdot\hat{\mathbf{J}}\right)^2\right)\left(\hat{1} - \frac{i\theta}{\hbar}\mathbf{m}\cdot\hat{\mathbf{J}} - \frac{\theta^2}{2\hbar^2}\left(\mathbf{m}\cdot\hat{\mathbf{J}}\right)^2\right) \\
&= \left(\hat{1} + \frac{i\theta}{\hbar}\mathbf{m}\cdot\hat{\mathbf{J}} + \frac{i\varphi}{\hbar}\mathbf{n}\cdot\hat{\mathbf{J}} - \frac{\theta^2}{2\hbar^2}\left(\mathbf{m}\cdot\hat{\mathbf{J}}\right)^2 + \frac{i\varphi}{\hbar}\mathbf{n}\cdot\hat{\mathbf{J}}\frac{i\theta}{\hbar}\mathbf{m}\cdot\hat{\mathbf{J}} - \frac{\varphi^2}{2\hbar^2}\left(\mathbf{n}\cdot\hat{\mathbf{J}}\right)^2\right) \\
&\quad \times\left(\hat{1} - \frac{i\theta}{\hbar}\mathbf{m}\cdot\hat{\mathbf{J}} - \frac{i\varphi}{\hbar}\mathbf{n}\cdot\hat{\mathbf{J}} - \frac{\theta^2}{2\hbar^2}\left(\mathbf{m}\cdot\hat{\mathbf{J}}\right)^2 + \frac{i\varphi}{\hbar}\mathbf{n}\cdot\hat{\mathbf{J}}\frac{i\theta}{\hbar}\mathbf{m}\cdot\hat{\mathbf{J}} - \frac{\varphi^2}{2\hbar^2}\left(\mathbf{n}\cdot\hat{\mathbf{J}}\right)^2\right) \\
&= \hat{1} - \frac{i\theta}{\hbar}\mathbf{m}\cdot\hat{\mathbf{J}} - \frac{i\varphi}{\hbar}\mathbf{n}\cdot\hat{\mathbf{J}} - \frac{\theta^2}{2\hbar^2}\left(\mathbf{m}\cdot\hat{\mathbf{J}}\right)^2 + \frac{i\varphi}{\hbar}\mathbf{n}\cdot\hat{\mathbf{J}}\frac{i\theta}{\hbar}\mathbf{m}\cdot\hat{\mathbf{J}} - \frac{\varphi^2}{2\hbar^2}\left(\mathbf{n}\cdot\hat{\mathbf{J}}\right)^2 \\
&\quad + \left(\frac{i\theta}{\hbar}\mathbf{m}\cdot\hat{\mathbf{J}} + \frac{i\varphi}{\hbar}\mathbf{n}\cdot\hat{\mathbf{J}}\right)\left(\hat{1} - \frac{i\theta}{\hbar}\mathbf{m}\cdot\hat{\mathbf{J}} - \frac{i\varphi}{\hbar}\mathbf{n}\cdot\hat{\mathbf{J}}\right) \\
&\quad - \frac{\theta^2}{2\hbar^2}\left(\mathbf{m}\cdot\hat{\mathbf{J}}\right)^2 + \frac{i\varphi}{\hbar}\mathbf{n}\cdot\hat{\mathbf{J}}\frac{i\theta}{\hbar}\mathbf{m}\cdot\hat{\mathbf{J}} - \frac{\varphi^2}{2\hbar^2}\left(\mathbf{n}\cdot\hat{\mathbf{J}}\right)^2 \\
&= \hat{1} - \frac{i\theta}{\hbar}\mathbf{m}\cdot\hat{\mathbf{J}} + \frac{i\theta}{\hbar}\mathbf{m}\cdot\hat{\mathbf{J}} + \frac{i\varphi}{\hbar}\mathbf{n}\cdot\hat{\mathbf{J}} - \frac{i\varphi}{\hbar}\mathbf{n}\cdot\hat{\mathbf{J}} \\
&\quad + \left(\frac{\theta^2}{\hbar^2} - \frac{\theta^2}{2\hbar^2} - \frac{\theta^2}{2\hbar^2}\right)\left(\mathbf{m}\cdot\hat{\mathbf{J}}\right)^2 \\
&\quad + \left(\frac{\varphi^2}{\hbar^2} - \frac{\varphi^2}{2\hbar^2} - \frac{\varphi^2}{2\hbar^2}\right)\left(\mathbf{n}\cdot\hat{\mathbf{J}}\right)^2 \\
&\quad - \frac{i\varphi}{\hbar}\mathbf{n}\cdot\hat{\mathbf{J}}\frac{i\theta}{\hbar}\mathbf{m}\cdot\hat{\mathbf{J}} + \frac{i\varphi}{\hbar}\mathbf{n}\cdot\hat{\mathbf{J}}\frac{i\theta}{\hbar}\mathbf{m}\cdot\hat{\mathbf{J}} + \frac{i\varphi}{\hbar}\mathbf{n}\cdot\hat{\mathbf{J}}\frac{i\theta}{\hbar}\mathbf{m}\cdot\hat{\mathbf{J}} - \frac{i\theta}{\hbar}\mathbf{m}\cdot\hat{\mathbf{J}}\frac{i\varphi}{\hbar}\mathbf{n}\cdot\hat{\mathbf{J}} \\
&= \hat{1} - \frac{\varphi\theta}{\hbar^2}\left(\mathbf{n}\cdot\hat{\mathbf{J}}\mathbf{m}\cdot\hat{\mathbf{J}} - \mathbf{m}\cdot\hat{\mathbf{J}}\mathbf{n}\cdot\hat{\mathbf{J}}\right)
\end{aligned}$$

we may identify

$$i\hbar\chi l_k \hat{J}_k = \varphi\theta n_i m_j \left[\hat{J}_i, \hat{J}_j\right]$$

This must hold for all $\varphi, \theta, n_i, m_j$. Introducing coefficients, c_{ijk}, such that

$$\left[\hat{J}_i, \hat{J}_j\right] = i\hbar \sum_{k=1}^{3} c_{ijk} \hat{J}_k$$

we have

$$\chi l_k = c_{ijk}\varphi\theta n_i m_j$$

telling us how the parameters χl_k for the resultant rotation is related to the the parameters, φn_i and θm_j, of the of the original rotations.

We may find the coefficients c_{ijk} by computing in any particular case. For the real, 3-dimensional rotations above, we expand infinitesimally. With $\cos\theta \approx 1$, $\sin\theta \approx \theta$, we find,

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix} \approx \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \theta \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$\begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix} \approx \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \theta \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} \cos\theta & \sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \approx \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \theta \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Defining

$$\bar{J}_x = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$\bar{J}_y = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}$$

$$\bar{J}_z = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

we easily compute

$$\left[\bar{J}_x, \bar{J}_y\right] = \bar{J}_z$$

and in general,

$$\begin{aligned} \left[\bar{J}_i, \bar{J}_j\right] &= \sum_{k=1}^{3} \varepsilon_{ijk} \bar{J}_k \\ &= \varepsilon_{ijk} \bar{J}_k \end{aligned}$$

for i, j, k each ranging over all three indices. This shows that c_{ijk} is proportional to the Levi-Civita tensor, so with a suitable normalization of our general generators, J_i, we have

$$\left[\hat{J}_i, \hat{J}_j\right] = i\hbar\varepsilon_{ijk} \hat{J}_k$$

where we include a factor of i to make the real, anti-symmetric matrices $\bar{J}$ hermitian. This final relationship holds for *any linear representation* of rotations. Given any three objects, J_i, satisfying this set of commutation relations, we may use them to generate any finite rotation by taking the limit

$$\begin{aligned} \mathscr{D}(\mathbf{n}, \varphi) &= \lim_{n \to \infty} \left(\hat{1} - \frac{i\varepsilon}{\hbar} \mathbf{n} \cdot \hat{\mathbf{J}} \right)^n \\ &= \exp\left(-\frac{i\varphi}{\hbar} \mathbf{n} \cdot \hat{\mathbf{J}} \right) \end{aligned}$$

where $\varphi = \lim_{n \to \infty} n\varepsilon$.