One of the most important problems in quantum mechanics is the simple harmonic oscillator, in part because its properties are directly applicable to field theory. The treatment in Dirac notation is particularly satisfying.

1 Hamiltonian

Writing the potential $\frac{1}{2}kx^2$ in terms of the classical frequency, $\omega = \sqrt{\frac{k}{m}}$, puts the classical Hamiltonian in the form

$$H = \frac{p^2}{2m} + \frac{m\omega^2 x^2}{2}$$

Since there are no products of non-commuting operators, there is no ambiguity in the resulting Hamiltonian operator,

$$\hat{H} = \frac{1}{2m}\hat{P}^2 + \frac{m\omega^2}{2}\hat{X}^2$$

We make no choice of basis.

2 Raising and lowering operators

Notice that

$$\begin{aligned}
\left(x + \frac{ip}{m\omega}\right)\left(x - \frac{ip}{m\omega}\right) &= x^2 + \frac{p^2}{m^2\omega^2} \\
&= \frac{2}{m\omega^2}\left(\frac{1}{2}m\omega^2 x^2 + \frac{p^2}{2m}\right)
\end{aligned}$$

so that we may write the classical Hamiltonian as

$$H = \frac{m\omega^2}{2}\left(x + \frac{ip}{m\omega}\right)\left(x - \frac{ip}{m\omega}\right)$$

We can write the quantum Hamiltonian in a similar way. Choosing our normalization with a bit of foresight, we define two conjugate operators,

$$\begin{aligned}
\hat{a} &= \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{X} + \frac{i}{m\omega}\hat{P}\right) \\
\hat{a}^\dagger &= \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{X} - \frac{i}{m\omega}\hat{P}\right)
\end{aligned}$$

The operator $\hat{a}^\dagger$ is called the *raising operator* and $\hat{a}$ is called the *lowering operator*. Notice that they are *not* Hermitian. In taking the product of these, we must be careful with ordering since $\hat{X}$ and $\hat{P}$

$$\begin{aligned}
\hat{a}^\dagger\hat{a} &= \frac{m\omega}{2\hbar}\left(\hat{X} - \frac{i\hat{P}}{m\omega}\right)\left(\hat{X} + \frac{i\hat{P}}{m\omega}\right) \\
&= \frac{m\omega}{2\hbar}\left(\hat{X}^2 + \frac{i}{m\omega}\hat{X}\hat{P} - \frac{i}{m\omega}\hat{P}\hat{X} + \frac{\hat{P}^2}{m^2\omega^2}\right) \\
&= \frac{m\omega}{2\hbar}\left(\hat{X}^2 + \frac{i}{m\omega}\left[\hat{X},\hat{P}\right] + \frac{\hat{P}^2}{m^2\omega^2}\right)
\end{aligned}$$

Using the commutator, $\left[\hat{X}, \hat{P}\right] = i\hbar\hat{1}$, this becomes

$$\begin{aligned}
\hat{a}^\dagger \hat{a} &= \left(\frac{1}{\hbar\omega}\right)\left(\frac{1}{2}m\omega^2\right)\left(\hat{X}^2 - \frac{\hbar}{m\omega} + \frac{\hat{P}^2}{m^2\omega^2}\right) \\
&= \frac{1}{\hbar\omega}\left(\frac{1}{2}m\omega^2\hat{X}^2 - \frac{1}{2}\hbar\omega + \frac{\hat{P}^2}{2m}\right) \\
&= \frac{1}{\hbar\omega}\left(\hat{H} - \frac{1}{2}\hbar\omega\right)
\end{aligned}$$

and therefore,

$$\hat{H} = \hbar\omega\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right)$$

3 The number operator

This turns out to be a very convenient form for the Hamiltonian because $\hat{a}$ and $a^\dagger$ have very simple properties. First, their commutator is simply

$$\begin{aligned}
\left[\hat{a}, \hat{a}^\dagger\right] &= \frac{m\omega}{2\hbar}\left[\left(\hat{X} + \frac{i\hat{P}}{m\omega}\right), \left(\hat{X} - \frac{i\hat{P}}{m\omega}\right)\right] \\
&= \frac{m\omega}{2\hbar}\left(\left[\hat{X}, -\frac{i}{m\omega}\hat{P}\right] + \left[\frac{i}{m\omega}\hat{P}, \hat{X}\right]\right) \\
&= -\frac{2i}{m\omega}\frac{m\omega}{2\hbar}\left[\hat{X}, \hat{P}\right] \\
&= -\frac{i}{\hbar}i\hbar \\
&= 1
\end{aligned}$$

Consider one further set of commutation relations. Defining $\hat{N} \equiv \hat{a}^\dagger\hat{a} = \hat{N}^\dagger$, called the *number operator*, we have

$$\begin{aligned}
\left[\hat{N}, \hat{a}\right] &= \left[\hat{a}^\dagger\hat{a}, \hat{a}\right] \\
&= \hat{a}^\dagger\left[\hat{a}, \hat{a}\right] + \left[\hat{a}^\dagger, \hat{a}\right]\hat{a} \\
&= -\hat{a}
\end{aligned}$$

and

$$\begin{aligned}
\left[\hat{N}, \hat{a}^\dagger\right] &= \left[\hat{a}^\dagger\hat{a}, \hat{a}^\dagger\right] \\
&= \hat{a}\left[\hat{a}^\dagger, \hat{a}^\dagger\right] + \hat{a}^\dagger\left[\hat{a}, \hat{a}^\dagger\right] \\
&= \hat{a}^\dagger
\end{aligned}$$

Notice that $\hat{N}$ is Hermitian, hence observable, and that $\hat{H} = \hbar\omega\left(\hat{N} + \frac{1}{2}\right)$.

4 Energy eigenstates

4.1 Positivity of the energy

Prove that expectation values of Hamiltonian, hence all energies, are positive.

Q1: Consider an arbitrary expectation value of the Hamiltonian,

$$\begin{aligned}\langle\psi|\,\hat{H}\,|\psi\rangle &= \langle\psi|\,\hbar\omega\left(\hat{a}^\dagger\hat{a}+\frac{1}{2}\right)|\psi\rangle \\ &= \hbar\omega\left(\langle\psi|\,\hat{a}^\dagger\hat{a}\,|\psi\rangle+\frac{1}{2}\,\langle\psi\mid\psi\rangle\right)\end{aligned}$$

Use the fact that the norm of any state is positive, $\langle\psi\,|\psi\rangle > 0$ to show that the energy of any SHO state is positive:

$$\langle\psi|\,\hat{H}\,|\psi\rangle \;=\; E \;>\; 0$$

Hint: It helps to define $|\beta\rangle \equiv \hat{a}\,|\psi\rangle$.

4.2 The lowest energy state

Suppose that $|E\rangle$ is any energy eigenket with

$$\hat{H}\,|E\rangle = E\,|E\rangle$$

Consider the new ket formed by acting on $|E\rangle$ with the lowering operator $\hat{a}$.

Q2: Show that $\hat{a}\,|E\rangle$ is also an energy eigenket with energy $E-\hbar\omega$,

$$\hat{H}\,(\hat{a}\,|E\rangle) = (E-\hbar\omega)\,(\hat{a}\,|E\rangle)$$

Since $\hat{a}\,|E\rangle$ is an energy eigenket, we may repeat this procedure to show that $\hat{a}^2\,|E\rangle$ is an energy eigenket

with energy $E-2\hbar\omega$. Continuing in this way, we find that $\hat{a}^k\,|E\rangle$ will have energy $E-k\hbar\omega$. This process cannot continue indefinitely, because the energy must remain positive. Let k be the largest integer for which $E-k\hbar\omega$ is positive,

$$\hat{H}\hat{a}^k\,|E\rangle = (E-k\hbar\omega)\,\hat{a}^k\,|E\rangle$$

with corresponding state $\hat{a}^k\,|E\rangle$. Then applying the lowering operator one more time cannot give a new state. The only other possibility is zero. Rename the lowest energy state $|0\rangle = A_0\hat{a}^k\,|E\rangle$, where we choose A_0 so that $|0\rangle$ is normalized. We then must have

$$\hat{a}\,|0\rangle = 0$$

This is the lowest energy state of the oscillator, called the *ground state*.

4.3 The complete spectrum

Q3: Now that we have the ground state, we reverse the process, acting instead with the raising operator. Show that, acting any energy eigenket, $|E\rangle$, the raising operator gives another energy eigenket, $\hat{a}^\dagger\,|E\rangle$, with energy $E+\hbar\omega$,

$$\hat{H}\left(\hat{a}^\dagger\,|E\rangle\right) \;=\; (E+\hbar\omega)\left(\hat{a}^\dagger\,|E\rangle\right)$$

Q4: Find the energy of the state $\hat{a}^\dagger |0\rangle$.

Q5: Define the normalized state to be $|1\rangle \equiv A_1 \hat{a}^\dagger |0\rangle$. Find the normalization constant, A_1. There is nothing to prevent us continuing this procedure indefintely. Continuing n times, we have states

$$|n\rangle = A_n \left(\hat{a}^\dagger\right)^n |0\rangle$$

satisfying

$$\hat{H} |n\rangle = \left(n + \frac{1}{2}\right) \hbar\omega |n\rangle$$

This gives the complete set of energy eigenkets.

4.4 Normalization

Now, consider the expectation of $\hat{N}$ in the n^{th} state, $|n\rangle = A_n \left(\hat{a}^\dagger\right)^n |0\rangle$.

Q6: Develop a recursion relation for the normalization constants, A_n, starting from $1 = \langle n \mid n\rangle$. Specifically, show that $|A_{n-1}|^2 = n |A_n|^2$. Iterating the recursion relation gives

$$\begin{aligned} |A_n|^2 &= \frac{1}{n} |A_{n-1}|^2 \\ &= \frac{1}{n(n-1)} |A_{n-2}|^2 \\ &\vdots \\ &= \frac{1}{n!} |A_1|^2 \end{aligned}$$

so that, choosing all of the coefficients real, we have the complete set of normalized states

$$|n\rangle = \frac{1}{\sqrt{n!}} \left(\hat{a}^\dagger\right)^n |0\rangle$$

Q7: Show that $\hat{a}^\dagger |n\rangle = \sqrt{n+1} |n+1\rangle$ and find the corresponding relation for $\hat{a} |n\rangle$.

5 Wave function

Choosing a coordinate basis to express the condition

$$\hat{a} |0\rangle = 0$$

gives

$$\begin{aligned} 0 &= \langle x| \hat{a} |0\rangle \\ &= \sqrt{\frac{m\omega}{2\hbar}} \langle x| \left(\hat{X} + \frac{i}{m\omega}\hat{P}\right) |0\rangle \\ &= \sqrt{\frac{m\omega}{2\hbar}} \left(\langle x| \hat{X} |0\rangle + \frac{i}{m\omega}\hat{P} |0\rangle\right) \\ &= \sqrt{\frac{m\omega}{2\hbar}} \left(x \langle x \mid 0\rangle + \frac{i}{m\omega} \langle x| \hat{P} |0\rangle\right) \end{aligned}$$

and since we have shown that $\langle x|\hat{P}|0\rangle = -i\hbar\frac{d}{dx}\langle x \mid 0\rangle$ this becomes a differential equation,

$$\frac{d}{dx}\langle x \mid 0\rangle + \frac{m\omega x}{\hbar}\langle x \mid 0\rangle = 0$$

Setting $\psi_0(x) = \langle x \mid 0\rangle$ and integrating,

$$\begin{aligned}
\frac{d\psi_0}{\psi_0} &= -\frac{m\omega x}{\hbar}dx \\
\ln\left(\frac{\psi_0(x)}{\psi_0(0)}\right) &= -\frac{m\omega x^2}{2\hbar} \\
\psi_0(x) &= \psi_0(0)\, e^{-\frac{m\omega x^2}{2\hbar}}
\end{aligned}$$

we find that the wave function of the ground state is a Gaussian, with the magnitude of the remaining coefficient $\psi_0(0)$ determined by normalizing the Gaussian,

$$\psi_0(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-\frac{m\omega x^2}{2\hbar}}$$

Now consider the wave function, $\psi_n(x)$, for the eigenstates. We have already found that the ground state is given by a normalized Gaussian,To find the wave functions of the higher energy states, consider

$$\begin{aligned}
\psi_n(x) &= \langle x \mid n\rangle \\
&= \langle x|\frac{1}{\sqrt{n!}}\left(\hat{a}^\dagger\right)^n|0\rangle \\
&= \frac{1}{\sqrt{n}}\langle x|\hat{a}^\dagger\frac{1}{\sqrt{(n-1)!}}\left(\hat{a}^\dagger\right)^{n-1}|0\rangle \\
&= \frac{1}{\sqrt{n}}\sqrt{\frac{m\omega}{2\hbar}}\langle x|\left(\hat{X} - \frac{i}{m\omega}\hat{P}\right)|n-1\rangle \\
&= \frac{1}{\sqrt{n}}\sqrt{\frac{m\omega}{2\hbar}}\left(x\langle x \mid n-1\rangle - \frac{i}{m\omega}\left(-i\hbar\frac{d}{dx}\right)\langle x \mid n-1\rangle\right) \\
&= \sqrt{\frac{m\omega}{2n\hbar}}\left(x - \frac{\hbar}{m\omega}\frac{d}{dx}\right)\psi_{n-1}(x)
\end{aligned}$$

Therefore, we can find all states by iterating this operator,

$$\psi_n(x') = \left(\frac{m\omega}{2n\hbar}\right)^{\frac{n}{2}}\left(x - \frac{\hbar}{m\omega}\frac{\partial}{\partial x}\right)^n \psi_0(x)$$

The result is a series of polynomials, the Hermite polynomials, times the Gaussian factor.

Q8: Find $\psi_1(x)$ and $\psi_2(x)$.

6 Time evolution of a mixed state of the oscillator

Consider the time evolution of the most general superposition of the lowest two eigenstates

$$|\psi\rangle = \cos\theta\,|0\rangle + e^{i\varphi}\sin\theta\,|1\rangle$$

Applying the time translation operator with $t_0 = 0$,

$$\begin{aligned}
|\psi, t\rangle &= \mathcal{U}(t)|\psi\rangle \\
&= e^{-\frac{i}{\hbar}\hat{H}t}|\psi\rangle \\
&= \cos\theta e^{-\frac{i}{\hbar}\hat{H}t}|0\rangle + e^{i\varphi}\sin\theta e^{-\frac{i}{\hbar}\hat{H}t}|1\rangle \\
&= \cos\theta e^{-\frac{i}{2}\omega t}|0\rangle + e^{i\varphi}\sin\theta e^{-\frac{3}{2}i\omega t}|1\rangle \\
&= e^{-\frac{i}{2}\omega t}\left(\cos\theta|0\rangle + e^{i\varphi}\sin\theta e^{-i\omega t}|1\rangle\right)
\end{aligned}$$

Now look at the time dependence of the expectation value of the position operator, which we write in terms of raising and lowering operators as $\hat{X} = \sqrt{\frac{\hbar}{2m\omega}}\left(\hat{a} + \hat{a}^\dagger\right)$. Remembering that the states are orthonormal,

$$\begin{aligned}
\langle\psi, t|\hat{X}|\psi, t\rangle &= \left(\cos\theta\langle 0| + e^{-i\varphi}\sin\theta e^{i\omega t}\langle 1|\right)e^{\frac{i}{2}\omega t}\hat{X}e^{-\frac{i}{2}\omega t}\left(\cos\theta|0\rangle + e^{i\varphi}\sin\theta e^{-i\omega t}|1\rangle\right) \\
&= \sqrt{\frac{\hbar}{2m\omega}}\left(\cos\theta\langle 0| + e^{-i\varphi}\sin\theta e^{i\omega t}\langle 1|\right)\left(\hat{a} + \hat{a}^\dagger\right)\left(\cos\theta|0\rangle + e^{i\varphi}\sin\theta e^{-i\omega t}|1\rangle\right) \\
&= \sqrt{\frac{\hbar}{2m\omega}}\left(\cos\theta\langle 0| + e^{-i\varphi}\sin\theta e^{i\omega t}\langle 1|\right)\left(\cos\theta|1\rangle + e^{i\varphi}\sin\theta e^{-i\omega t}\left(|0\rangle + \sqrt{2}|2\rangle\right)\right) \\
&= \sqrt{\frac{\hbar}{2m\omega}}\left(\cos\theta\sin\theta e^{-i(\omega t-\varphi)} + \sin\theta\cos\theta e^{i(\omega t-\varphi)}\right) \\
&= \sqrt{\frac{\hbar}{2m\omega}}\sin 2\theta\cos(\omega t - \varphi)
\end{aligned}$$

where we have used $\hat{a}|0\rangle = 0, \hat{a}|1\rangle = |0\rangle, \hat{a}^\dagger|0\rangle = |1\rangle$ and $\hat{a}^\dagger|1\rangle = \sqrt{2}|2\rangle$. We see that the expected position oscillates back and forth between $\pm\sqrt{\frac{\hbar}{2m\omega}}\sin 2\theta$ with frequency ω. Superpositions involving higher excited states will bring in harmonics, $n\omega$, and will then allow for varied traveling waveforms.

Q9: Find the expectation value of the momentum, $\hat{P} = \frac{1}{2i}\sqrt{2\hbar m\omega}\left(\hat{a} - \hat{a}^\dagger\right)$

Q10: Find the uncertainty in position, $(\Delta X)^2 = \left\langle\left(\hat{X} - \left\langle\hat{X}\right\rangle\right)^2\right\rangle = \left\langle\hat{X}^2\right\rangle - \left\langle\hat{X}\right\rangle^2$

7 Coherent states and the correspondence principle

Any alternative physical theory must approach, in some suitable limit, the previously tested and accepted standard model. Such a limit was stated concretely for quantum mechanics by Niels Bohr in 1920. The *correspondence principle* states that in the limit of large numbers of quanta, quantum systems should replicate classical behavior. We can see this for the simple harmonic oscillator by finding states that oscillate with the natural frequency ω and its overtones. Schrödinger found that such states are given by minimum uncertainty wave packets called *coherent states*.

7.1 The uncertainty in position and momentum

Schrödinger asked for simple harmonic oscillator states, $|\lambda\rangle$, for with the dimensionless forms of the position and momentum operators,

$$\hat{X}_0 \equiv \sqrt{\frac{m\omega}{2\hbar}}\hat{X} = \frac{1}{2}\left(\hat{a} + \hat{a}^\dagger\right)$$

$$\hat{P}_0 \equiv \sqrt{\frac{1}{2\hbar m\omega}}\,\hat{P} = \frac{1}{2i}\left(\hat{a} - \hat{a}^\dagger\right)$$

have equal uncertainty,

$$\Delta\hat{X}_0 = \Delta\hat{P}_0$$

and which achieve the minimum value in the uncertainty relation,

$$\Delta\hat{X}_0\Delta\hat{P}_0 \geq \frac{\hbar}{2}$$

The uncertainty in any operator in a state $|\lambda\rangle$ is defined to be $\Delta\hat{A}$ where

$$\begin{aligned}
\left(\Delta\hat{A}\right)^2 &\equiv \langle\lambda|\left(\hat{A} - \left\langle\hat{A}\right\rangle\right)^2|\lambda\rangle \\
&= \langle\lambda|\left(\hat{A}^2 - 2\hat{A}\left\langle\hat{A}\right\rangle + \left\langle\hat{A}\right\rangle^2\right)|\lambda\rangle \\
&= \langle\lambda|\,\hat{A}^2\,|\lambda\rangle - 2\langle\lambda|\,\hat{A}\,|\lambda\rangle\left\langle\hat{A}\right\rangle + \left\langle\hat{A}\right\rangle^2 \\
&= \left\langle\hat{A}^2\right\rangle - \left\langle\hat{A}\right\rangle^2
\end{aligned}$$

where we define $\left\langle\hat{A}\right\rangle \equiv \langle\lambda|\,\hat{A}\,|\lambda\rangle$. The uncertainty for position is then found from the expection value of $\hat{X}_0$

$$\begin{aligned}
\left\langle\hat{X}_0\right\rangle &= \frac{1}{2}\langle\lambda|\left(\hat{a} + \hat{a}^\dagger\right)|\lambda\rangle \\
&= \frac{1}{2}\left(\langle\hat{a}\rangle + \langle\hat{a}^\dagger\rangle\right)
\end{aligned}$$

and the expectation value of $\hat{X}_0^2$,

$$\begin{aligned}
\left\langle\hat{X}_0^2\right\rangle &= \left\langle\frac{1}{2}\left(\hat{a} + \hat{a}^\dagger\right)^2\right\rangle \\
&= \frac{1}{4}\left\langle\left(\hat{a}^2 + \hat{a}\hat{a}^\dagger + \hat{a}^\dagger\hat{a} + \hat{a}^{\dagger 2}\right)\right\rangle \\
&= \frac{1}{4}\left\langle\left(\hat{a}^2 + \left[\hat{a}, \hat{a}^\dagger\right] + 2\hat{N} + \hat{a}^{\dagger 2}\right)\right\rangle \\
&= \frac{1}{4}\left\langle\left(\langle\hat{a}^2\rangle + 2\left(\hat{N} + \frac{1}{2}\right) + \langle\hat{a}^{\dagger 2}\rangle\right)\right\rangle \\
&= \frac{1}{4}\left(\langle\hat{a}^2\rangle + \frac{2}{\hbar\omega}\left\langle\hat{H}\right\rangle + \langle\hat{a}^{\dagger 2}\rangle\right)
\end{aligned}$$

to be given by

$$\begin{aligned}
\left(\Delta\hat{X}_0\right)^2 &= \frac{1}{4}\left(\langle\hat{a}^2\rangle + \frac{2}{\hbar\omega}\left\langle\hat{H}\right\rangle + \langle\hat{a}^{\dagger 2}\rangle\right) - \frac{1}{4}\left(\langle\hat{a}\rangle^2 + 2\langle\hat{a}\rangle\langle\hat{a}^\dagger\rangle + \langle\hat{a}^\dagger\rangle^2\right) \\
&= \frac{1}{4}\left(\langle\hat{a}^2\rangle + \frac{2}{\hbar\omega}\left\langle\hat{H}\right\rangle + \langle\hat{a}^{\dagger 2}\rangle - \langle\hat{a}\rangle^2 - 2\langle\hat{a}\rangle\langle\hat{a}^\dagger\rangle - \langle\hat{a}^\dagger\rangle^2\right) \\
&= \frac{1}{4}\left(\frac{2}{\hbar\omega}\left\langle\hat{H}\right\rangle + (\Delta\hat{a})^2 + \left(\Delta\hat{a}^\dagger\right)^2 - 2\langle\hat{a}\rangle\langle\hat{a}^\dagger\rangle\right)
\end{aligned}$$

Similarly, we find $\left\langle\hat{P}_0\right\rangle, \left\langle\hat{P}_0^2\right\rangle$ and $\left(\Delta\hat{P}_0\right)^2$,

$$\left\langle\hat{P}_0\right\rangle = \frac{1}{2i}\langle\lambda|\left(\hat{a} - \hat{a}^\dagger\right)|\lambda\rangle$$

$$
\begin{aligned}
&= \frac{1}{2i}\langle \hat{a}\rangle - \frac{1}{2i}\left\langle \hat{a}^\dagger\right\rangle \\
\left\langle \hat{P}_0^2\right\rangle &= -\frac{1}{4}\langle\lambda|\left(\hat{a}-\hat{a}^\dagger\right)^2|\lambda\rangle \\
&= -\frac{1}{4}\langle\lambda|\left(\hat{a}\hat{a}-2\hat{N}-1+\hat{a}^\dagger\hat{a}^\dagger\right)|\lambda\rangle \\
&= \frac{1}{4}\left(\frac{2}{\hbar\omega}\left\langle\hat{H}\right\rangle - \left\langle\hat{a}^2\right\rangle - \left\langle\hat{a}^{\dagger 2}\right\rangle\right) \\
\left(\Delta\hat{P}_0\right)^2 &= \frac{1}{4}\left(\frac{2}{\hbar\omega}\left\langle\hat{H}\right\rangle - \left\langle\hat{a}^2\right\rangle - \left\langle\hat{a}^{\dagger 2}\right\rangle\right) + \frac{1}{4}\left(\langle\hat{a}\rangle^2 - 2\langle\hat{a}\rangle\left\langle\hat{a}^\dagger\right\rangle + \left\langle\hat{a}^\dagger\right\rangle^2\right) \\
&= \frac{1}{4}\left(\frac{2}{\hbar\omega}\left\langle\hat{H}\right\rangle - \left\langle\hat{a}^2\right\rangle - \left\langle\hat{a}^{\dagger 2}\right\rangle + \langle\hat{a}\rangle^2 - 2\langle\hat{a}\rangle\left\langle\hat{a}^\dagger\right\rangle + \left\langle\hat{a}^\dagger\right\rangle^2\right) \\
&= \frac{1}{4}\left(\frac{2}{\hbar\omega}\left\langle\hat{H}\right\rangle - \left\langle\hat{a}^2\right\rangle + \langle\hat{a}\rangle^2 - \left\langle\hat{a}^{\dagger 2}\right\rangle + \left\langle\hat{a}^\dagger\right\rangle^2 - 2\langle\hat{a}\rangle\left\langle\hat{a}^\dagger\right\rangle\right) \\
&= \frac{1}{4}\left(\frac{2}{\hbar\omega}\left\langle\hat{H}\right\rangle - (\Delta\hat{a})^2 - \left(\Delta\hat{a}^\dagger\right)^2 - 2\langle\hat{a}\rangle\left\langle\hat{a}^\dagger\right\rangle\right)
\end{aligned}
$$

7.2 Equality of uncertainties and the minimum uncertainty state

We demand equality of the uncertainties, $\left(\Delta\hat{X}_0\right)^2 = \left(\Delta\hat{P}_0\right)^2$. Imposing this,

$$
\begin{aligned}
\frac{2}{\hbar\omega}\left\langle\hat{H}\right\rangle + (\Delta\hat{a})^2 + \left(\Delta\hat{a}^\dagger\right)^2 - 2\langle\hat{a}\rangle\left\langle\hat{a}^\dagger\right\rangle &= \frac{2}{\hbar\omega}\left\langle\hat{H}\right\rangle - (\Delta\hat{a})^2 - \left(\Delta\hat{a}^\dagger\right)^2 - 2\langle\hat{a}\rangle\left\langle\hat{a}^\dagger\right\rangle \\
(\Delta\hat{a})^2 + \left(\Delta\hat{a}^\dagger\right)^2 &= -(\Delta\hat{a})^2 - \left(\Delta\hat{a}^\dagger\right)^2
\end{aligned}
$$

and since both $(\Delta\hat{a})^2$ and $\left(\Delta\hat{a}^\dagger\right)^2$ are nonnegative, both must vanish. Setting

$$
\begin{aligned}
(\Delta\hat{a})^2 &= 0 \\
\left\langle\hat{a}^2\right\rangle &= \langle\hat{a}\rangle^2
\end{aligned}
$$

There are many ways to accomplish this equality, but the simplest choice is to let $|\lambda\rangle$ be an eigenstate of $\hat{a}$,

$$\hat{a}|\lambda\rangle = \lambda|\lambda\rangle$$

since this always makes the dispersion vanish. It is easy to show that this choice also gives miniumum uncertainty.

We also need $\left(\Delta\hat{a}^\dagger\right)^2 = 0$, but this follows automatically,

$$
\begin{aligned}
\left\langle\hat{a}^{\dagger 2}\right\rangle - \left\langle\hat{a}^\dagger\right\rangle^2 &= \langle\lambda|\left(\hat{a}^\dagger - \left\langle\hat{a}^\dagger\right\rangle\right)^2|\lambda\rangle \\
&= \langle\lambda|\hat{a}^\dagger\hat{a}^\dagger|\lambda\rangle - \left\langle\hat{a}^\dagger\right\rangle^2 \\
&= \langle\lambda|\hat{a}\hat{a}|\lambda\rangle^* - \left\langle\hat{a}^\dagger\right\rangle^2 \\
&= \lambda^{*2} - \langle\hat{a}\rangle^{*2} \\
&= \lambda^{*2} - \lambda^{*2} \\
&= 0
\end{aligned}
$$

The condition

$$\hat{a}|\lambda\rangle = \lambda|\lambda\rangle$$

is therefore sufficient to guarantee that the uncertainties in position and momentum will be equal.

When this condition holds, the uncertainties are

$$\left(\Delta\hat{X}_0\right)^2 = \frac{1}{2}\left(\frac{1}{\hbar\omega}\left\langle\hat{H}\right\rangle - \langle\hat{a}\rangle\left\langle\hat{a}^\dagger\right\rangle\right)$$
$$\left(\Delta\hat{P}_0\right)^2 = \frac{1}{2}\left(\frac{1}{\hbar\omega}\left\langle\hat{E}\right\rangle - \langle\hat{a}\rangle\left\langle\hat{a}^\dagger\right\rangle\right)$$

with product

$$\left(\Delta\hat{X}_0\right)^2\left(\Delta\hat{P}_0\right)^2 = \frac{1}{4}\left(\lambda^*\lambda + \frac{1}{2} - \lambda^*\lambda\right)^2$$
$$\Delta\hat{X}_0\Delta\hat{P}_0 = \frac{1}{4}$$

and putting the units back in terms of position and momentum,

$$\Delta\hat{X}\Delta\hat{P} = \frac{\hbar}{2}$$

so this condition also gives a minimum uncertainty wave packet.

7.3 Coherent states

We define a coherent state of the harmonic oscillator to be an eigenstate of the lowering operator,

$$\hat{a}\left|\lambda\right\rangle = \lambda\left|\lambda\right\rangle$$

To find such states explicitly, expand in the complete number basis,

$$\left|\lambda\right\rangle = \sum_{n=0}^{\infty} c_n\left|n\right\rangle$$

Q11: Find the form of coherent states. Writing

$$\hat{a}\sum_n c_n\left|n\right\rangle = \lambda\sum_n c_n\left|n\right\rangle$$

work out the effect of $\hat{a}$ on the left to show that

$$\sum_{n=1}^{\infty} c_n\sqrt{n}\left|n-1\right\rangle = \sum_{m=0}^{\infty} c_{m+1}\sqrt{m+1}\left|m\right\rangle$$

This gives a recursion relation for the coefficients c_n. Shifting the summation index to $m = n-1$ on the left,

$$\sum_{n=1}^{\infty} c_n\sqrt{n}\left|n-1\right\rangle = \sum_{m=0}^{\infty} c_{m+1}\sqrt{m+1}\left|m\right\rangle$$

and renaming $m = n$ on the right,

$$\sum_{m=0}^{\infty} c_{m+1}\sqrt{m+1}\left|m\right\rangle = \lambda\sum_{m=0}^{\infty} c_m\left|m\right\rangle$$

so that term by term equality gives

$$c_{m+1} = \frac{\lambda}{\sqrt{m+1}}c_m$$

Iterating this recursion relationship, we find

$$c_n = \frac{\lambda^n}{\sqrt{n!}}$$

for all n. The coherent state is therefore given by

$$|\lambda\rangle = \sum_n \frac{\lambda^n}{\sqrt{n!}} |n\rangle \tag{1}$$

Finally, substituting the expression for the n^{th} number state, $|n\rangle = \frac{1}{\sqrt{n!}} \left(\hat{a}^\dagger\right)^n |0\rangle$, we express the eigenstates of the lowering operator as

$$\begin{aligned} |\lambda\rangle &= \sum_n \frac{\lambda^n}{n!} \left(\hat{a}^\dagger\right)^n |0\rangle \\ &= e^{\lambda \hat{a}^\dagger} |0\rangle \end{aligned} \tag{2}$$

These are the coherent states of the simple harmonic oscillator. They correspond to a Poisson distribution of number states.

Q12: Normalize the states. Show using eq.(1) that normalized coherent states are given by

$$e^{-\frac{1}{2}|\lambda|^2} e^{\lambda \hat{a}^\dagger} |0\rangle$$

7.4 Time dependence of coherent states

Q13: Show that the time dependence of a coherenet state, given by acting with $\hat{U}(t, t_0)$.

Setting $t_0 = 0$ and $|\lambda, t_0\rangle = |\lambda\rangle$ show that

$$|\lambda, t\rangle = e^{-\frac{i\omega t}{2}} \left|\lambda e^{-i\omega t}\right\rangle$$

that is, up to an overall phase the complex parameter λ is just replaced by $\lambda e^{-i\omega t}$ in the original state.

Q14: Show that the expectation values of $\hat{X}$ and $\hat{P}$ oscillate harmonically just as their classical counterparts. (If you like, show that the wave function is a Gaussian centered on $\left\langle \hat{X} \right\rangle$!)