Neutrino oscillations

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Energy is produced in the sun through a series of nuclear interactions in which two protons collide to produce deuterium. This requires one of the protons to emit a positron and a neutrino. Subsequent reactions produce helium3, then helium. This results in a huge neutrino flux, which was first detected and measured by R. Davis, Jr and J. N. Bahcall in 1970. The *solar neutrino problem* was an immediate outcome: the number of neutrinos detected was only $\frac{1}{3}$ the number predicted from solar models.

The problem was only resolved definitively after decades of careful neutrino experiments. The Davis-Bahcall experiment was sensitive only to electron neutrinos, but we have now detected three distinct types of neutrino, associated with three leptons – the electron, the muon, and the taon. The key to understanding the factor of $\frac{1}{3}$ is that these three can oscillate into one another. Raymond Davis shared the 2002 Nobel Prize in Physics with Masatoshi Koshiba and Riccardo Giacconi for their contributions to neutrino astrophysics.

The key to neutrino oscillations is that the *flavor* eigenstates (electron, mu, tau) are different from the *energy* eigenstates. In deuterium production in the sun, a positron (anti-electron) is produced, so the neutrino is produced in the flavor eigenstate of the electron, $|\nu_e\rangle$. If neutrinos had equal (or zero) mass, this is what would be detected at Earth. However, each flavor eigenstate is a linear combination of the three energy eigenstates so in the time evolution of the initial $|\nu_e\rangle$ state the combination of energy eigenstates changes, giving an admixture of the other two flavors.

Concretely, the time evolution of three types of neutrino with flavor eigenstates$|\nu_i\rangle = (|\nu_e\rangle, |\nu_\mu\rangle, |\nu_\tau\rangle)$ is governed by the energy eigenstates $|m_i\rangle = (|m_1\rangle, |m_2\rangle, |m_3\rangle)$. In the standard model, the mixing between these states, $|\nu_i\rangle = M_{ij}|m_j\rangle$is given as a 3×3 unitary matrix, the Pontecorvo–Maki–Nakagawa–Sakata matrix (PMNS matrix), written in the form of a rotation in each of three orthogonal planes and a phase:

$$\begin{aligned} M &= R_x(\theta_{23}) R_y(\theta_{13}, \delta_{CP}) R_z(\theta_{12}) \\ &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & c_{23} & s_{23} \\ 0 & -s_{23} & c_{23} \end{bmatrix} \begin{bmatrix} c_{13} & 0 & s_{13}e^{-i\delta_{CP}} \\ 0 & 1 & 0 \\ -s_{13}e^{i\delta_{CP}} & 0 & c_{13} \end{bmatrix} \begin{bmatrix} c_{12} & s_{12} & 0 \\ -s_{12} & c_{12} & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta_{CP}} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta_{CP}} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta_{CP}} & s_{23}c_{13} \\ s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta_{CP}} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\delta_{CP}} & c_{23}c_{13} \end{bmatrix} \end{aligned}$$

Because neutrinos have different masses, the different mass eigenstates have different kinetic energies and therefore propagate at slightly different velocities (all very close to c, since the masses are tiny).

We restrict our discussion to the case for two neutrinos; the addition of a third is analogous. For more detail, read http://en.wikipedia.org/wiki/Neutrino_oscillation.

1 Neutrino flavor eigenstates

Consider the time evolution of just a pair of neutrinos. The flavor eigenstates and mass eigenstates are related by a submatrix of the PMNS matrix,

$$\begin{aligned} |\nu_e\rangle &= \cos\theta\,|m_1\rangle + \sin\theta\,|m_2\rangle \\ |\nu_\mu\rangle &= -\sin\theta\,|m_1\rangle + \cos\theta\,|m_2\rangle \end{aligned}$$

The angle θ is called a mixing angle.

2 Time dependence of the states

The time evolution of states is governed by the Hamiltonian. For each energy eigenstate

$$\hat{H}\,|m_i\rangle = E_i\,|m_i\rangle$$

Taking the spatial dependence to be a plane wave, $\langle \mathbf{x} \mid m_i, 0\rangle = e^{\frac{i}{\hbar}\mathbf{p}_i\cdot\mathbf{x}}$, the time evolution is given by

$$\begin{aligned} |m_i, \mathbf{p}_i, t\rangle &= e^{\frac{i}{\hbar}\mathbf{p}_i\cdot\mathbf{x}} e^{-\frac{i}{\hbar}\hat{H}t}\,|m_i, 0\rangle \\ &= e^{\frac{i}{\hbar}\mathbf{p}_i\cdot\mathbf{x}} e^{-\frac{i}{\hbar}E_i t}\,|m_i, 0\rangle \end{aligned}$$

Here, even though we choose a position basis for the state, there remain the flavor or mass degrees of freedom in the final ket $|m_1, 0\rangle$.

Neutrinos are ultra-relativistic $p \gg mc$, so the mass-energy relation may be approximated,

$$\begin{aligned} E &= \sqrt{p^2c^2 + m^2c^4} \\ &= pc\sqrt{1 + \frac{m^2c^2}{p^2}} \\ &\approx pc + \frac{m^2c^3}{2p} \\ &\approx E + \frac{m^2c^4}{2E} \end{aligned}$$

so the evolution over a distance $L \approx ct$ becomes

$$\begin{aligned} |m_i, t\rangle &= e^{\frac{i}{\hbar}\left(\frac{E_i L}{c} - \left(E_i + \frac{m_i^2c^4}{2E_i}\right)t\right)}\,|m_i, 0\rangle \\ &= e^{\left(\frac{i}{\hbar}\frac{E_i L}{c} - \frac{i}{\hbar}E_i\frac{L}{c} - \frac{i}{\hbar}\frac{m_i^2c^4}{2E_i}\frac{L}{c}\right)}\,|m_i, 0\rangle \\ &= \exp\left(-\frac{im_i^2c^3L}{2\hbar E_i}\right)|m_i, 0\rangle \end{aligned}$$

3 Mixing

Now consider the mixing. The original electron neutrino state evolves (in distance traveled) as

$$\begin{aligned} |\nu_e, t\rangle &= e^{-\frac{i}{\hbar}\hat{H}t}\,|\nu_e, 0\rangle \\ & \cos\theta e^{\frac{i}{\hbar}\mathbf{p}_1\cdot\mathbf{x}} e^{-\frac{i}{\hbar}\hat{H}t}\,|m_1, 0\rangle + \sin\theta e^{\frac{i}{\hbar}\mathbf{p}_2\cdot\mathbf{x}} e^{-\frac{i}{\hbar}\hat{H}t}\,|m_2, 0\rangle \\ &= \cos\theta e^{\frac{i}{\hbar}\mathbf{p}_i\cdot\mathbf{x}} e^{-\frac{im_1^2c^3L}{2\hbar E}}\,|m_1, 0\rangle + \sin\theta e^{\frac{i}{\hbar}\mathbf{p}_2\cdot\mathbf{x}} e^{-\frac{im_2^2c^3L}{2\hbar E}}\,|m_2, 0\rangle \end{aligned}$$

so the probability amplitude for detecting a muon neutrino after a distance L is

$$\begin{aligned}
\langle \nu_\mu \mid \nu_e, t\rangle &= \langle \nu_\mu | \left(\cos\theta e^{-\frac{i m_1^2 c^3 L}{2\hbar E}} |m_1, 0\rangle + \sin\theta e^{-\frac{i m_2^2 c^3 L}{2\hbar E}} |m_2, 0\rangle \right) \\
&= \left(-\sin\theta \langle m_1, 0| + \cos\theta \langle m_2, 0|\right) \left(\cos\theta e^{-\frac{i m_1^2 c^3 L}{2\hbar E}} |m_1, 0\rangle + \sin\theta e^{-\frac{i m_2^2 c^3 L}{2\hbar E}} |m_2, 0\rangle \right) \\
&= -\sin\theta\cos\theta e^{-\frac{i m_1^2 c^3 L}{2\hbar E}} + \cos\theta\sin\theta e^{-\frac{i m_2^2 c^3 L}{2\hbar E}} \\
&= \sin\theta\cos\theta \left(e^{-\frac{i m_2^2 c^3 L}{2\hbar E}} - e^{-\frac{i m_1^2 c^3 L}{2\hbar E}} \right)
\end{aligned}$$

The probability is therefore,

$$\begin{aligned}
|\langle \nu_\mu \mid \nu_e, t\rangle|^2 &= \sin^2\theta\cos^2\theta \left(e^{-\frac{i m_2^2 c^3 L}{2\hbar E}} - e^{-\frac{i m_1^2 c^3 L}{2\hbar E}} \right) \left(e^{\frac{i m_2^2 c^3 L}{2\hbar E}} - e^{\frac{i m_1^2 c^3 L}{2\hbar E}} \right) \\
&= \sin^2\theta\cos^2\theta \left(e^{-\frac{i m_2^2 c^3 L}{2\hbar E}} e^{\frac{i m_2^2 c^3 L}{2\hbar E}} - e^{-\frac{i m_2^2 c^3 L}{2\hbar E}} e^{\frac{i m_1^2 c^3 L}{2\hbar E}} - e^{-\frac{i m_1^2 c^3 L}{2\hbar E}} e^{\frac{i m_2^2 c^3 L}{2\hbar E}} + e^{-\frac{i m_1^2 c^3 L}{2\hbar E}} e^{\frac{i m_1^2 c^3 L}{2\hbar E}} \right) \\
&= \sin^2\theta\cos^2\theta \left(2 - e^{\frac{i\left(m_1^2 - m_2^2\right) c^3 L}{2\hbar E}} - e^{\frac{-i\left(m_1^2 - m_2^2\right) c^3 L}{2\hbar E}} \right) \\
&= 2\sin^2\theta\cos^2\theta \left(1 - \cos\frac{\left(m_1^2 - m_2^2\right) c^3 L}{2\hbar E} \right) \\
&= \sin 2\theta \left(1 - \cos\frac{L}{2\hbar c E}\left(m_1^2 c^4 - m_2^2 c^4\right) \right)
\end{aligned}$$

If all neutrinos had the same, or zero, mass, this probability would vanish and solar neutrino detections at Earth would consist entirely of electron neutrinos. Similarly, if we had $\theta = 0$, the flavor and energy eigenstates would be identical and again, only electron neutrinos would be seen.

For the probability of detecting both neutrinos at Earth to be substantial, the cosine term must become appreciable, and for complete mixing, the argument of the cosine must become large, $\frac{L}{2\hbar c E}\left(m_1^2 c^4 - m_2^2 c^4\right) \gg 2\pi$. Since the energy of solar neutrinos is of order $10^6 eV$ and the Earth-sun distance is about 1.5×10^{11}meters, the difference in the squares of the neutrino masses needs to satisfy

$$\begin{aligned}
\frac{L}{2\hbar c E}\left(m_1^2 c^4 - m_2^2 c^4\right) &\gg 2\pi \\
m_1^2 c^4 - m_2^2 c^4 &\gg \frac{2\pi \times 2 \times 200 \times 10^{-15} MeV m 10^6 eV}{1.5 \times 10^{11}} \\
m_1^2 c^4 - m_2^2 c^4 &\gg 8.38 \times 10^{-12} eV^2
\end{aligned}$$

so that extremely small masses may still lead to substantial mixing.

The experimental value of the squared mass difference is $\left|m_1^2 c^4 - m_2^2 c^4\right| = 0.000079 eV^2$, while the energy is of order $10^6 eV$. Using the Earth-sun distance of $1.5 \times 10^{11} m$, we have

$$\begin{aligned}
\frac{\left(m_1^2 - m_2^2\right) c^3 L}{2\hbar E} &\approx \frac{\left(8\times 10^{-5}\right)\left(1.5\times 10^{11}\right)\left(1.6\times 10^{-19}\right)}{2\left(6.6\times 10^{-34}\right)\left(10^6\right)} \frac{(eV)\left(kgm^2/s^2\right)(m)(s)}{\left(m^2 kg/s\right)(eV)(m)} \\
&\sim 1.5 \times 10^{15}
\end{aligned}$$

so there are many oscillations between production in the sun and arrival at Earth.

The summed masses of the three neutrinos is known to satisfy

$$\sum_{i=1}^{3} m_i < 0.3 eV$$

and the squared-mass difference between the first two is

$$\left|m_2^2c^4 - m_1^2c^4\right| = 0.000079eV^2$$

If $m_2 = .1\,eV \lesssim .3\,eV$ then for m_1 we find

$$\begin{aligned} m_2^2c^4 - m_1^2c^4 &= 0.000079eV^2 \\ .01 - m_1^2c^4 &= 0.000079eV^2 \\ m_1^2c^4 &= (.01 - 0.000079)\,eV^2 \\ m_1c^2 &= .0996 \end{aligned}$$

At the other extreme, if m_1 is massless, then

$$\begin{aligned} m_2^2c^4 &= 0.000079eV^2 \\ m_2c^2 &= 0.0089eV \end{aligned}$$

placing m_1, m_2 in the ranges

$$\begin{aligned} 0 &\leq m_1 \lesssim 0.1\,eV \\ .0089\,eV &\leq m_2 \lesssim 0.1\,eV \end{aligned}$$